# Fixed Point Theorems in Right Dislocated and Left Dislocated Metric Spaces

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***Abstract*—Some interesting properties of Right dislocated and Left dislocated metrics are obtained. Using these properties the existence and uniqueness of fixed points for contractions and Kannan mappings over these spaces are derived.**

***Index Terms*—Metric space, dislocated metric space, fixed point, contractions, Kannan mapping.**

***MSC 2000 Codes* – 47H10, 54H25.**

## I. Introduction

THE increasing applications of fixed point theorems of metric spaces and their generalizations in various branches of engineering and other sciences attracted several researchers to work on them in recent past. For instance, the generalization of well known Banach Contraction Principle of metric space to the dislocated metric space proved by Pascal Hitzler and A. K. Seda, played a key role in the development of logic programming semantics(see [4]). This concept of dislocated metric space was further generalized into dislocated quasi, right dislocated, left dislocated metric spaces by M.A. Ahmed et.al (see [2],[3]). The purpose of this paper is to prove some interesting properties of right dislocated, left dislocated metric spaces and using these properties we derive right and as well as left dislocated metric versions of Banach's Contraction principle and Kannan's fixed point theorems.

## II. Basic Definitions, Examples and Preliminary Results

In what follows $\mathbb{R}$, $\mathbb{N}$ denote the sets of real, positive numbers respectively and $X$ a nonempty set. For undefined terms and basic results refer to ([1], [2]).

**Definition** *2.1:* ([5]) A real function $d: X \times X \to [0, \infty)$ is called a **distance** on $X$ and the corresponding pair $(X, d)$ is called a **distance space**.

**Definition** *2.2:* ([3],[1]) Let a distance $d$ on $X$ satisfy the following conditions:

(D1) $d(x, y) = d(y, x) = 0$ implies $x = y$,

(D2) $d(x, y) = d(y, x)$,

(DQ) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a **Dislocated metric** (d-metric for short) on $X$. If $d$ satisfies conditions (D1) and (DQ) then $d$ is called a **Dislocated quasi-metric** (dq-metric for short) on $X$.

**Definition** *2.3:* ([2]) A distance $d$ on $X$ satisfies conditions (D1) and

(RD) $d(x, y) \le d(x, z) + d(y, z)$ for all $x, y, z \in X$,

is called **Right dislocated metric** (rd-metric in short) on $X$ and the pair $(X, d)$ is called a **Right dislocated metric space** (in short rd-metric space).

Instead of the condition (RD) if $d$ satisfies

(LD) $d(x, y) \le d(z, x) + d(z, y)$ for all $x, y, z \in X$, then $d$ is called **Left dislocated metric** (ld-metric in short) on $X$ and the pair $(X, d)$ is called **Left dislocated metric space** (in short ld-metric space).

**Definition** *2.4:* ([3]) A sequence $(x_n)$ in a distance space $(X, d)$ **rd-coverges** (resp. **ld-converges**) to $x$ in $X$ if $\lim d(x, x_n) = 0$ (resp. $\lim d(x_n, x) = 0$).

In this case $x$ is called a rd-limit (resp. ld-limit) of $(x_n)$ and we write $\lim x_n = x$ in $(X, d)$.

**Definition** *2.5:* ([2]) A sequence $(x_n)$ in a rd-metric (resp. ld-metric) space $(X, d)$ is called **Cauchy** if for each $\epsilon > 0$ there exists a positive integer $N_\epsilon$ such that for all $m, n \ge N_\epsilon$, $d(x_m, x_n) < \epsilon$. $(X, d)$ is called a complete rd-metric (resp. ld-metric) space if every Cauchy sequence in $X$ is a rd-converges (resp. ld-converges) in $X$.

**Definition** *2.6:* A self map $T$ on a distance space $(X, d)$ is called

(i) a **Kannan mapping** on $X$ if

$$d(Tx, Ty) \le \alpha\{d(x, Tx) + d(y, Ty)\} \tag{1}$$

for all $x, y \in X$, and $0 \le \alpha < \frac{1}{2}$. $\alpha$ is called a Kannan contracting constant (in short KC constant).

(ii) a **Conjugate Kannan mapping** on $X$ if

$$d(Tx, Ty) \le \alpha\{d(Tx, x) + d(Ty, y)\} \tag{2}$$

for all $x, y \in X$, and $0 \le \alpha < \frac{1}{2}$. $\alpha$ is called a Conjugate Kannan contracting constant (in short CKC constant).

**Theorem** *2.7:* Let $d$ be a distance on $X$. If we define $d^*$ on $X \times X$ by $d^*(x, y) = d(y, x)$, then

(i) $d$ is a rd-metric on $X$ if and only if $d^*$ is a ld-metric on $X$.

(ii) $\lim d(x, x_n) = 0$ if and only if $\lim d^*(x_n, x) = 0$.

(iii) $(X, d)$ is a complete rd-metric space if and only if $(X, d^*)$ is a complete ld-metric space.

(iv) $T$ is a contraction on $(X, d)$ if and only if $T$ is a contraction on $(X, d^*)$.

(v) $T$ is a Kannan mapping on $(X, d)$ if and only if $T$ is a Conjugate Kannan map on $(X, d^*)$.

*Proof:* Routine. □

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**Examples** *2.8:* 1) It is clear that every metric space and d-metric spaces are rd-metric space and ld-metric spaces.

2) Let $X = \{0\} \cup \{\frac{1}{n}: n \in \mathbb{N}\}$. Define

$$d(0,0) = 0, d(0, \frac{1}{n}) = 0, d(\frac{1}{n}, 0) = 1, d(\frac{1}{n}, \frac{1}{m}) = 1$$

for all $n, m \in \mathbb{N}$. Then $(X, d)$ is a complete rd-metric space in which the sequence $(\frac{1}{n})$ rd-converges to 0 and it is not a Cauchy sequence in $X$. Also, $d$ is not a d-metric on $X$.

3) Let $X = \mathbb{N}$. Define $d(n, m) = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $(X, d)$ is a rd-metric space in which $(n)$ is Cauchy and it is not a rd-convergent sequence in $X$.

4) Let $d$ be as in example 2.8 (2) and $d^*$ defined as in Theorem 2.7. Then $d^*$ is a ld-metric on $X$. □

**Remark** *1:* If $x$ is a ld-limit (resp. rd-limit) of $(x_n)$ in a ld-metric (resp. rd-metric) space $(X, d)$, then $(x_n)$ is Cauchy in $X$ if and only if $\lim d(x, x_n) = 0$ (resp. $\lim d(x_n, x) = 0$). □

**Notation** *1:* In a distance space $(X, d)$ we will denote the set $\{x \in X: d(x, x) = 0\}$ by $X_0$. □

**Example** *2.9:* The set $X_0$ in Example 2.8 (2) is $\{0\}$ whereas in Example 2.8 (3) it is empty. □

**Result** *2.10:* Let $(X, d)$ be a rd-metric space and $(x_n)$ a sequence in $X$. Then
(i) if $x \in X$ is a rd-limit of $(x_n)$, then $x \in X_0$,
(ii)$(X_0, d)$ is a metric space if $X_0$ is a nonempty set.

*Proof:* (ii) is clear. We prove (i): Let $x$ be a rd-limit of a sequence $(x_n)$ in $X$. Let $\epsilon > 0$. Then there exists a $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$, $0 \leq d(x, x) \leq d(x, x_n) + d(x, x_n) < \epsilon$. Since $\epsilon > 0$ is arbitrary, $d(x, x) = 0$. □

In a similar manner, One can easily prove the ld-metric analogue of Result 2.10. □

## III. Main Results

Unless specified otherwise throughout this section $(X, d)$ will denote a complete rd-metric space.

**Lemma** *3.1:* If $X_0$ is a nonempty set, then $(X_0, d)$ is a complete metric space.

*Proof:* By (ii) of Result 2.10, $d$ is a metric on $X_0$. Let $(x_n)$ be a Cauchy sequence in $X_0$. Then there exists $u \in X$ such that $(x_n)$ rd-converges to $u$ in $X$. By (i) of Result 2.10, $u \in X_0$. Hence $(X_0, d)$ is complete. □

**Lemma** *3.2:* Let $T$ be a contraction on $X$ with a contracting constant $\lambda$. Then for each $x \in X$ the sequence $(T^n x)$ is a Cauchy sequence in $X$.

*Proof:* Let $x \in X$ and $x_n = T^n x$ for $n \in \mathbb{N}$. Then $d(x_n, x_{n+1}) = d(T^n x, T^{n+1} x) \leq \lambda d(T^{n-1} x, T^n x) \leq ... \leq \lambda^n d(x, Tx)$.
Now $d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ..... + d(x_{n+k-1}, x_{n+k}) + d(x_{n+k}, x_{n+k})$
$\leq (\lambda^n + \lambda^{n+1} + ... + \lambda^{n+k-1}) d(x, Tx) + \lambda^{n+k} d(x, x)$
$< \frac{\lambda^n}{1-\lambda} \max\{d(x, x), d(x, Tx)\}$.
Assume that $d(x, Tx) > 0$. Letting $n \to \infty$, $(x_n)$ is a Cauchy sequence in $X$. Also, if $d(x, Tx) = 0$ then $d(x, x) \leq d(x, Tx) + d(x, Tx) = 0$. Therefore $d(x_n, x_{n+p}) = 0$ for all $n, p > 0$ and hence $(x_n)$ is a Cauchy sequence in $X$. □

We now present a simple proof of rd-metric version of Banach's Contraction principle.

**Theorem** *3.3:* Let $T$ be a contraction on $X$ with contracting constant $\lambda$. Then $T$ has a unique fixed point in $X$.

*Proof:* Let $x \in X$ and $x_n = T^n x$ for $n \in \mathbb{N}$.

By Lemma 3.2, $(x_n)$ is Cauchy in $X$. Since $X$ is complete, there exists $u \in X$ such that $\lim x_n = u$.

By Result 2.10 (i) and Lemma 3.1, $u \in X_0$ and $(X_0, d)$ is a complete metric space.

Let $x \in X_0$. Then $d(T(x), T(x)) \leq \lambda d(x, x) = 0$ and hence $T(X_0) \subseteq X_0$. Therefore $T$ is a contraction on $X_0$ with respect to $d$. Hence by Banach contraction principle for metric spaces, $T$ has a unique fixed point $x$ in $X_0$.

If $y$ is a fixed point of $T$ in $X$, then $d(x, y) = d(Tx, Ty) \leq \lambda d(x, y)$. Since $0 \leq \lambda < 1$, $d(x, y) = 0$. By symmetry, $d(y, x) = 0$. Therefore $x = y$. □

**Corollary** *3.4:* (ld-metric version of Banach Contraction Priniciple): If $T$ is a contraction on a complete ld-metric space $(X, d)$ with a contracting constant $\lambda$, then $T$ has a unique fixed point in $X$.

*Proof:* This is a straightforward consequence of (iii) and (iv) of Theorem 2.7 followed by Theorem 3.3. □

We now prove rd-metric version of Kannan's mapping theorem.

**Theorem** *3.5:* Let $T$ be a Kannan mapping on $X$ with KC constant $\alpha$. Then $T$ has a unique fixed point in $X$.

*Proof:* Let $x \in X$ and write $x_n = T^n x$ for $n \in \mathbb{N}$.
Then $d(x_1, x_2) = d(Tx, Tx_1) \leq \alpha\{d(x, Tx) + d(x_1, Tx_1)\}$
$\Rightarrow (1-\alpha) d(x_1, x_2) \leq \alpha d(x, x_1)$
$\Rightarrow d(x_1, x_2) \leq \frac{\alpha}{(1-\alpha)} d(x, x_1)$
An inductive argument yields

$$d(x_n, x_{n+1}) \leq \lambda^n d(x, x_1) \tag{3}$$

where $\lambda = \frac{\alpha}{(1-\alpha)}$.
Also,

$$d(x_n, x_n) \leq 2\alpha\lambda^n d(x, x_1) \tag{4}$$

where $\lambda = \frac{\alpha}{(1-\alpha)}$.
Now $d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ..... + d(x_{n+k-1}, x_{n+k}) + d(x_{n+k}, x_{n+k}) \leq (\lambda^n + \lambda^{n+1} + ... + \lambda^{n+k-1}) d(x, x_1) + \lambda^{n+k} d(x, x_1) < \frac{\lambda^n}{1-\lambda} d(x, x_1)$ for $k > n \geq 1$.
Since $0 \leq \lambda < 1$, $(x_n)$ is a Cauchy sequence in $X$ and hence $\lim d(u, x_n) = 0$ for some $u \in X$.
We show that $u$ is a fixed point of $T$.
Now, $d(u, Tu) \leq d(u, x_n) + d(Tu, x_n) = d(u, x_n) + d(Tu, Tx_{n-1}) \leq d(u, x_n) + \alpha\{d(x_{n-1}, x_n) + d(u, Tu)\} \leq d(u, x_n) + \alpha d(u, Tu) + \alpha\lambda^{n-1} d(x, x_1)$.
$\Rightarrow 0 \leq d(u, Tu) \leq \frac{1}{1-\alpha} d(u, x_n) + \lambda^n d(x, x_1)$.
Letting $n \to \infty$, we get $d(u, Tu) = 0$.
Since $d(u, Tu) = 0$, $d(Tu, u) \leq d(Tu, Tu) + d(u, Tu) \leq \alpha\{d(u, Tu) + d(u, Tu)\} = 0$. Therefore, $d(Tu, u) = d(u, Tu) = 0$ and hence $Tu = u$.
Uniqueness follows as in Theorem 5 of [1]. □

**Corollary** *3.6:* Let $T$ be a Conjugate Kannan mapping on a complete ld-metric space $(X, d^*)$ with CKC constant $\alpha$. Then $T$ has a unique fixed point in $X$.

*Proof:* It follows from (iii), (v) of Theorem 2.7 and Theorem 3.5. □

**Theorem** *3.7:* Let $T$ be Kannan mapping on a complete ld-metric space $(X, d)$ with KC constant $\alpha$. Then $T$ has a unique fixed point in $X$.

*Proof:* Let $x \in X$ and write $x_n = T^n x$ for $n \in \mathbb{N}$. Then

$$d(x_n, x_{n+1}) \leq \lambda^n d(x, x_1) \tag{5}$$

and

$$d(x_n, x_n) \leq 2\alpha\lambda^n d(x, x_1) \tag{6}$$

where $\lambda = \frac{\alpha}{(1-\alpha)}$.
Also, $d(x_n, x_{n+k}) < \frac{\lambda^{n-1}}{1-\lambda} d(x, x_1)$ for $k > n \geq 1$.
Since $0 \leq \lambda < 1$, $\{T^n x\}$ is Cauchy in $X$ and since $X$ is complete, there exists a $u \in X$ such that $\lim d(T^n x, u) = 0$.
Now, $d(u, Tu) \leq d(T^n x, u) + d(T^n x, Tu) = d(T^n x, u) + d(T(T^{n-1}x), Tu) \leq d(T^n x, u) + \alpha\{d(T^{n-1}x, T(T^{n-1}x)) + d(u, Tu)\} \leq d(T^n x, u) + \alpha d(u, Tu) + \alpha\lambda^{n-1} d(x, Tx)$.
$\Rightarrow 0 \leq d(u, Tu) \leq \frac{1}{1-\alpha} d(T^n x, u) + \lambda^n d(x, Tx)$.
Letting $n \to \infty$, we get $d(u, Tu) = 0$.
Also, $d(Tu, u) \leq d(u, u) + d(u, Tu) = 0$ (since $u$ is a ld-limit of $(x_n)$, $d(u, u) = 0$). Therefore, $d(Tu, u) = d(u, Tu) = 0$ and hence $Tu = u$.
Uniqueness is obvious.□

In view of (iii), (v) of Theorem 2.7 and Theorem 3.7, we have the following result.

**Corollary** *3.8:* Let $T$ be a Conjugate Kannan mapping on $(X, d)$ with CKC constant $\alpha$. Then $T$ has a unique fixed point in $X$. □

**Remark** *2:* Theorem 2.7 is useful in deriving the existence and uniqueness of fixed points for a self map $T$ on a complete rd-metric (resp. ld-metric) space $(X, d)$ satisfying a certain contractive inequality, as a consequence of corresponding theorems for $T$ on a complete ld-metric (resp. rd-metric) space $(X, d^*)$ satisfying its conjugate contractive inequality. This fact has been demonstrated in Corollary 3.6 and Corollary 3.8. However, we can also provide a direct proof to these corollaries by slight modification in proofs of Theorems 3.7 and 3.5 respectively. □

Finally, we provide examples to illustrate that the conditions (DQ), (RD) and (LD) of a distance on $X$ satisfying (D1) are independent.

**Example** *3.9:* Let $X = \{0, 1\}$, define $d(0, 0) = 0$, $d(0, 1) = 1$, $d(1, 0) = 0$, $d(1, 1) = 0$. Then $d$ is a dq-metric and it is not a rd-metric and ld-metrics on $X$. For this, $d(0, 1) > d(1, 0) + d(1, 1)$, $d(0, 1) > d(0, 0) + d(1, 0)$. □

**Example** *3.10:* Let $X = \{0, 1, 2\}$, define $d(0, 0) = d(1, 1) = d(1, 2) = 0$, $d(0, 1) = d(1, 0) = d(0, 2) = d(2, 0) = d(2, 1) = 1$, $d(2, 2) = 2$. Then $d$ is a rd-metric on $X$ and $d$ does not satisfy conditions (DQ) and (LD). To see this, $d(2, 2) > d(2, 1) + d(1, 2)$, $d(2, 2) > d(1, 2) + d(1, 2)$. Also, if we define $d^*$ on $X \times X$ by $d^*(x, y) = d(y, x)$, then $d^*$ is a ld-metric but it is not a rd-metric and dq-metric on $X$, which is evident from $d^*(2, 2) > d^*(2, 1) + d^*(2, 1)$, $d^*(2, 2) > d^*(2, 1) + d^*(1, 2)$. □

## IV. Conclusion

The fixed point theorems proved in this paper are generalizations of corresponding metric and d-metric versions. Finally, one can conclude from the Example 3.9 and Example 3.10 that these theorems are independent of results established by I.R. Sarma et al [1].

## Acknowledgment

The author is grateful to Professor I. Ramabhadra Sarma for his valuable comments and suggestions.

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