Solutions of Circulant Geometric Linear Systems

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***Abstract*—In this paper, we present the solutions of the two types of circulant geometric linear systems (CGLS): right circulant geometric linear systems (RCGLS) and left circulant geometric linear systems (LCGLS).**

***Index Terms*—right circulant geometric linear system, left circulant geometric linear system.**

***MSC 2010 Codes* – 65F05, 93C05**

I. Introduction

LET $\{ar^k\}_{k=0}^{+\infty}$ be the geometric sequence $\{a,\ ar,\ ar^2,\ldots\}$ where $a \neq 0$ and $r \neq 0,\ 1$ then the two types of circulant geometric linear systems are the following:

A right circulant geometric linear system (RCGLS) is a linear system the form

$$R\vec{x}=\vec{b}$$

where

$$R = \begin{pmatrix} a & ar & ar^2 & ... & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & ... & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & ... & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & ... & a & ar \\ ar & ar^2 & ar^3 & ... & ar^{n-1} & a \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}$$

A left circulant geometric linear system (LCGLS) is a linear system the form

$$L\vec{x}=\vec{b}$$

where

$$L = \begin{pmatrix} a & ar & ar^2 & ... & ar^{n-2} & ar^{n-1} \\ ar & ar^2 & ar^3 & ... & ar^{n-1} & ar^{n-2} \\ ar^2 & ar^3 & ar^4 & ... & ar^{n-2} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^{n-2} & ar^{n-1} & ar^{n-2} & ... & ar^2 & ar \\ ar^{n-1} & ar^{n-2} & ar^{n-3} & ... & ar & a \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}$$

In [3], it has been shown that the matrices R and L are related by the following equation:

$$L=\Pi R \tag{1}$$

where $\Pi = \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \end{pmatrix}$ which is an orthonormal matrix meaning $\Pi=\Pi^T=\Pi^{-1}$. Furthermore, left multipliction by Π fixes the first row and does a horizontal flip on the remaining rows. Also, the right multiplication by Π fixes the first column and does a vertical flip on the remaining columns.

In [2], it has been established that

$$R^{-1}=\frac{1}{a(r^n-1)}\begin{pmatrix} -1 & r & 0 & ... & 0 & 0 \\ 0 & -1 & r & ... & 0 & 0 \\ 0 & 0 & -1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & -1 & r \\ r & 0 & 0 & ... & 0 & -1 \end{pmatrix} \tag{2}$$

Hence,

$$\begin{aligned} L^{-1} &= (\Pi R)^{-1} \\ &= R^{-1}\Pi \end{aligned}$$

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$$= \frac{1}{a(r^n-1)} \begin{pmatrix} -1 & 0 & 0 & \dots & 0 & r \\ 0 & 0 & 0 & \dots & r & -1 \\ 0 & 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & r & -1 & \dots & 0 & 0 \\ r & -1 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (3)$$

We will use equations (2) and (3) to prove our main results.

II. Main Results

Theorem 2.1: The solutions of RCGLS are the following:

- For k=1,2, ..., n-1

$$x_k = \frac{b_{k+1}r - b_k}{a(r^n-1)}$$

- and for k=n

$$x_n = \frac{b_1 r - b_n}{a(r^n-1)}$$

.

Proof:
Solving for $\vec{x}$, we have have $\vec{x} = R^{-1}\vec{b}$. Using (2), this will yield

$$\vec{x} = \begin{pmatrix} \frac{b_2 r - b_1}{a(r^n-1)} \\ \frac{b_3 r - b_2}{a(r^n-1)} \\ \frac{b_4 r - b_3}{a(r^n-1)} \\ \vdots \\ \frac{b_n r - b_{n-1}}{a(r^n-1)} \\ \frac{b_1 r - b_n}{a(r^n-1)} \end{pmatrix}$$

which is as desired.

Theorem 2.2: The solutions of LCGLS are the following:

- For k=1,

$$x_1 = \frac{b_n r - b_1}{a(r^n-1)}$$

- and for k=2,3, ..., n

$$x_k = \frac{b_{n-(k-1)}r - b_{n-(k-2)}}{a(r^n-1)}$$

Proof: Solving for $\vec{x}$, we have $\vec{x} = L^{-1}\vec{b}$. Using (3), the result will be

$$\vec{x} = \begin{pmatrix} \frac{b_n r - b_1}{a(r^n-1)} \\ \frac{b_{n-1} r - b_n}{a(r^n-1)} \\ \frac{b_{n-2} r - b_{n-1}}{a(r^n-1)} \\ \vdots \\ \frac{b_2 r - b_3}{a(r^n-1)} \\ \frac{b_1 r - b_2}{a(r^n-1)} \end{pmatrix}$$

which is as desired. □

III. Conclusion

In this paper, the solutions of RCGLS and LCGLS were obtained. The solutions are dependent on the first term a, common ratio r and number of terms n of the geometric sequence $\{ar^k\}_{k=0}^{+\infty}$.

References

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