Generalized Fixed Point Results in Dislocated and Dislocated Quasi-Metric Spaces

Mujeeb Ur Rahman and Muhammad Sarwar

***Abstract*—The aim of this paper is to study some fixed point results in dislocated and dislocated quasi-metric spaces. We have proved that for existence of fixed point the condition of continuity is not necessary. Our obtained results generalize some existing fixed point results in the literature. A fixed point theorem for a pair of continuous mappings is established which modify the result of Shrivastava et al. [1] in dislocated quasi-metric space.**

***Index Terms*—Complete dislocated and dislocated quasi-metric space, contraction mapping, Cauchy sequence, self mapping, fixed point.**

***MSC 2010 Codes* – 76A05, 76M55, 54H15**

I. INTRODUCTION

FIXED point theory is one of the most dynamic research subject in the development of non-linear analysis. In this field the first important and significant result was proved by Banach [2] in 1922 for contraction mapping in complete metric space. The well known Banach contraction theorem may be stated as follows: "Every contraction mapping on a complete metric space X into itself has a unique fixed point"(Bonsall 1962). Where Dass and Gupta [3] generalized the Banach contraction principle in metric space for rational type contraction conditions. Rohades[4] introduced a partial ordering for various definitions contractive mapping.

Hitzler and Seda ([5], [6]) introduced the notion of dislocated metric space and generalized the Banach contraction principle in such a space. Zeyada et al. [7] generalized the result of Hitzler and Seda and introduced the concept of complete dislocated quasi-metric space. Aage and Salunke [8] proved some fixed point results in dislocated quasi-metric space. Isufati [9] proved some fixed point results for continuous contractive conditions with rational expression in dislocated quasi-metric space. Zoto [10] give some new results in dislocated and dislocated quasi-metric spaces. Also Shrivastava et al. [1] proved a fixed point theorem for continuous mapping in dislocated quasi-metric spaces. Recently, Sharma and Thakur [11] established some fixed point theorems without considering the continuity of self-mappings.

In this paper we have established fixed point results by omitting the condition of continuity of the self-mappings in the context of dislocated and dislocated quasi-metric spaces, which generalize the results of Zeyada et al. [7] and Shrivastava et al. [1].

Mujeeb Ur Rahman Research Scholar in Department of Mathematics, University of Malakand, Dir(L), Pakistan. (E-mail: mujeeb846@yahoo.com)

Muhammad Sarwar Assistant Professor in Department of Mathematics, University of Malakand, Dir(L), Pakistan. (E-mail: sarwar@uom.edu.pk).

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II. PRELIMINARIES

Throughout this paper $\mathbb{R}^+$ will represent the set of non-negative real numbers.

We begin with the following definition as a recall from [7].

Definition 2.1. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a distance function satisfying the conditions,

d_1) $d(x, x) = 0$;

d_2) $d(x, y) = d(y, x) = 0$ implies that $x = y$;

d_3) $d(x, y) = d(y, x)$;

d_4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfy the conditions from d_1 to d_4 then it is called metric on X, if d satisfy conditions d_2 to d_4 then it is called dislocated metric (d-metric) on X, and if d satisfy conditions d_2 and d_4 only then it is called dislocated quasi-metric(dq-metric) on X.

Clearly every metric space is a dislocated metric space but the converse is not necessarily true, as clear from the following example.

Example 2.2. Let X=$\mathbb{R}^+$ define the distance function $d : X \times X \to \mathbb{R}^+$ as

$$d(x, y) = \max\{x, y\}.$$

Clearly the define function is dislocated metric space but not a metric space.

Also every metric space is dislocated quasi-metric space but the converse is not true and every dislocated metric space is dislocated quasi-metric space but the converse is not true, as clear from the following example.

Example 2.3. Let $X = \mathbb{R}$ we define the function $d : X \times X \to \mathbb{R}^+$ as,

$$d(x, y) = |x - y| + |x| \quad \text{ for all } x, y \in X.$$

Clearly the defined function is dq-metric space but not a metric space nor dislocated metric space.

In our main work we will use the following definitions which can be found in [7].

Definition 2.4. A sequence $\{x_n\}$ in dq-metric space (X, d) is called Cauchy sequence if for $\epsilon > 0$ there exists a positive integer $n_0 \in N$ such that for $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.

Definition 2.5. A sequence $\{x_n\}$ is called dq-convergent in (X, d) if for $n \in N$ we have

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

In this case x is called the dq-limit of the sequence $\{x_n\}$.

Definition 2.6. A dq-metric space (X, d) is said to be complete if every Cauchy sequence in X converge to a point of X.

Definition 2.7. Let (X, d) be a dq-metric space, a mapping

$T : X \to X$ is called contraction if there exists $0 \le \alpha < 1$ such that

$$d(Tx, Ty) \le \alpha \cdot d(x, y) \quad \text{for all} \quad x, y \in X.$$

Lemma 2.8[7]. Limit in dq-metric space is unique.
Theorem 2.9.[7]. Let (X, d) be a complete dq-metric space $T : X \to X$ be a contraction, then T has a unique fixed point.

III. Main Results

Theorem 3.1. Let (X, d) be a complete dq-metric space and $T : X \to X$ be a mapping satisfying the condition

$$d(Tx, Ty) \le a \cdot d(x, y) + b \cdot \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + c \cdot \frac{d(y, Ty) + d(y, Tx)}{1 + d(y, Ty) \cdot d(y, Tx)} \quad (1)$$

where $a, b, c \ge 0$ with $0 \le a + b + 3c < 1$ and for all x, y in X. Then T has a unique fixed point in X.
Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ by the rule,

$$x_0, x_1 = Tx_0, ..., x_{n+1} = Tx_n \text{ for } n \in N.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X then consider,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

From equation (1) we have

$$d(x_n, x_{n+1}) \le a \cdot d(x_{n-1}, x_n) + b \cdot \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + c \cdot \frac{d(x_n, x_{n+1}) + d(x_n, x_n)}{1 + d(x_n, x_{n+1}).d(x_n, x_n)}$$

$$\le a \cdot d(x_{n-1}, x_n) + b \cdot \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + c \cdot [d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})].$$

Therefore

$$d(x_n, x_{n+1}) \le \frac{a + c}{1 - (b + 2c)} \cdot d(x_{n-1}, x_n) = h \cdot d(x_{n-1}, x_n)$$

where

$$h = \frac{a + c}{1 - (b + 2c)}$$

with $h < 1$, because $a + b + 3c < 1$.

$$d(x_n, x_{n+1}) \le h^2 \cdot d(x_{n-2}, x_{n-1}).$$

Similarly proceeding we have

$$d(x_n, x_{n+1}) \le h^n \cdot d(x_0, x_1).$$

Since $0 \le h < 1$ as $n \to \infty$, $h^n \to 0$. Therefore

$$\Rightarrow \; d(x_n, x_{n+1}) \to 0.$$

Thus $\{x_n\}$ is a Cauchy sequence in complete dq-metric space. So there must exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now to show that z is the fixed point of T for this consider,

$$d(Tx_n, Tz) \le a \cdot d(x_n, z) + b \cdot \frac{d(z, Tz)[1 + d(x_n, Tx_n)]}{1 + d(x_n, z)} + c \cdot \frac{d(z, Tz) + d(z, Tx_n)}{1 + d(z, Tz).d(z, Tx_n)}. \quad (2)$$

From the construction it is clear that $Tx_n = x_{n+1}$ and also x_n is a Cauchy sequence converges to z. Therefore taking limit $n \to \infty$ equation (2) become,

$$d(z, Tz) \le (b + c) \cdot d(z, Tz)$$

which is possible only if $d(z, Tz) = 0$ similarly we can show that $d(Tz, z) = 0$. Which implies that $Tz = z$. Hence z is the fixed point of T.

Uniqueness: Suppose that T has two fixed points z and w for $z \ne w$. Consider,

$$d(z, w) = d(Tz, Tw) \le a \cdot d(z, w) + b \cdot \frac{d(w, Tw)[1 + d(z, Tz)]}{1 + d(z, w)} + c \cdot \frac{d(w, Tw) + d(w, Tz)}{1 + d(w.Tw).d(w, Tz)}$$

if x is fixed point of T than $d(x, x) = 0$ by use of (1). So we have the following inequality

$$d(z, w) \le a \cdot d(z, w) + c \cdot d(w, z). \quad (3)$$

Similarly we have

$$d(w, z) \le a \cdot d(w, z) + c \cdot d(z, w). \quad (4)$$

From equation (3) and (4) we have,

$$|d(z, w) - d(w, z)| \le |a - c| \cdot |d(z, w) - d(w, z)|. \quad (5)$$

Since $|a - c| < 1$, so inequality (5) is possible if

$$|d(z, w) - d(w, z)| = 0$$

which implies that

$$d(z, w) = d(w, z).$$

Putting in equation (3) we have $d(z, w) = 0$ similarly $d(w, z) = 0 \Rightarrow z = w$. Hence T has a unique fixed point.
The above theorem yield the following corollaries.
Corollary 3.2. Let (X, d) be a complete dq-metric space. Suppose that $T : X \to X$ be a continuous mapping and all other conditions of Theorem 3.1 are satisfied then T has a unique fixed point.
Corollary 3.3. Let (X, d) be a complete dq-metric space. Suppose that $T : X \to X$ be a continuous mapping satisfying the conditions of Theorem 3.1 and the constants $b = c = 0$ then T has a unique fixed point.
Example 3.4. Let $X = [0, 6]$ and d be usual metric on X and T is defined by

$$Tx = \begin{cases} 0 & \text{if} \quad 0 \le x < 2 \\ 1 & \text{if} \quad 2 \le x \le 6. \end{cases}$$

Let $a = 1/4, b = 1/6, c = 1/8$, clearly T is discontinuous at $x = 2$ and satisfy all the conditions of Theorem 3.1 having

$x = 0$ is its unique fixed point.

Theorem 3.5. Let (X, d) be a complete d-metric space. Suppose that $S, T : X \to X$ be self-mappings satisfying the condition

$$d(Sx, Ty) \leq a \cdot d(x, y) + b \cdot d(x, Sx) + c \cdot d(y, Ty) \quad (6)$$

where $a, b, c \geq 0$ with $0 \leq a+b+c < 1$ and for all $x, y \in X$. Then S and T have a unique common fixed point.

Proof. Let x_0 be arbitrary point in X, we define a sequence $\{x_n\}$ in X by the rule,

$$x_0, x_1 = Sx_0, ...x_{2n+1} = Sx_{2n}$$

and

$$x_2 = Tx_1, ..., x_{2n} = Tx_{2n-1} \text{ for all } n \in N.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X. Consider

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

Now using equation (6) and the defined construction of the sequence we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq (\frac{a+b}{1-c}) \cdot d(x_{2n}, x_{2n+1}) \\ &= h.d(x_{2n}, x_{2n+1}). \end{aligned}$$

Where

$$h = \frac{a+b}{1-c}$$

so $0 \leq h < 1$, as $0 \leq a+b+c < 1$ and similarly we can show that

$$d(x_{2n}, x_{2n+1}) \leq h \cdot d(x_{2n-1}, x_{2n}).$$

Thus

$$d(x_{2n+1}, x_{2n+2}) \leq h^2 \cdot d(x_{2n-1}, x_{2n}).$$

Similarly we proceeding to get

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n) \leq h^2 \cdot d(x_{n-2}, x_{n-1}).$$

Finally we have

$$d(x_n, x_{n+1}) \leq h^n \cdot d(x_0, x_1).$$

Since $h < 1$, therefore as $n \to 0$, then $h^n \to 0$. This prove that $\{x_n\}$ is a Cauchy sequence in complete d-metric space so there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now to show that actually this limit point x is the fixed point of S and T for this consider

$$d(Sx_{2n}, Tx) \leq a \cdot d(x_{2n}, x) + b \cdot d(x_{2n}, Sx_{2n}) + c \cdot d(x, Tx)$$

since the sub-sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ also converges to x. Therefore by taking limit $n \to 0$ we have

$$d(x, Tx) \leq c \cdot d(x, Tx)$$

which is possible only if $d(x, Tx) = 0$ because $0 \leq a + b + c < 1$. Hence $Tx = x$. Similarly we can show that $Sx = x$. Therefore x is the common fixed point of S and T.

Uniqueness: Suppose that S and T have two common fixed points x and y for $x \neq y$. Since x and y are the fixed points of S and T. So replacing x by y in (6) we get $d(y, y) = 0$ similarly we can get $d(x, x) = 0$ using these in (6) we have the following inequality

$$d(x, y) = d(Sx, Ty) \leq a \cdot d(x, y) + b \cdot d(x, Sx) + c \cdot d(y, Ty)$$
$$= a \cdot d(x, y) + b \cdot d(x, x) + c \cdot d(y, y)$$
$$d(x, y) \leq a \cdot d(x, y)$$

but $0 \leq a+b+c < 1$. So the above inequality is possible only if $d(x, y) = 0$. Similarly we have $d(y, x) = 0$ Which implies that $x = y$. Hence S and T have a unique common fixed point.

Theorem 3.6. Let(X, d) be a complete dq-metric space. Suppose there exists non-negative constants a, b and c with $0 \leq a + b + 3c < 1$. Let $S, T : X \to X$ be two continuous self-mappings satisfying the condition

$$d(Sx, Ty) \leq a \cdot d(x, y) + b \cdot \frac{d(y, Ty)[1 + d(x, Sx)]}{1 + d(x, y)} + c \cdot \frac{d(y, Ty) + d(y, Sx)}{1 + d(y, Ty).d(y, Sx)} \quad (7)$$

for all x, y in X. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ by the rule,

$$x_0, x_1 = Sx_0, x_3 = Sx_2, ..., x_{2n+1} = Sx_{2n}$$

and

$$x_2 = Tx_1, x_4 = Tx_3, ..., x_{2n} = Tx_{2n-1} \text{ for } \ n \in N.$$

Now we have to show that $\{x_n\}$ is a Cauchy sequence in X. For this consider

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

so by condition (7) we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq a \cdot d(x_{2n}, x_{2n+1}) + \\ & b \cdot \frac{d(x_{2n+1}, x_{2n+2})[1 + d(x_{2n}, x_{2n+1})]}{1 + d(x_{2n}, x_{2n+1})} + \\ & c \cdot \frac{d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{1 + d(x_{2n+1}, x_{2n+2}) \cdot d(x_{2n+1}, x_{2n+1})} \\ &\leq a \cdot d(x_{2n}, x_{2n+1}) + b \cdot d(x_{2n+1}, x_{2n+2}) + \\ & c \cdot [d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]. \end{aligned}$$

Hence we have,

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{a+c}{1-(b+2c)} \cdot d(x_{2n}, x_{2n+1}).$$

Let

$$h = \frac{a+c}{1-(b+2c)}$$

then clearly $0 \leq h < 1$ as $0 \leq a + b + 3c < 1$. So

$$d(x_{2n+1}, x_{2n+2}) \leq h \cdot d(x_{2n}, x_{2n+1}).$$

Similarly we can show that,

$$d(x_{2n}, x_{2n+1}) \leq h.d(x_{2n-1}, x_{2n}).$$

Thus

$$d(x_{2n+1}, x_{2n+2}) \le h^2 \cdot d(x_{2n-1}, x_{2n}).$$

In the same procedure we have

$$d(x_n, x_{n+1}) \le h.d(x_{n-1}, x_n) \le h^2 \cdot d(x_{n-2}, x_{n-1}).$$

Hence

$$d(x_n, x_{n+1}) \le h^n.d(x_0, x_1),$$

but $0 \le h < 1$ and as $n \to \infty$, then $h^n \to 0$. Which implies that $\{x_n\}$ is a Cauchy sequence in complete dq-metric space so by completeness there exists z in X such that $x_n \to z$ as $n \to \infty$. Also the sub-sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converges to z.

$$\lim_{n\to\infty} Tx_{2n+1} = Tz, and \lim_{n\to\infty} Sx_{2n} = Sz.$$

Since it is clear that $Tx_{2n+1} = x_{2n+2}$ and $Sx_{2n} = x_{2n+1}$. So $Tz = z$ and $Sz = z$. Therefore z is the common fixed point of S and T.

Uniqueness: Suppose that S and T have two common fixed points z and w for $z \ne w$. Then

$$d(z,w) = d(Sz,Tw) \le a \cdot d(z,w) + b \cdot \frac{d(w,Tw)[1 + d(z,Sz)]}{1 + d(z,w)}$$

$$+c \cdot \frac{d(w,Tw) + d(w,Sz)}{1 + d(w,Tw).d(w,Sz)}$$

since z and w are the fixed points of S and T replacing z by w in the above equation we get $d(w,w) = 0$ similarly we can show that $d(z,z) = 0$. Using these in the above we have the following inequality

$$d(z,w) \le a \cdot d(z,w) + c \cdot d(w,z). \quad (8)$$

Similarly we have

$$d(w,z) \le a \cdot d(w,z) + c \cdot d(z,w). \quad (9)$$

From (8) and (9) we have

$$|d(z,w) - d(w,z)| \le |a - c| \cdot |d(z,w) - d(w,z)|.$$

Since $|a - c| < 1$ so the above inequality is possible if

$$|d(z,w) - d(w,z)| = 0$$

which implies that $d(z,w) = d(w,z)$ using this in (8) and (9) we get

$$d(z,w) = 0 = d(w,z)$$

$\Rightarrow z = w$. Which contradict our assumption that $z \ne w$. Therefore S and T have a unique common fixed point.

The following corollary can be deduced from Theorem (3.6).

Corollary 3.7. Let(X,d) be a complete dq-metric space and suppose there exists non-negative constants a, b and c with $0 \le a+b+3c < 1$. Let $T : X \to X$ be a continuous mapping satisfying the condition,

$$d(Tx,Ty) \le a \cdot d(x,y) + b \cdot \frac{d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)} +$$

$$c \cdot \frac{d(y,Ty) + d(y,Tx)}{1 + d(y,Ty).d(y,Tx)}$$

for all x, y in X. Then T has a unique fixed point.

Proof. Putting $S = T$ in Theorem 3.6 one can get the required result easily.

IV. Conclusion

Our established results generalize and modify the results of Sharivastava et al. [1], Zeyada et al. [7] and Aage and Salunke [8] and many other similar type of fixed point results.

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