Solutions of Circulant Geometric Linear Systems

A.C.F. Bueno

Abstract—In this paper, we present the solutions of the two types of circulant geometric linear systems (CGLS): right circulant geometric linear systems (RCGLS) and left circulant geometric linear systems (LCGLS).

Index Terms—right circulant geometric linear system, left circulant geometric linear system.

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I. INTRODUCTION

Let \( \{ar^k\}_{k=0}^{\infty} \) be the geometric sequence \( \{a, ar, ar^2, \ldots\} \) where \( a \neq 0 \) and \( r \neq 0, 1 \) then the two types of circulant geometric linear systems are the following:

A right circulant geometric linear system (RCGLS) is a linear system the form

\[
R\vec{x} = \vec{b}
\]

where

\[
R = \begin{pmatrix}
ar & ar^2 & \ldots & ar^{n-2} & ar^{n-1} \\
ar^2 & ar^3 & \ldots & ar^{n-3} & ar^{n-2} \\
r^3 & ar^4 & \ldots & ar^{n-4} & ar^{n-3} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
ar^{n-2} & ar^{n-3} & \ldots & a & ar \\
ar^{n-1} & ar^n & \ldots & ar & a \\
\end{pmatrix}
\]

\[
\vec{x} = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1} \\
x_n \\
\end{pmatrix}
\]

\[
\vec{b} = \begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_{n-1} \\
b_n \\
\end{pmatrix}
\]

A left circulant geometric linear system (LCGLS) is a linear system the form

\[
L\vec{x} = \vec{b}
\]

where

\[
L = \begin{pmatrix}
a & ar & ar^2 & \ldots & ar^{n-2} & ar^{n-1} \\
ar & ar^2 & ar^3 & \ldots & ar^{n-3} & ar^{n-2} \\
ar^2 & ar^3 & ar^4 & \ldots & ar^{n-4} & ar^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
ar^{n-2} & ar^{n-3} & ar^{n-4} & \ldots & a & ar \\
ar^{n-1} & ar^n & \ldots & ar & a \\
\end{pmatrix}
\]

\[
\vec{x} = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1} \\
x_n \\
\end{pmatrix}
\]

\[
\vec{b} = \begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_{n-1} \\
b_n \\
\end{pmatrix}
\]

In [3], it has been shown that the matrices \( R \) and \( L \) are related by the following equation:

\[
L = \Pi R
\]

(1)

where \( \Pi = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\end{pmatrix} \]

which is an orthonormal matrix meaning \( \Pi = \Pi^T = \Pi^{-1} \). Furthermore, left multiplication by \( \Pi \) fixes the first row and does a horizontal flip on the remaining rows. Also, the right multiplication by \( \Pi \) fixes the first column and does a vertical flip on the remaining columns.

In [2], it has been established that

\[
R^{-1} = \frac{1}{ar^n - 1} \begin{pmatrix}
-1 & r & 0 & \ldots & 0 & 0 \\
0 & -1 & r & \ldots & 0 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & -1 & r \\
0 & 0 & 0 & \ldots & 0 & -1 \\
\end{pmatrix}
\]

(2)

Hence,

\[
L^{-1} = (\Pi R)^{-1} = R^{-1} \Pi
\]
We will use equations (2) and (3) to prove our main results.

II. MAIN RESULTS

Theorem 2.1: The solutions of RCGLS are the following:

- For \( k=1,2, \ldots, n-1 \)
  \[
  x_k = \frac{b_{k+1} r - b_k}{a(r^n - 1)}
  \]
- and for \( k=n \)
  \[
  x_n = \frac{b_1 r - b_n}{a(r^n - 1)}
  \]

Proof: Solving for \( \bar{x} \), we have \( \bar{x} = R^{-1}\bar{b} \). Using (2), this will yield

\[
\bar{x} = \begin{pmatrix}
\frac{b_2}{a(r^n - 1)} & \frac{b_3}{a(r^n - 1)} & \ldots & \frac{b_n}{a(r^n - 1)} \\
\frac{b_3}{a(r^n - 1)} & \frac{b_4}{a(r^n - 1)} & \ldots & \frac{b_n}{a(r^n - 1)} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{b_n}{a(r^n - 1)} & \frac{b_2}{a(r^n - 1)} & \ldots & \frac{b_3}{a(r^n - 1)}
\end{pmatrix}
\]

which is as desired.

Theorem 2.2: The solutions of LCGLS are the following:

- For \( k=1 \)
  \[
  x_1 = \frac{b_n r - b_1}{a(r^n - 1)}
  \]
- and for \( k=2,3, \ldots, n \)
  \[
  x_k = \frac{b_{n-(k-1)} r - b_{n-(k-2)}}{a(r^n - 1)}
  \]

Proof: Solving for \( \bar{x} \), we have \( \bar{x} = L^{-1}\bar{b} \). Using (3), the result will be

\[
\bar{x} = \begin{pmatrix}
\frac{b_2}{a(r^n - 1)} & \frac{b_3}{a(r^n - 1)} & \ldots & \frac{b_n}{a(r^n - 1)} \\
\frac{b_3}{a(r^n - 1)} & \frac{b_4}{a(r^n - 1)} & \ldots & \frac{b_n}{a(r^n - 1)} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{b_n}{a(r^n - 1)} & \frac{b_2}{a(r^n - 1)} & \ldots & \frac{b_3}{a(r^n - 1)}
\end{pmatrix}
\]

which is as desired.

III. CONCLUSION

In this paper, the solutions of RCGLS and LCGLS were obtained. The solutions are dependent on the first term \( a \), common ratio \( r \) and number of terms \( n \) of the geometric sequence \( \{a^k\}_{k=0}^{\infty} \).

REFERENCES

