Compact Operators on Hilbert Spaces

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***Abstract*—In this paper, we obtain some results on compact operators. In particular, we prove that if T is an unitary operator on a Hilbert space H, then it is compact if and only if H has finite dimension. As the main theorem we prove that if T be a hypercyclic operator on a Hilbert space, then T^n ($n \in \mathbb{N}$) is noncompact.**

***Index Terms*—Compact operator, Linear Projections, Heine-Borel Property.**

***MSC 2010 Codes* – 46B50**

I. Introduction

Surely, the operator theory is the heart of functional analysis. This means that if one wish to work on functional analysis, he/she must study the operator theory. In operator theory, we study operators and connection between it with other mathematical subjects. Recently, many mathematicians have studied compact operators; See [1], [3] for more details. One of the important applications of these operators is in solving the integral equations. In this paper, we obtain some results on compact operators. First of all we recall some definition and result that be needed in the sequel. Any unexplained notion and definitions can be found in [4].

Definition I.1. Let T be a linear operator on a Hilbert space H. T is called to be compact if for every bounded subset M of H, $\overline{T(M)}$ is compact.

Notation I.2. The set of all compact operators on H denoted by $K(H)$. It is easy to see that $K(H)$ is a linear subspace of $BL(H)$.

Lemma I.3. *Let H be an infinite dimensional Hilbert space. Then the identity operatot on H is not compact.*

Theorem I.4. *Any bounded operator with finite rank is compact.*

Definition I.5. An operator P on a normed space X is called a projection if $P^2 = P$.

Proposition I.6. *Let P be a linear projection on H. Then P is compact if and only if it has finite rank.*

Definition I.7. [5]: We say that a topological vector space X has property Heine-Borel when every bounded and closed subset of X is compact.

II. Compact Operators on Hilbert Spaces

We begin the section with the following theorem.

Theorem II.1. *Let H be an infinite-dimensional Hilbert space and T a compact operator on H. Then T is not invertible.*

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Proof: Let T is invertible. It is well-known and easy to show that the composition of two operator which at least one of them be compact is also compact ([4], Th. 11.5). Therefore $TT^{-1} = I$ is compact which contradicts to Lemma I.3. ■

Corollary II.2. *If T is an invertible operator on an infinite dimensional Hilbert space, then it is not compact.*

Corollary II.3. *Let T be a bounded operator with finite rank on an infinite-dimensional Hilbert space H. Then T is not invertible.*

Proof: By Theorem I.4, T is compact. Now the proof is completed by Theorem II.1. ■

Corollary II.4. *Let P be a linear projection on H with finite rank. Then it is not invertible.*

Proof: Use the Proposition I.6 together with Theorem II.1. ■

Theorem II.5. *Let H be an infinite-dimensional Hilbert space and T a compact and self-adjoint operator on H. Then T^n is not invertible, for each $n \in \mathbb{N}$.*

Proof: First note that T^* is compact. On the other hand, theorem implies that TT^* is compact ([4], Th. 9.2 (ii)). Now, since T is self-adjoint, so T^2 is compact and by Theorem II.1 is not invertible. In a similar manner, we conclude that T^n is not invertible, for each $n \in \mathbb{N}$. ■

Theorem II.6. *Let T be an unitary operator on H. Then T is compact if and only if H has finite dimension.*

Proof: ($\Rightarrow$)We have $T^*T = TT^* = I$. Since T is compact, so I is compact. Therefore H has finite dimension (Lemma I.3).
($\Leftarrow$)Since the identity operator I on H is compact (Lemma I.3), so T^*T (and also TT^*) is compact. On the other hand, $TT^*T = T$ which implies T is compact. ■

Theorem II.7. *Let H be a real or complex finite dimensional Hilbert space. Then*

$$\mathrm{card}(K(H)) \geq c$$

Where c is the cardinal of continuum.

Proof: Define

$$A = \{\lambda I : \ \lambda \in \mathbb{R} \ or \ \mathbb{C}\}$$

Note that

$$\mathrm{card}(\mathbb{R} \ or \ \mathbb{C}) = c$$

Hence

$$\mathrm{card}(A) = c$$

H has finite dimension and so $A \subset K(H)$. It follows that

$$\mathrm{card}(K(H)) \geq c$$

Thus the proof is finished. ■

Remark II.8. *If H is a finite dimensional Hilbert space, then any bounded linear operator on H is compact. Although, in general, a bounded linear operator on a Hilbert space H need not be compact; But in the following we give a sufficient condition for a bounded linear operator to be compact.*

Theorem II.9. *Let H be a Hilbert space with Heine-Borel property. Then $K(H) = BL(H)$.*

Proof: It is sufficient to show that $BL(H) \subset K(H)$. For this, let $T \in BL(H)$. Then since H has the Heine-Borel property, it is easy to show that $T(M)$ is relatively compact, for every bounded set M in X, as required. ■

Corollary II.10. *H be a real or complex Hilbert space with Heine-Borel property. Then*

$$\mathrm{card}(K(H)) \geq c$$

Where c is the cardinal of continuum.

Proof: This immediately follows the fact that the cardinal of the set

$$A = \{\lambda I : \ \lambda \in \mathbb{R} \ or \ \mathbb{C}\}$$

Is equal to c. ■

Corollary II.11. *There is no infinite dimensional Hilbert space with property Heine-Borel.*

Proof: By contrary, let H be an infinite dimensional Hilbert space with property Heine-Borel. Then the identity operator on H is compact (Theorem II.9) which contradicts to Lemma I.3. ■

Now we reach at the important following result.

Corollary II.12. *A Hilbert space has property Heine-Borel if and only if it has finite dimension.*

Proof: Use Corollary II.11 together with the well-known fact that finite dimensional spaces have property Heine-Borel. ■

Finally we prove the main theorem; Indeed, we introduce a class of noncompact operators.

Theorem II.13. *If T be a hypercyclic operator on a Hilbert space, then T^n ($n \in \mathbb{N}$) is noncompact.*

Proof: The proof follows immediately from the fact that if T^n is compact for some $n \in \mathbb{N}$, then it is not hypercyclic. ■

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