Estimation of Reduced t Intercept and Maximum Lifespan

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***Abstract*—In the absence of age-specific mortality data we estimate the reduced $'t'$ intercept and maximum lifespan permitting a new representation for the age-dependent stretched exponent in the extended Weibull model.**

***Index Terms*—Age-dependent stretched exponent (Shape parameter), Characteristic life, Vertex point.**

***MSC 2010 Codes* – 62N05, 62F10, 62P10, 62N02, 26D20**

I. Introduction

BIOLOGISTS and gerontologists are hunting for a variety of useful ways to prolong life in animals including mice and worms [1]. Their research suggests that human lifespan may be remarkably pliable [1]. A long-standing question in human aging-how long can human live?-remains to be resolved in interdisciplinary aging researches [2,4]. Fundamental studies of the aging process have lately attracted the interest of researches in a variety of disciplines, linking ideas and theories from such diverse fields as biochemistry of mathematics [9]. General functions for human survival and mortality may support a possibility of general mechanisms in human aging [5]. Frequently, a mathematical model fits to observed mortality rates and hypothesis concerning parameter values among treatment population are investigated [3,7]. Many models have been suggested to mathematically describe survival and mortality curves [8].

Aging (or) senescence is a decline in state at later ages that is manifested through a reduction in survival and fecundity. It means that reproductive prospects and hence the life history options (trade offs) open to the organism decline [6].

The maximum lifespan (called ω), which is defined as the maximum lifespan of a species, is determined mainly by biological factors. The maximum human lifespan is generally postulated to be around 125 years [11,14] and [16], while the observed oldest ages at death and the life expectancy are increasing today [10].

Many mathematical models for survival curves have been proposed [8,24]. One of the fundamental mortality laws is the Gompertz law [23], in which the mortality rate increases roughly exponentially with increasing age at senescence. This model is most commonly employed to compare mortality rates between different populations [12].

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For a better estimation of the maximum human lifespan, it is thus desirable to find a valid mathematical model which overcomes the uncertainty of old-age mortality pattern. For ages above 90 years [8], found that the classical Weibull model is inferior to a model described by [21]. For this reason, Weon et al. [15] modified the classical Weibull model by exchanging the mathematical nature of the stretched exponent as a function of age. With this modification, they made an Extended Weibull model adoptable to any shape of the empirical human survival curves. The age-dependence of shape parameter must be associated with the fundamental mechanism of human aging [18].

In our previous work in the absence of age-specific mortality data, we provided the estimation of ratio of vertex point (The age at which stretched exponent attains its maximum value) to the characteristic life and maximum lifespan in the (extended Weibull model) Weon model.

In this paper we give a estimation of reduced t intercept and maximum lifespan using a new representation for the age-dependent stretched exponent in the (extended Weibull model) Weon model in the absence of age-specific mortality data.

II. Mathemtical Model: Weon Model

Weon et al. put forward a general expression for human survival and mortality rate[15]. A new mathematical model, namely "Weon model" is derived from the Weibull survival function. It is simply described by two parameters, the age dependent shape parameter $\beta(t)$ and the characteristic life (α). The age dependent shape parameter enables us to model the survival and mortality functions.

$$S(t) = e^{(-\frac{t}{\alpha})^{\beta(t)}}, \tag{1}$$

where $S(t)$ denotes the survival probability at any age t. After graphically determining the characteristic life (α) value, an adequate mathematical expression for the age dependent shape parameter $\beta(t)$ can be given by,

$$\beta(t) = \frac{\ln(-\ln S(t))}{\ln(\frac{t}{\alpha})}. \tag{2}$$

The mortality function is described by the mathematical relationship with the survival function as follows

$$\mu(t) = \frac{-d\ln S(t)}{dt}.$$

On account of (1), we get

$$\mu(t) = \left(\frac{t}{\alpha}\right)^{\beta(t)} \left[\frac{\beta(t)}{t} + \ln\left(\frac{t}{\alpha}\right)\frac{d\beta(t)}{dt}\right]. \tag{3}$$

The characteristic life indicates the duration of survival to be $t = \alpha$ when $S = exp(-1) \approx 36.79\%$. The original idea was obtained as follows: typical human survival curves show i) a rapid decline in survival in the first few years of life and ii) a relatively steady decline and then an abrupt decline near death thereafter (see: Fig:1a). Interestingly, the former behaviour resembles the Weibull survival function with $\beta < 1$ and the latter behaviour seems to follow the case of $\beta >> 1$.

In empirical practice, a polynomial expression of $\beta(t)$ as a function of age: $\beta(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + ...$, where the associated coefficients are determined by a regression analysis in the plot of shape parameter curve. And, then the derivative of $\beta(t)$ is $\beta_1 + 2\beta_2 t + ...$ which indicates again that the shape parameter is a function of age. Particularly, used the quadratic pattern $\beta(t) = \beta_0 + \beta_1 t + \beta_2 t^2$ is the mathematically valid before an age limit (the vertex is $\nu = -\frac{\beta_1}{\beta_2}$).

In principle, the survival probability $S(t)$ is mathematically a monotonic decay function of age ($\frac{dS(t)}{dt} < 0$) between $S(0) = 1$ and $S(\omega) = 0$ for a maximum age (ω). For this reason, the slope of the stretched exponent with age can be given by

$$\frac{d\beta(t)}{dt} \begin{cases} < +\epsilon(t), t < \alpha \\ > -\epsilon(t), t > \alpha \end{cases} \tag{4}$$

where $\epsilon(t) = \left| -\frac{\beta(t)}{t\ln(\frac{t}{\alpha})} \right|$ is the mathematical constraint of $\beta(t)$.

According to the Weon model, the quadratic expression for the shape parameter at ages $80 - 109$ results in that the mortality function is zero at the maximum longevity by the mortality dynamics in nature [13]. The maximum human lifespan (ω) per each survival curve might be estimated at the mathematical limit of

$$\frac{d\beta(t)}{dt} = -\frac{\beta(t)}{t\ln(t/\alpha)}. \tag{5}$$

Remark: In this paper we follow the usual notation t_m for maximum lifespan.

III. Estimation of Reduced t Intercept and Maximum Lifespan

We observe that, the quadratic form $\beta(t)$ can be expressed by

$$\beta(t) = \beta(\nu) - c(\nu - t)^2. \tag{6}$$

Here $\beta(\nu)$ and c are unknown constants.

Letting $\beta(t) = 0$, we get

$$t = \nu - \sqrt{\frac{\beta(\nu)}{c}} \;\; and \;\; t = \nu + \sqrt{\frac{\beta(\nu)}{c}}. \tag{7}$$

In [17] (see: Fig:1b), the curvature and the vertex point give an upper $'t'$-intercept, as can be defined as the "q" point.

$$q = \nu + \sqrt{\frac{\beta(\nu)}{c}}. \tag{8}$$

Using (2), it can be shown E.S.Lakshminarayanan et.al. [19] that,

$$\beta'(t)_{/t=\alpha} = \frac{\beta(\alpha)}{2\alpha}. \tag{9}$$

(i) First, we determine the reduced $'t'$ intercept, $\frac{\beta(\nu)}{c}$. Using (6), for $t = \alpha$ equation (9) gives

$$4\alpha(\nu - \alpha) + (\nu - \alpha)^2 = \frac{\beta(\nu)}{c}. \tag{10}$$

In our previous work [20], we have shown that

$$\frac{\nu}{\alpha} \leq 1.18523. \tag{11}$$

On account of (11), equation (10) gives the reduced t intercept

$$\frac{\beta(\nu)}{c} \leq 0.77523\;\alpha^2. \tag{12}$$

For a Comparison with the numerical data [17] in Table 1 ($Mathematica\;\; version$ 6.0) we list the numerical values of (8), using(12) for a given ν and α.

TABLE I
(Reprinted from [20])

$\alpha(years)$	$\nu(years)$	numerical data q	estimated q $\leq$
88.07717	92.06	157.23	169.61
87.57264	96.30	162.70	173.41
85.47782	95.36	165.86	170.62
90.37934	95.20	159.00	174.78
83.79004	96.27	169.96	170.05
91.59045	98.17	161.51	178.81
83.82117	95.46	165.53	169.26
81.29562	98.59	155.26	170.17
87.42881	94.93	157.52	171.91
86.15168	100.50	159.68	176.35
85.86844	95.55	160.55	171.16
86.30050	94.58	162.12	170.57
86.53163	94.78	158.11	170.97
86.91140	94.57	158.95	171.09
87.34920	94.04	160.56	170.95

(ii) Next, we estimate the maximum human lifespan (t_m). Substitution of (6) into (5) gives

$$2c(\nu - t) = \frac{-[\beta(\nu) - c(\nu - t)^2]}{t\ln(t/\alpha)}.$$

Let $t > \nu > \alpha$. Rearranging the last equation, we get

$$(t - \nu)^2 + 2(t - \nu)t\ln(t/\alpha) = \frac{\beta(\nu)}{c}. \tag{13}$$

Since $\frac{\beta(\nu)}{c} \leq 0.77523\;\alpha^2$, (13) gives

$$(t - \nu)^2 + 2(t - \nu)t\ln(t/\alpha) \leq 0.77523\;\alpha^2. \tag{14}$$

Letting $\frac{t}{\alpha} = z > 1$, equation (14) becomes

$$(z - (\nu/\alpha))^2 + 2(z - (\nu/\alpha))z\ln(z) \leq 0.77523. \tag{15}$$

Recalling the lower bound obtained in [20] for maximum human lifespan

$$\frac{t_m}{\alpha} \geq 0.5\left[-\eta + \sqrt{\eta^2 + 4\left(\frac{\nu}{\alpha}\right)\left(\left(\frac{\nu}{\alpha}\right) + \sqrt{\left(\frac{\nu}{\alpha}\right)^2 + \eta^2 - 3 + 2\left(\frac{\nu}{\alpha}\right)}\right)}\right], \tag{16}$$

where $\eta = \frac{\nu}{\alpha}\ln\frac{\nu}{\alpha}$ and setting

$$c^* = 0.5\left[-\eta + \sqrt{\eta^2 + 4\left(\frac{\nu}{\alpha}\right)\left(\left(\frac{\nu}{\alpha}\right) + \sqrt{\left(\frac{\nu}{\alpha}\right)^2 + \eta^2 - 3 + 2\left(\frac{\nu}{\alpha}\right)}\right)}\right],$$

we solve (15) for z, subject to $z \geq c^*$. Notice that $\ln(z) \geq \ln(c^*)$. Hence (15) satisfies,

$$\left(z - \left(\frac{\nu}{\alpha}\right)\right)^2 + 2\left(z - \left(\frac{\nu}{\alpha}\right)\right) z \ln(c^*) \leq 0.77523. \quad (17)$$

Simplifying further, we get

$$z^2(1+2\ln(c^*)) - 2\left(\frac{\nu}{\alpha}\right)(1+\ln(c^*))z + \left(\frac{\nu}{\alpha}\right)^2 - 0.77523 \leq 0.$$

Solving the above inequality for z, we get

$$z = \frac{t_m}{\alpha} \leq \frac{1}{(1+2\ln(c^*))}\left[\left(\frac{\nu}{\alpha}\right)(1+\ln(c^*)) + u\right], \quad (18)$$

where $u = \left[\sqrt{\left(\frac{\nu}{\alpha}\ln(c^*)\right)^2 + 0.77523 + 1.55046\ln(c^*)}\right]$.

For a Comparison with the numerical data [15,17] in Table 2 ($Mathematica \;\; version$ 6.0) we list the numerical values of (18) for a given ν and α.

TABLE II
(REPRINTED FROM [18,20])

$\alpha(years)$	$\nu(years)$	ν/α	$\ln c^*$	t_m	estimated $t_m \leq$
88.1960	95.9	1.08735	0.27114	125.57	143.80
87.6732	95.8	1.09269	0.27901	125.26	142.82
87.4026	96.1	1.09951	0.28866	125.64	142.26
87.4567	96.9	1.10798	0.30013	123.44	142.25
84.2641	96.0	1.13928	0.33923	127.27	137.06
86.6991	97.2	1.12112	0.31720	125.50	140.96
86.1580	96.9	1.12468	0.32165	122.14	139.81
86.7532	95.6	1.10198	0.29208	126.17	141.17
86.8074	94.5	1.08862	0.27308	124.69	141.50
85.8874	95.4	1.11074	0.30385	126.54	139.67
84.9134	94.7	1.11525	0.30972	129.35	138.07
86.1039	92.6	1.07544	0.25283	129.46	140.76
86.3203	91.9	1.06464	0.23483	131.99	141.61
87.8355	93.0	1.0588	0.22444	126.08	148.45
88.07717	92.06	1.04522	0.19797	126.16	145.93
85.47782	95.36	1.11561	0.31019	129.85	138.98
90.37934	95.20	1.05334	0.21422	128.33	149.02
83.79004	96.27	1.14894	0.35036	131.53	136.38
91.59045	98.17	1.07184	0.24699	130.31	149.89
87.57264	96.30	1.09966	0.28871	129.24	142.54
83.82117	95.46	1.13885	0.33873	129.20	136.34
87.43102	103.37	1.1823	0.38623	128.74	142.90

Here
$$c^* = 0.5\left[-\eta + \sqrt{\eta^2 + 4\left(\frac{\nu}{\alpha}\right)\left(\left(\frac{\nu}{\alpha}\right) + \sqrt{\left(\frac{\nu}{\alpha}\right)^2 + \eta^2 - 3 + 2\left(\frac{\nu}{\alpha}\right)}\right)}\right].$$

IV. CONCLUSION

Thus the estimation of reduced t intercept $\frac{\beta(\nu)}{c}$ enables us to find the upper bound for maximum human lifespan.

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Reprinted from [22,17]

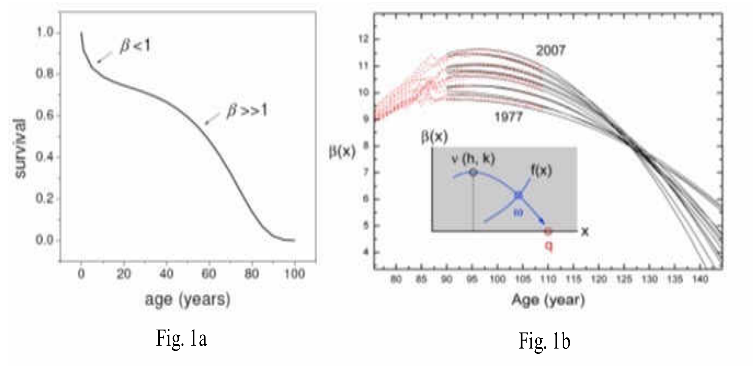

Fig:1a Typical human survival curve.
Fig:1b The evolution of Sweden females survival curves (dashed lines) from 1977 to 2007.

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