Co-isolated Locating Domination Number for the Complement of a Doubly Connected Graph

S. Muthammai and N. Meenal

Abstract—Let $G(V, E)$ be a simple, finite, undirected connected graph. A non-empty set $S \subseteq V$ of a graph $G$ is a dominating set, if every vertex in $V - S$ is adjacent to at least one vertex in $S$. A dominating set $S \subseteq V$ is called a locating dominating set, if for any two vertices $v, w \in V - S$, $N(v) \cap S \neq N(w) \cap S$. A locating dominating set $S \subseteq V$ is called a co-isolated locating dominating set, if there exists at least one isolated vertex in $< V - S >$. The co-isolated locating domination number $\gamma_{cild}(G)$ is the minimum cardinality of a co-isolated locating dominating set. A locating dominating set of minimum cardinality is called a $\gamma(G)$-set. $\gamma_{cild}(G)$-set is defined likewise. A graph $G$ is called doubly-connected, if both $G$ and its complement $\overline{G}$ are connected.

In this paper, the co-isolated locating domination number for the complement of a doubly connected graph is studied. Also, the bounds for Nordhaus-Gaddum type result are studied.

Index Terms—Dominating set, locating dominating set, co-isolated locating dominating set, doubly connected graph.

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I. INTRODUCTION

Let $G = (V, E)$ be a simple, finite graph. For $v \in V(G)$, the neighborhood $N_G(v)$ (or simply $N(v)$) of $v$ is the set of all vertices adjacent to $v$ in $G$. The complement of a graph $G$, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. If $G$ and $H$ are simple graphs, an isomorphism from $G$ to $H$ is a bijection $f : V(G) \rightarrow V(H)$ such that $u$ and $v$ are adjacent in $G$ if and only if $f(u)$ and $f(v)$ are adjacent in $H$. In this case, $G$ is said to be isomorphic to $H$ and is written as $G \cong H$. The graph $G$ is said to be self complementary, if $G$ is isomorphic to its complement.

The concept of domination in graphs was introduced by Ore [5]. A non-empty set $S \subseteq V(G)$ of a graph $G$ is a dominating set, if every vertex in $V(G) - S$ is adjacent to some vertex in $S$. A special case of dominating set $S$ is called a locating dominating set. A dominating set $S$ in a graph $G$ is called a locating dominating set in $G$, if for any two vertices $v, w \in V(G) - S$, $N_G(v) \cap S, N_G(w) \cap S$ are distinct. The locating dominating number $\gamma_l(G)$ of $G$ is the minimum cardinality of a locating dominating set in $G$. Locating domination was introduced by Slater [9], [10]. For further studies on locating domination [1] and [2] are referred. We introduce the concept of co-isolated locating domination number in graphs. A locating dominating set $S \subseteq V(G)$ is called a co-isolated locating dominating set, if $< V - S >$ contains at least one isolated vertex. The co-isolated locating domination number $\gamma_{cild}(G)$ is the minimum cardinality of a co-isolated locating dominating set. A locating dominating set of minimum cardinality is called a $\gamma(G)$-set. $\gamma_{cild}(G)$-set is defined likewise. A graph $G$ is called doubly-connected, if both $G$ and its complement $\overline{G}$ are connected.

A Bull Graph is the graph obtained by attaching exactly one pendant edge at each of the two vertices of a cycle on three vertices.

In this paper, the co-isolated locating domination number for the complement of a doubly connected graph is studied.

II. PRIOR RESULTS

The following results are obtained in [7] & [8].

Theorem 2.1: [7] For every non-trivial simple connected graph $G$ on $n$ vertices, $1 \leq \gamma_{cild}(G) \leq n - 1$.

Theorem 2.2: [7] $\gamma_{cild}(G) = 1$ if and only if $G \cong K_2$.

Theorem 2.3: [7] $\gamma_{cild}(K_n) = n - 1$, where $K_n$ is a complete graph on $n$ vertices.

Theorem 2.4: [8] $\gamma_{cild}(K_n - e) = n - 1$, where $e \in E(K_n)$.

Observation 2.5: [8] If $S$ is an co-isolated locating dominating set of a graph $G$ with $n$ vertices and if $|S| = k$, then $V(G) - S$ contains at most $nC_1 + nC_2 + \cdots + nC_k$ vertices.

Theorem 2.6: [7] $\gamma_{cild}(G) = 2$ if and only if $G$ is one of the following graphs

(a) $P_p$ $(p = 3, 4, 5)$, where $P_p$ is a path on $p$ vertices
(b) $C_p$ $(p = 3, 5)$, where $C_p$ is a cycle on $p$ vertices
(c) $C_5$ with a chord.
(d) $G$ is the graph obtained by attaching a pendant edge at a vertex of $C_3$ (or) at a vertex of degree 2 in $K_4 - e$.
(e) $G$ is the graph obtained by attaching a path of length 2 at a vertex of $C_3$
(f) $G$ is the Bull Graph.

Theorem 2.7: [8] $\gamma_{cild}(G) = p - 1$ $(p \geq 4)$ if and only if $V(G)$ can be partitioned into two sets $X$ and $Y$ such that one of the sets $X$ and $Y$ say, $Y$ is independent and each vertex in $X$ is adjacent to each in $Y$ and the subgraph $< X >$ of $G$ induced by $X$ is one of the following,

(a) $< X >$ is a complete graph
(b) $< X >$ is totally disconnected
(c) Any two non-adjacent vertices in $V(< X >)$ have common neighbors in $< X >$.

Theorem 2.8: [8] For a path $P_p$, on $p$ vertices,

$p \equiv 5 \pmod{5}$

\[ \gamma_{cild}(P_p) = \begin{cases} 2 \left( \frac{p}{5} \right), & \text{if } n \equiv 0 \pmod{5} \\ 2 \left( \frac{p}{5} \right) + 1, & \text{if } n \equiv 1, 2 \pmod{5} \\ 2 \left( \frac{p}{5} \right) + 2, & \text{if } n \equiv 3, 4 \pmod{5} \end{cases} \]

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III. MAIN RESULTS

In the following, co-isolated locating domination number for the complements of some standard graphs are found.

**Theorem 3.1**: Let $G$ be a doubly connected graph of order $p \geq 5$ such that $diam(G) = diam(G) = 2$. Then $G$ contains a co-isolated locating dominating set of cardinality $p - 3$.

**Proof**: Let $P$ be an induced path of order 4 in $G$. Since the complement of every non-trivial $P_k$-free is disconnected, such a path $P$ exist in $G$. Let $V(P) = \{v_1, v_2, v_3, v_4\}$ and $E(P) = \{v_1v_2, v_2v_3, v_3v_4\}$. Since, $diam(G) = 2$, there exists a vertex $v_5 \in V(G)$ such that $d_G(v_1, v_5) = d_G(v_4, v_5) = 1$.

Case (i): $v_2v_5, v_3v_5 \notin E(G)$.

Then the set $\{v_1, v_2, v_3, v_4, v_5\}$ induces a cycle $C$ in $G$, and the set $S = V(G) - \{v_2, v_4, v_5\}$ is a co-isolated locating dominating set of $G$, since $V - S = \{v_2, v_4, v_5\}$ and $N(v_2) \cap S$ and $N(v_4) \cap S$ and $N(v_5) \cap S$ are non-empty and distinct. Also, $v_5$ is isolated in $< V - S >$.

Case (ii): $v_5$ is adjacent to exactly one vertex of $\{v_2, v_3\}$

Let $v_2v_5 \in E(G)$ and $v_3v_5 \notin E(G)$. Then the set $S = V(G) - \{v_1, v_3, v_4\}$ is a co-isolated locating dominating set of $G$.

Case (iii): $v_2v_5, v_3v_5 \in E(G)$.

Since $diam(G) = diam(G) = 2$, there exist a vertex $v_6 \in V(G) - \{v_1, v_2, v_3, v_4, v_5\}$ such that $v_6$ is not adjacent to at least one of $v_1, v_2, v_3, v_4$. Since $G \cong P_4 + mK_1$, $m \geq 1$, then $G$ is disconnected. Let $v_7$ be not adjacent to $v_6$. Then the set $S = V(G) - \{v_1, v_3, v_4\}$ is a co-isolated locating dominating set of $G$.

From case (i) $\rightarrow$ case (iii), $G$ has a co-isolated locating dominating set of cardinality $p - 3$.

**Observation 3.2**: 

(i) If $S$ is a co-isolated locating dominating set of $G$, then $S$ will not be co-isolated locating dominating set of $G$.

(ii) Let $S$ be a co-isolated set of $G$ such that $< V - S >$ has exactly one isolated vertex, say $v$. Let there exist a vertex $u \in S$ such that $N(u) \cap S \subset S$ and $N(u) \cap V - S = (V - S) - \{v\}$.

(a) If there exists no vertex $w \in V - S$ such that $S \subset N_G(w)$, then $(S - \{u\}) \cup \{v\}$ is a co-isolated locating dominating set of $G$. Hence, $\gamma_{isol}(G) \leq \gamma_{isol}(G)$.

(b) If there exists a vertex $w \in V - S$ such that $S \subset N_G(w)$, then $S \cup \{v\}$ is a co-isolated locating dominating set of $G$. Hence, $\gamma_{isol}(G) \leq \gamma_{isol}(G) + 1$.

**Lemma 3.3**: If $G$ is a connected graph, then $\delta(G) \leq \gamma_{isol}(G)$, where $\delta(G)$ is the minimum degree of $G$.

**Proof**: If $S$ is a co-isolated set of $G$, then $< V - S >$ contains an isolated vertex say, $u$. Then $N(u) \subset S$ and $|N(u)| \leq |S|$.

That is, $\delta_G(v) \leq |S|$. Hence, $\delta(G) \leq \gamma_{isol}(G)$.

**Theorem 3.4**: For a path $P_p$ on $p$ vertices $\gamma_{isol}(P_p) = p - 3, (p \geq 5)$.

**Proof**: Let $G \cong P_p$, $p \geq 5$. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$ and let $d_G(v_1) = d_G(v_p) = 1$ and $d_G(v_2) = d_G(v_3) = d_G(v_4) = d_G(v_{p-1}) = 2$. Therefore, $d_G(v_1) = d_G(v_p) = p - 2$ and $d_G(v_2) = d_G(v_3) = \cdots = d_G(v_{p-1}) = p - 3$. Let $S = N_G(v_2)$. Then, $|S| = p - 3, v_2 \in V - S$ is the isolated vertex in $< V - S >$. Also, $N(u) \cap S \neq N(u) \cap S$ for any $u, v \in V - S$. Hence the set $S = N_G(v_2)$ is a $\gamma_{isol}$-set of $P$. Therefore, $\gamma_{isol}(P_p) \leq p - 3$.

By Lemma 3.3, $\delta(G) \leq \gamma_{isol}(G)$.

Therefore, $\gamma_{isol}(P_p) \geq p - 3$. Hence, $\gamma_{isol}(P_p) = p - 3$.

**Corollary 3.5**: $\gamma_{isol}(P_p) = \gamma_{isol}(P_p) = p - 3$. Since, $\gamma_{isol}$-set of $P$ is also a $\gamma_{isol}$-set of $P$.

**Definition 3.6**: $P^+_p$ is the tree obtained from the path $P_p$ on $p$ vertices, by attaching a pendant edge to each vertex of the path.

**Theorem 3.7**: $\gamma_{isol}(P^+_p) = 2p - 4, p \geq 5$.

**Proof**: Let $T \cong P^+_p$ and $V(T) = \{v_1, v_2, \ldots, v_p\}$ where $v_1, v_2, \ldots, v_p$ are the supports and $u_1, u_2, \ldots, u_p$ are the pendant vertices. Therefore, $d_T(v_1) = d_T(v_p) = 2$, $d_T(v_2) = d_T(v_3) = \cdots = d_T(v_{p-1}) = 2$ and $d_T(v_1) = d_T(v_2) = d_T(v_3) = \cdots = d_T(v_{p-1}) = 2$.

Also, $d_T(v_1) = d_T(v_2) = d_T(v_3) = \cdots = d_T(v_{p-1}) = 2$.

Therefore, $\gamma_{isol}(T) \geq 2p - 4$. Hence, $\gamma_{isol}(T) \geq 2p - 4$.

**Definition 3.8**: Given two positive integers $r$ and $s$, let $K_{2r}$ denote the double star, obtained after joining the central vertices of the stars $K_1r$ and $K_r$.

**Theorem 3.9**: $\gamma_{isol}(K_{2r}) = r + s - 2, (r, s \geq 2)$.

**Proof**: Let $G \cong K_{2r}$ and $V(G) = \{v_1, \ldots, v_r, u_1, u_2, \ldots, u_s\}$ with $v_1$ and $u_1$ as the central vertices and $v_i u_j \in E(G)$. Then $d_G(v_1) = r$ and $d_G(u_1) = s$. Also, $d_G(v_2) = d_G(v_3) = \cdots = d_G(v_r) = 1$ and $d_G(u_2) = d_G(u_3) = \cdots = d_G(u_s) = 1$.

Therefore, $d_G(v_1) = s - 1$ and $d_G(u_1) = r - 1$. Also, $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_r) = r + s - 2$. Let $S = N_G(v_2)$.

Then $S \subset N_G(v_2)$.

Therefore, $\gamma_{isol}(K_{2r}) = r + s - 2$.

**Theorem 3.10**: $\gamma_{isol}(K_{m,n} - e) = m + n - 2, (m, n \geq 2)$.

**Proof**: Let $G \cong K_{m,n} - e$ with vertex set $V(G) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$. Let $u_1, u_2 \notin E(G)$. Then $d_G(u_1) = n - 1$ and $d_G(v_1) = m - 1$. Also, $d_G(u_2) = d_G(u_3) = \cdots = d_G(u_m) = n$ and $d_G(v_2) = d_G(v_3) = \cdots = d_G(v_n) = m$.

Therefore, $d_G(u_1) = m$ and $d_G(v_1) = n$. Also, $d_G(u_1) = m - 1$ and $d_G(v_1) = n - 1$. Also, $d_G(u_1) = m - 1$ and $d_G(v_1) = n - 1$. Let $S = N_G(v_1)$.

Then, $S = \{u_1, u_2, \ldots, u_m, v_2, \ldots, v_n-1\}$ and $V - S = \{v_1, v_n\}$.

Also, $N_G(v_1) \cap S \neq N_G(v_1) \cap S$. Hence, $S$ forms a co-isolated locating dominating set of $G$. Therefore, $\gamma_{isol}(K_{m,n} - e) \leq |S| = m + n - 2$. Let $\gamma_{isol}(K_{m,n} - e) = m + n - 2$.

By Lemma 3.3, the set $S = N_G(v_1)$ and $|S| = n$. Also, $V - S = \{v_1, v_2, \ldots, v_n\}$ and $N_G(v_1) \cap S = N_G(v_1) \cap S$.
for $i \neq j$ and $i, j = 2, 3, \ldots, n$. Therefore, $S$ cannot be a $\gamma_{cild}$-set of $\overline{G}$. Hence, $\gamma_{cild}(K_{m,n} - e) = m + n - 2$.

Definition 3.11: $K_{1,r}^s$ represents the graph obtained by adding a new vertex to $s$ leaves of the star $K_{1,r}$; $s \leq r$ and $s \geq 1$.

Theorem 3.12: Let $G \cong K_{1,r}^s$, then $\gamma_{cild}(\overline{G}) = r - 1$, where $r \geq 3$ and $s < r$.

Proof: Let $G \cong K_{1,r}^s$, where $1 \leq s < r$ and $r \geq 3$. Let $u$ be the central vertex of $K_{1,r}^s$ and $v$ is the vertex adjacent to $s$ pendant vertices of $K_{1,r}$. Then $deg_{\overline{G}}(u) = 1$ and $v$ is its support in $\overline{G}$. The subgraph of $\overline{G}$ induced by $r$ pendant vertices in $G$ is complete in $\overline{G}$. Since $s < r$, there exists two vertices say $w, x \in V(G)$ such that $vw \in E(\overline{G})$ and $vx \notin E(\overline{G})$ where $w, x$ are pendant vertices in $G$. Let $S' = \{u, w, x\}$ and $S = V - S'$. Then $S$ is a co-isolated locating dominating set of $\overline{G}$. Therefore, $\gamma_{cild}(\overline{G}) \leq |S| = r - 1$. Also, $K_{1,r}$ is an induced subgraph of $\overline{G}$ and hence, $\gamma_{cild}(\overline{G}) \geq \gamma_{cild}(K_{1,r}) = r - 1$. Therefore, $\gamma_{cild}(\overline{G}) = r - 1$.

Lemma 3.13: If $G \cong mK_2$, then $\gamma_{cild}(\overline{G}) = 2m - 1$, $m \geq 1$.

Proof: Let $G \cong mK_2$ and let $V(G) = \{u_1, v_1, u_2, v_2, \ldots, u_m, v_m\}$ where $u_i, v_i \in E(K_2)$ for $i = 1, 2, \ldots, m$. Let $Y = \{u_1, v_1\}$ and $X = V(G) - Y$. $Y$ is an independent subset of $\overline{G}$ and each vertex in $X$ is adjacent to each in $Y$ in $\overline{G}$. Also, any two non-adjacent vertices of the induced subgraph $< X >$ of $\overline{G}$ have common neighbors in $< X >$. Then by Theorem 2.7, $\gamma_{cild}(\overline{G}) = 2m - 1$.

Theorem 3.14: For any doubly connected graph $G$ of order $p \geq 4$.

(i) $4 \leq \gamma_{cild}(G) + \gamma_{cild}(\overline{G}) \leq 2p - 4$ and

(ii) $4 \leq \gamma_{cild}(G) \gamma_{cild}(\overline{G}) \leq (p - 2)^2$.

Proof: By Theorem 2.2, $\gamma_{cild}(G) = 1$ if and only if $G \cong K_2$ and its complement $\overline{G}$ is $2K_1$. Therefore, every doubly-connected graph $G$ of order at least 4 satisfies $2 \leq \gamma_{cild}(G)$. That is, for any non-trivial doubly-connected graph $G$, $4 \leq \gamma_{cild}(G) + \gamma_{cild}(\overline{G})$. Also by Theorem 2.1, for every non-trivial simple connected graph $G$, $\gamma_{cild}(G) \leq p - 1$. In Theorem 2.7, the graphs for which $\gamma_{cild}(G) = p - 1$ are characterized. For those graphs, $\overline{G}$ is disconnected. Therefore, $\gamma_{cild}(G) \leq p - 2$ and $\gamma_{cild}(\overline{G}) \leq p - 2$ and hence $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) \leq 2p - 4$. Result (ii) follows easily.

Theorem 3.15: Let $G$ be a doubly connected graph with $p \geq 4$. Then $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) = 4$ if and only if $G$ is one of the following graphs $P_4, P_5, C_5, C_5$ with a chord and the Bull graph.

Proof: In Theorem 2.6, the graphs for which $\gamma_{cild}(G) = 2$ are characterized. Since $p \geq 4$, $G \cong P_4, P_5, C_5$.

(i) If $G \cong P_4, C_5$ or the Bull Graph, then $\overline{G}$ is self complementary. Therefore, $\gamma_{cild}(G) = \gamma_{cild}(\overline{G}) = 2$.

(ii) If $G \cong P_5$, then $\overline{G}$ is the cycle $C_5$ with a chord and $\gamma_{cild}(\overline{G}) = 2$ and vice versa.

(iii) If $G$ is the graph obtained by attaching a pendant edge at a vertex of $C_3$ (or) at a vertex of degree 2 in $K_4 - e$, then $\overline{G}$ is disconnected (or) $\gamma_{cild}(\overline{G}) = 3$.

(iv) If $G$ is the graph as in (e) of Theorem 2.6, then $\gamma_{cild}(\overline{G}) = 3$.

Therefore, $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) = 4$, if $G \cong P_4, P_5, C_5$, with a chord and the Bull graph.

Observation 3.16: For a doubly connected graph $G$ with $p \geq 4$, $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) = 5$ if and only if either $G$ is the graph obtained by attaching a pendant edge at a vertex of degree 2 in $K_4 - e$ or $G$ is the graph obtained by attaching a path of length 2 at a vertex of $C_3$.

IV. CONCLUSION

Here, co-isolated locating domination numbers of complements of Path, Cycle, Double Star, $K_{m,n} - e, K_{1,r}^s$ and $mK_2$ are studied. Also, the graphs for which $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) = 4$ are found.

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