

Co-isolated Locating Domination Number for the Complement of a Doubly Connected Graph

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Abstract—Let $G(V, E)$ be a simple, finite, undirected connected graph. A non-empty set $S \subseteq V$ of a graph G is a dominating set, if every vertex in $V - S$ is adjacent to atleast one vertex in S . A dominating set $S \subseteq V$ is called a locating dominating set, if for any two vertices $v, w \in V - S$, $N(v) \cap S \neq N(w) \cap S$. A locating dominating set $S \subseteq V$ is called a co-isolated locating dominating set, if there exists atleast one isolated vertex in $\langle V - S \rangle$. The co-isolated locating domination number γ_{cild} is the minimum cardinality of a co-isolated locating dominating set. A graph G is called doubly-connected if both G and its complement \overline{G} are connected.

In this paper, the co-isolated locating domination number for the complement of a doubly connected graph is studied. Also, the bounds for Nordhaus-Gaddum type result are studied

Index Terms—Dominating set, locating dominating set, co-isolated locating dominating set, doubly connected graph.

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I. INTRODUCTION

LET $G = (V, E)$ be a simple, finite graph. For $v \in V(G)$, the neighborhood $N_G(v)$ (or simply $N(v)$) of v is the set of all vertices adjacent to v in G . The complement of a graph G , denoted by \overline{G} , is the graph on the same vertices such that two vertices are adjacent in \overline{G} if and only if they are not adjacent in G . If G and H are simple graphs, an isomorphism from G to H is a bijection $f : V(G) \rightarrow V(H)$ such that u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H . In this case, G is said to be isomorphic to H and is written as $G \cong H$. The graph G is said to be self complementary, if G is isomorphic to its complement.

The concept of domination in graphs was introduced by Ore [5]. A non-empty set $S \subseteq V(G)$ of a graph G is a dominating set, if every vertex in $V(G) - S$ is adjacent to some vertex in S . A special case of dominating set S is called a locating dominating set. A dominating set S in a graph G is called a locating dominating set in G , if for any two vertices $v, w \in V(G) - S$, $N_G(v) \cap S$, $N_G(w) \cap S$ are distinct. The locating dominating number $\gamma_L(G)$ of G is the minimum cardinality of a locating dominating set in G . Locating domination was introduced by Slater [9], [10]. For further studies on locating domination [1] and [2] are referred. We introduce the concept of co-isolated locating domination number in graphs. A locating dominating set $S \subseteq V(G)$ is called a co-isolated locating dominating set, if $\langle V - S \rangle$

contains atleast one isolated vertex. The co-isolated locating domination number $\gamma_{cild}(G)$ is the minimum cardinality of a co-isolated locating dominating set. A locating dominating set of minimum cardinality is called a $\gamma_L(G)$ -set. $\gamma_{cild}(G)$ -set is defined likewise. A graph G is called doubly-connected, if both G and its complement \overline{G} are connected.

A Bull Graph is the graph obtained by attaching exactly one pendent edge at each of the two vertices of a cycle on three vertices.

In this paper, the co-isolated locating domination number for the complement of a doubly connected graph is studied.

II. PRIOR RESULTS

The following results are obtained in [7] & [8].

Theorem 2.1: [7] For every non-trivial simple connected graph G on n vertices, $1 \leq \gamma_{cild}(G) \leq n - 1$.

Theorem 2.2: [7] $\gamma_{cild}(G) = 1$ if and only if $G \cong K_2$.

Theorem 2.3: [7] $\gamma_{cild}(K_n) = n - 1$, where K_n is a complete graph on n vertices.

Theorem 2.4: [8] $\gamma_{cild}(K_n - e) = n - 1$, where $e \in E(K_n)$.

Observation 2.5: [8] If S is an co-isolated locating dominating set of a graph G with n vertices and if $|S| = k$, then $V(G) - S$ contains atleast $nC_1 + nC_2 + \dots + nC_k$ vertices.

Theorem 2.6: [7] $\gamma_{cild}(G) = 2$ if and only if G is one of the following graphs

- P_p ($p = 3, 4, 5$), where P_p is a path on p vertices
- C_p ($p = 3, 5$), where C_p is a cycle on p vertices
- C_5 with a chord.
- G is the graph obtained by attaching a pendant edge at a vertex of C_3 (or) at a vertex of degree 2 in $K_4 - e$.
- G is the graph obtained by attaching a path of length 2 at a vertex of C_3
- G is the Bull Graph.

Theorem 2.7: [8] $\gamma_{cild}(G) = p - 1$ ($p \geq 4$) if and only if $V(G)$ can be partitioned into two sets X and Y such that one of the sets X and Y say, Y is independent and each vertex in X is adjacent to each in Y and the subgraph $\langle X \rangle$ of G induced by X is one of the following,

- $\langle X \rangle$ is a complete graph
- $\langle X \rangle$ is totally disconnected
- Any two non-adjacent vertices in $V(\langle X \rangle)$ have common neighbors in $\langle X \rangle$.

Theorem 2.8: [8] For a path P_p on p vertices,

$$\gamma_{cild}(P_p) = \begin{cases} 2 \left\lfloor \frac{p}{5} \right\rfloor, & \text{if } n \equiv 0 \pmod{5} \\ 2 \left\lfloor \frac{p}{5} \right\rfloor + 1, & \text{if } n \equiv 1, 2 \pmod{5} \\ 2 \left\lfloor \frac{p}{5} \right\rfloor + 2, & \text{if } n \equiv 3, 4 \pmod{5} \end{cases}$$

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III. MAIN RESULTS

In the following, co-isolated locating domination number for the complements of some standard graphs are found.

Theorem 3.1: Let G be a doubly connected graph of order $p \geq 5$ such that $\text{diam}(G) = \text{diam}(\overline{G}) = 2$. Then G contains a co-isolated locating dominating set of cardinality $p - 3$.

Proof: Let P be an induced path of order 4 in G . Since the complement of every non-trivial P_4 -free is disconnected, such a path P exist in G . Let $V(P) = \{v_1, v_2, v_3, v_4\}$ and $E(P) = \{v_1v_2, v_2v_3, v_3v_4\}$. Since, $\text{diam}(G) = 2$, there exists a vertex $v_5 \in V(G)$ such that $d_G(v_1, v_5) = d_G(v_4, v_5) = 1$.

Case (i): $v_2v_5, v_3v_5 \notin E(G)$.

Then the set $\{v_1, v_2, v_3, v_4, v_5\}$ induces a cycle C in G . and the set $S = V(G) - \{v_2, v_4, v_5\}$ is a co-isolated locating dominating set of G , since $V - S = \{v_2, v_4, v_5\}$ and $N(v_2) \cap S$ and $N(v_4) \cap S$ and $N(v_5) \cap S$ are non-empty and distinct. Also, v_2 is isolated in $\langle V - S \rangle$.

Case (ii): v_5 is adjacent to exactly one vertex of $\{v_2, v_3\}$

Let $v_2v_5 \in E(G)$ and $v_3v_5 \notin E(G)$. Then the set $S = V(G) - \{v_1, v_3, v_4\}$ is a co-isolated locating dominating set of G .

Case (iii): $v_2v_5, v_3v_5 \in E(G)$

Since $\text{diam}(G) = \text{diam}(\overline{G}) = 2$, there exist a vertex $v_6 \in V(G) - \{v_1, v_2, v_3, v_4, v_5\}$ such that v_6 is not adjacent to atleast one of v_1, v_2, v_3, v_4 . Since if $G \cong P_4 + mK_1$, $m \geq 1$, then \overline{G} is disconnected. Let v_2 to be not adjacent to v_6 . Then the set $S = V(G) - \{v_1, v_3, v_4\}$ is a co-isolated locating dominating set of G .

From case (i) \rightarrow case (iii), G has a co-isolated locating dominating set of cardinality $p - 3$. ■

Observation 3.2:

- (i) If S is a co-isolated locating dominating set of G , then S will not be co-isolated locating dominating set of \overline{G} .
- (ii) Let S be γ_{cild} -set of G such that $\langle V - S \rangle$ has exactly one isolated vertex, say v . Let there exist a vertex $u \in S$ such that $N(u) \cap S \subset S$ and $N(u) \cap V - S = (V - S) - \{v\}$.
 - (a) If there exists no vertex $w \in V - S$ such that $S \subseteq N_G(w)$, then $(S - \{u\}) \cup \{v\}$ is a co-isolated locating dominating set of \overline{G} . Hence, $\gamma_{cild}(\overline{G}) \leq \gamma_{cild}(G)$.
 - (b) If there exists a vertex $w \in V - S$ such that $S \subseteq N_G(w)$, then $S \cup \{w\}$ is a co-isolated locating dominating set of \overline{G} . Hence, $\gamma_{cild}(\overline{G}) \leq \gamma_{cild}(G) + 1$.

Lemma 3.3: If G is a connected graph, then $\delta(G) \leq \gamma_{cild}(G)$, where $\delta(G)$ is the minimum degree of G .

Proof: If S is a γ_{cild} -set of G , then $\langle V - S \rangle$ contains an isolated vertex say, u . Then $N(u) \subseteq S$ and $|N(u)| \leq |S|$. That is, $\deg_G(u) \leq |S|$. Hence, $\delta(G) \leq \gamma_{cild}(G)$. ■

Theorem 3.4: For a path P_p on p vertices $\gamma_{cild}(\overline{P}_p) = p - 3$, ($p \geq 5$).

Proof: Let $G \cong P_p$, $p \geq 5$. Let $V(G) = \{v_1, v_2, v_3, \dots, v_p\}$ and let $d_G(v_1) = d_G(v_p) = 1$ and $d_G(v_2) = d_G(v_3) = \dots = d_G(v_{p-1}) = 2$. Therefore, $d_{\overline{G}}(v_1) = d_{\overline{G}}(v_p) = p - 2$ and $d_{\overline{G}}(v_2) = d_{\overline{G}}(v_3) = \dots = d_{\overline{G}}(v_{p-1}) = p - 3$. Let $S = N_{\overline{G}}(v_2)$. Then, $|S| = p - 3$. $v_2 \in V - S$ is the isolated vertex in $\langle V - S \rangle$. Also,

$N(u) \cap S \neq N(v) \cap S$, for any $u, v \in V - S$. Hence the set $S = N_{\overline{G}}(v_2)$ is a γ_{cild} -set of \overline{P}_p . Therefore, $\gamma_{cild}(\overline{P}_p) \leq p - 3$. By Lemma 3.3, $\delta(G) \leq \gamma_{cild}(G)$.

Therefore, $\gamma_{cild}(\overline{P}_p) \geq p - 3$. Hence, $\gamma_{cild}(\overline{P}_p) = p - 3$. ■

Corollary 3.5: $\gamma_{cild}(\overline{P}_p) = \gamma_{cild}(\overline{C}_p) = p - 3$. Since, γ_{cild} -set of \overline{P}_p is also a γ_{cild} -set of \overline{C}_p .

Definition 3.6: P_p^+ is the tree obtained from the path P_p on p vertices, by attaching a pendant edge to each vertex of the path.

Theorem 3.7: $\gamma_{cild}(\overline{P}_p^+) = 2p - 4$, $p \geq 5$.

Proof: Let $T \cong P_p^+$ and $V(T) = \{v_1, v_2, \dots, v_p, u_1, u_2, \dots, u_p\}$ where v_1, v_2, \dots, v_p are the supports and u_1, u_2, \dots, u_p are the pendant vertices. Therefore, $d_T(v_1) = d_T(v_p) = 2$, $d_T(v_2) = d_T(v_3) = \dots = d_T(v_{p-1}) = 3$ and $d_T(u_1) = d_T(u_2) = \dots = d_T(u_p) = 1$. Therefore, $d_{\overline{T}}(v_1) = d_{\overline{T}}(v_p) = 2p - 3$, $d_{\overline{T}}(v_2) = d_{\overline{T}}(v_3) = \dots = d_{\overline{T}}(v_{p-1}) = 2p - 4$ and $d_{\overline{T}}(u_1) = d_{\overline{T}}(u_2) = \dots = d_{\overline{T}}(u_p) = 2p - 4$. The set $S = N_{\overline{T}}(v_2)$ is a co-isolated locating dominating set of \overline{T} . Therefore, $\gamma_{cild}(\overline{T}) \leq 2p - 4$. By Lemma 3.3, $\delta(\overline{P}_p^+) \leq \gamma_{cild}(\overline{P}_p^+)$. Hence, $\gamma_{cild}(\overline{T}) \geq 2p - 4$.

Therefore, $\gamma_{cild}(\overline{P}_p^+) = 2p - 4$. ■

Definition 3.8: Given two positive integers r and s , let $K_2(r, s)$ denote the double star, obtained after joining the central vertices of the stars $K_{1,r}$ and $K_{2,s}$.

Theorem 3.9: $\gamma_{cild}(K_2(r, s)) = r + s - 2$; ($r, s \geq 2$).

Proof: Let $G \cong K_2(r, s)$ and $V(G) = \{v_1, v_2, \dots, v_r, u_1, u_2, \dots, u_s\}$ with v_1 and u_1 as the central vertices and $u_1v_1 \in E(G)$. Then $d_G(v_1) = r$ and $d_G(u_1) = s$. Also, $d_G(v_2) = d_G(v_3) = \dots = d_G(v_r) = 1$ and $d_G(u_2) = d_G(u_3) = \dots = d_G(u_s) = 1$. Therefore, $d_{\overline{G}}(v_1) = s - 1$ and $d_{\overline{G}}(u_1) = r - 1$. Also, $d_{\overline{G}}(v_i) = d_{\overline{G}}(v_j) = r + s - 2$; $i = 2, 3, \dots, r$ and $j = 2, 3, \dots, s$. Let $S = N_{\overline{G}}(v_2)$. Then $\langle V - S \rangle$ contains only two vertices v_2 and v_1 . Also, $N_{\overline{G}}(v_1) \cap S \neq N_{\overline{G}}(v_2) \cap S$. Hence S forms a co-isolated locating dominating set of \overline{G} .

Therefore, $\gamma_{cild}(K_2(r, s)) \leq r + s - 2$.

Let $\gamma_{cild}(K_2(r, s)) < r + s - 2$.

Then by Lemma 3.3, the set $S = N_{\overline{G}}(v_1)$ then, $|S| = s - 1$. Also, $V - S = \{v_2, v_3, v_4, \dots, v_r\}$ and $N_{\overline{G}}(v_i) \cap S = N_{\overline{G}}(v_j) \cap S$ for $i \neq j$ and $i, j = 2, 3, \dots, r$.

Therefore, S cannot be a γ_{cild} -set of \overline{G} .

Hence, $\gamma_{cild}(K_2(r, s)) = r + s - 2$. ■

Theorem 3.10: $\gamma_{cild}(K_{m,n} - e) = m + n - 2$, for $m, n \geq 2$.

Proof: Let $G \cong K_{m,n} - e$ with vertex set $V(G) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$. Let $u_1v_1 \notin E(G)$. Then $d_G(u_1) = n - 1$ and $d_G(v_1) = m - 1$. Also, $d_G(u_2) = d_G(u_3) = \dots = d_G(u_m) = n$ and $d_G(v_2) = d_G(v_3) = \dots = d_G(v_n) = m$. Therefore, $d_{\overline{G}}(u_1) = m$ and $d_{\overline{G}}(v_1) = n$. Also, $d_{\overline{G}}(u_i) = m - 1$ and $d_{\overline{G}}(v_j) = n - 1$; $i = 2, 3, \dots, m$ and $j = 2, 3, \dots, n$. Let $S = N_{\overline{G}}(v_1)$. Then, $S = \{u_1, u_2, \dots, u_m, v_2, v_3, \dots, v_{n-1}\}$ and $V - S = \{v_1, v_n\}$ also, $N_{\overline{G}}(v_1) \cap S \neq N_{\overline{G}}(v_2) \cap S$. Hence, S forms a co-isolated locating dominating set of \overline{G} . Therefore, $\gamma_{cild}(K_{m,n} - e) \leq |S| = m + n - 2$. Let $\gamma_{cild}(K_{m,n} - e) < m + n - 2$. Then by Lemma 3.3, the set $S = N_{\overline{G}}(v_1)$ and $|S| = n$. Also $V - S = \{u_1, v_2, v_3, v_4, \dots, v_n\}$ and $N_{\overline{G}}(v_i) \cap S = N_{\overline{G}}(v_j) \cap S$

for $i \neq j$ and $i, j = 2, 3, \dots, n$. Therefore, S cannot be a γ_{cild} -set of \overline{G} .

Hence, $\gamma_{cild}(\overline{K_{m,n} - e}) = m + n - 2$. ■

Definition 3.11: $K_{1,r}^s$ represents the graph obtained by adding a new vertex to s leaves of the star $K_{1,r}$; $s < r$ and $s \geq 1$.

Theorem 3.12: Let $G \cong K_{1,r}^s$. Then $\gamma_{cild}(\overline{G}) = r - 1$, where $r \geq 3$ and $s < r$.

Proof: Let $G \cong K_{1,r}^s$ where $1 \leq s < r$ and $r \geq 3$. Let u be the central vertex of $K_{1,r}^s$ and v be the vertex adjacent to s pendant vertices of $K_{1,r}$. Then $deg_{\overline{G}}(u) = 1$ and v is its support in \overline{G} . The subgraph of \overline{G} induced by r pendant vertices in G is complete in \overline{G} . Since $s < r$, there exists two vertices say $w, x \in V(G)$ such that $vw \in E(\overline{G})$ and $vx \notin E(\overline{G})$ where w, x are pendant vertices in G . Let $S' = \{u, w, x\}$ and $S = V - S'$. Then S is a co-isolated locating dominating set of \overline{G} . Therefore, $\gamma_{cild}(\overline{G}) \leq |S| = r - 1$. Also, K_r is an induced subgraph of \overline{G} and hence, $\gamma_{cild}(\overline{G}) \geq \gamma_{cild}(K_r) = r - 1$. Therefore, $\gamma_{cild}(\overline{G}) = r - 1$. ■

Lemma 3.13: If $G \cong mK_2$, then $\gamma_{cild}(\overline{G}) = 2m - 1$, $m \geq 1$.

Proof: Let $G \cong mK_2$ and let $V(G) = \{u_1, v_1, u_2, v_2, \dots, u_m, v_m\}$ where $u_i v_i \in E(K_2)$ for $i = 1, 2, \dots, m$. Let $Y = \{u_1, v_1\}$ and $X = V(G) - Y$. Y is an independent subset of \overline{G} and each vertex in X is adjacent to each in Y in \overline{G} . Also, any two non-adjacent vertices of the induced subgraph $\langle X \rangle$ of \overline{G} have common neighbors in $\langle X \rangle$. Then by Theorem 2.7, $\gamma_{cild}(\overline{G}) = 2m - 1$. ■

Theorem 3.14: For any doubly connected graph G of order $p \geq 4$,

- (i) $4 \leq \gamma_{cild}(G) + \gamma_{cild}(\overline{G}) \leq 2p - 4$ and
- (ii) $4 \leq \gamma_{cild}(G) \cdot \gamma_{cild}(\overline{G}) \leq (p - 2)^2$.

Proof: By Theorem 2.2, $\gamma_{cild}(G) = 1$ if and only if $G \cong K_2$ and its complement \overline{G} is $2K_1$. Therefore, every doubly-connected graph G of order at least 4 satisfies $2 \leq \gamma_{cild}(G)$. That is, for any non-trivial doubly-connected graph G , $4 \leq \gamma_{cild}(G) + \gamma_{cild}(\overline{G})$. Also by Theorem 2.1, for every non-trivial simple connected graph G , $\gamma_{cild}(G) \leq p - 1$. In Theorem 2.7, the graphs for which $\gamma_{cild}(G) = p - 1$ are characterized. For those graphs, \overline{G} is disconnected. Therefore, $\gamma_{cild}(G) \leq p - 2$ and $\gamma_{cild}(\overline{G}) \leq p - 2$ and hence $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) \leq 2p - 4$. Result (ii) follows easily. ■

Theorem 3.15: Let G be a doubly connected graph with $p \geq 4$. Then $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) = 4$ if and only if G is one of the following graphs P_4, P_5, C_5, C_5 with a chord and the Bull graph.

Proof: In Theorem 2.6, the graphs for which $\gamma_{cild}(G) = 2$ are characterized. Since $p \geq 4$, $G \cong P_4, P_5, C_5$.

- (i) If $G \cong P_4, C_5$ or the Bull Graph, then \overline{G} is self complementary. Therefore, $\gamma_{cild}(G) = \gamma_{cild}(\overline{G}) = 2$.
- (ii) If $G \cong P_5$, then \overline{G} is the cycle C_5 with a chord and $\gamma_{cild}(\overline{G}) = 2$ and vice versa.
- (iii) If G is the graph obtained by attaching a pendant edge at a vertex of C_3 (or) at a vertex of degree 2 in $K_4 - e$, then \overline{G} is disconnected (or) $\gamma_{cild}(\overline{G}) = 3$.
- (iv) If G is the graph as in (e) of Theorem 2.6, then $\gamma_{cild}(\overline{G}) = 3$.

Therefore, $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) = 4$, if $G \cong P_4, P_5, C_5, C_5$ with a chord and the Bull graph. ■

Observation 3.16: For a doubly connected graph G with $p \geq 4$, $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) = 5$ if and only if either G is the graph obtained by attaching a pendant edge at a vertex of degree 2 in $K_4 - e$ or G is the graph obtained by attaching a path of length 2 at a vertex of C_3 .

IV. CONCLUSION

Here, co-isolated locating domination numbers of complements of Path, Cycle, Double Star, $K_{m,n} - e$, $K_{1,r}^s$ and mK_2 are studied. Also, the graphs for which $\gamma_{cild}(G) + \gamma_{cild}(\overline{G}) = 4$ are found.

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