Super Edge Bimagic Labeling Graphs

A. Amara Jothi*, N.G. David and J. Baskar Babujee

***Abstract*—A graph $G(V,E)$ with order p and size q is edge magic if there exists a bijection $g : V \cup E \rightarrow \{1,2,...,p+q\}$ such that $g(u)+g(v)+g(uv)$ is a constant c, for all $uv \in E$ and it is called super edge magic(SEM) if g also satisfies $g(V) = \{1,2,...,p\}$. G is called super edge bimagic(SEBM) if the bijective function $g : V \cup E \rightarrow \{1,2,3,...,p+q\}$ is such that $g(u)+g(v)+g(uv)$ is either c_1 or c_2 for all $uv \in E$ and $g(V) = \{1,2,...,p\}$. In this article, we exhibit the existence of edge bimagic labeling for switching of path, cycle, star, crown and helm graphs. Also examine whether operations on edge magic graphs results in edge bimagic graphs or not.**

***Index Terms*—Graph, magic labeling, bimagic labeling, bijective function.**

***MSC 2010 Codes* – 05C78**

I. Preliminary

In this article, we consider simple and undirected graphs $G(V,E)$ with $|V| = p$ and $|E| = q$. The concept of graph labeling was introduced in the early 1960s. In 1970, Kotzig and Rosa [1] defined a magic labeling of G as a bijection $g : V \cup E \rightarrow \{1,2,3,...,p+q\}$, such that $g(u)+g(uv)+g(v)$ is a constant c for all $uv \in E$. Inspired by Kotzig-Rosa notion, Enomoto, Llado, Nakamigawa and Ringel [2] called an edge magic total labeling as SEM labeling if the set of vertex labels are $\{1,2,...,p\}$. In [3] the authors conjectured that every tree admits SEM total labeling. In [4] investigated that some particular subclasses of trees, like every caterpillars, banana trees admit SEM labeling.

In 2004, the notion of edge bimagic labeling was introduced by Baskar Babujee [5]. A graph G is called edge bimagic if the bijection $g : V \cup E \rightarrow \{1,2,3,...,p+q\}$ is such that $g(u)+g(v)+g(uv)$ is either c_1 or c_2 for all $uv \in E$. An edge bimagic graph is called super edge bimagic(SEBM) if the set of vertex labels are $\{1,2,...,p\}$. The concept of edge bimagic total labeling was extensively studied by many authors. Further, it is shown that the graph $G_1 \ \hat{o} \ G_2$ admits edge bimagic total labeling if G_1 has superior edge magic labeling and G_2 has SEM labeling [6] and also it is proved that $G_1 \ \hat{o} \ G_2$ admits SEBM labeling if $G_1$and G_2 have super edge magic labeling. For further details, we refer to the dynamic survey of graph labeling by Gallian [7].

Definition 1.1 In a graph $G(V,E)$, vertex switching at $v \in V$, denoted by G_v is obtained by removing all edges $uv \in E$ incident to v and adding edges vw, whenever $vw \notin E$, $w \in V$.

Definition 1.2 [8] Duplication of a vertex $v \in V$ by an edge $v^{'}v^{''}$ in a graph G produces a new graph $G^{'}$ such that the neighborhood of $v^{'}$ and $v^{''}$ are respectively $N(v^{'}) = \{v, v^{''}\}$ and $N(v^{''}) = \{v, v^{'}\}$.

Definition 1.3 [8] Duplication of a vertex v of a graph G by a vertex $v^{'}$ produces a new graph $G^{'}$ by adding a new vertex $v^{'}$ in such a way that $N(v) = N(v^{'})$.

Definition 1.4 Given graph G_1 and G_2, $G_1 \ \hat{e} \ G_2$ is obtained by introducing an edge between an arbitrary vertex of G_1 and an arbitrary vertex of G_2. If G_1 has p_1 vertices and q_1 edges and G_2 has p_2 vertices and q_2 edges then $G_1 \ \hat{e} \ G_2$ will have (p_1+p_2) vertices and (q_1+q_2+1) edges.

A. Amara Jothi and N.G. David is in the Department of Mathematics, Madras Christian College, Chennai, India. (E-mail: amarajothia@gmail.com, ngdmcc@gmail.com)

J. Baskar Babujee is in the Department of Mathematics, Anna University, Chennai, India. (E-mail: baskarbabujee@yahoo.com)

II. Main Results

In this section, super edge bimagic graphs are obtained by using some operations on graphs.

Theorem 2.1 Switching a pendant vertex in path graph P_n, $n \geq 6$ admits SEBM total labeling.

Proof Let $P_n : v_1, v_2, ..., v_n$ be the path graph of order $n \geq 6$. We assume that the graph G_v is obtained from P_n by switching the vertex v_1. Let $E(G_v) = E_1 \cup E_2$ where $E_1 = \{v_iv_{i+1} : 2 \leq i \leq n-1\}$ and $E_2 = \{v_1v_i : 3 \leq i \leq n\}$ with $|E(G_v)| = 2n-4$. Define a bijective mapping $g : V(G_v) \cup E(G_v) \rightarrow \{1,2,...,3n-4\}$ in the following way.

Case (i): $n \equiv 0 \ (mod \ 2)$

$$g(v_i) = \begin{cases} \frac{n+i+1}{2} - 1, & \text{if } i \text{ is odd}, 3 \leq i \leq n-1, \\ \frac{i}{2}, & \text{if } i \text{ is even}, 2 \leq i \leq n. \end{cases}$$

$$g(v_1v_i) = \begin{cases} n + \frac{n-i-1}{2} + 1, & \text{if } i \text{ is odd}, 3 \leq i \leq n-1, \\ 2n - \frac{i}{2}, & \text{if } i \text{ is even}, 4 \leq i \leq n. \end{cases}$$

For $2 \leq i \leq n-1$; $g(v_iv_{i+1}) = 3n-2-i$, $g(v_1) = n$.

In the following, we verify the bimagic property of G_v, when $n \equiv 0$ (mod 2).

(i) For edges $v_iv_{i+1} \in E_1$,

$$g(v_i) + g(v_{i+1}) + g(v_iv_{i+1}) =$$
$$\begin{cases} (\frac{n+i+1}{2} - 1) + (\frac{i+1}{2}) + (3n-2-i) = \frac{7n-4}{2}, & \text{if } i \text{ is odd}, \\ (\frac{i}{2}) + (\frac{n+i+2}{2} - 1) + (3n-2-i) = \frac{7n-4}{2}, & \text{if } i \text{ is even}. \end{cases}$$

(ii) For edges $v_1v_i \in E_2$,

$$g(v_1) + g(v_i) + g(v_1v_i) =$$
$$\begin{cases} (n) + (\frac{n+i+1}{2} - 1) + (n + \frac{n-i-1}{2} + 1) = 3n, & \text{if } i \text{ is odd}, \\ (n) + (\frac{i}{2}) + (2n - \frac{i}{2}) = 3n, & \text{if } i \text{ is even}. \end{cases}$$

Case (ii): $n \equiv 1 \ (mod \ 2)$

$$g(v_i) = \begin{cases} \frac{n+i+2}{2} - 2, & \text{if } i \text{ is odd}, \ 3 \leq i \leq n, \\ \frac{i}{2}, & \text{if } i \text{ is even}, \ 2 \leq i \leq n-1. \end{cases}$$

$$g(v_1v_i) = \begin{cases} n + \frac{n-i}{2} + 1, & \text{if } i \text{ is odd, } 3 \leq i \leq n, \\ 2n - \frac{i}{2}, & \text{if } i \text{ is even, } 4 \leq i \leq n-1. \end{cases}$$

For $2 \leq i \leq n-1$; $g(v_iv_{i+1}) = 3n - 2 - i$, $g(v_1) = n$.

In the following, we verify the bimagic property of G_v, when $n \equiv 1 \pmod 2$.

(i) For edges $v_iv_{i+1} \in E_1$

$$g(v_i) + g(v_{i+1}) + g(v_iv_{i+1}) = \begin{cases} (\frac{n+i+2}{2} - 2) + (\frac{i+1}{2}) + (3n-2-i) = \frac{7n-5}{2}, & \text{if } i \text{ is odd,} \\ (\frac{i}{2}) + (\frac{n+i+3}{2} - 2) + (3n-2-i) = \frac{7n-5}{2}, & \text{if } i \text{ is even} \end{cases}$$

(ii) For edges $v_1v_i \in E_2$

$$g(v_1) + g(v_i) + g(v_1v_i) = \begin{cases} \{(n) + (\frac{n+i+2}{2} - 2) + (n + \frac{n-i}{2} + 1) = 3n\}, & \text{if } i \text{ is odd,} \\ (n) + (\frac{i}{2}) + (2n - \frac{i}{2}) = 3n, & \text{if } i \text{ is even.} \end{cases}$$

From the above cases we observe that the two bimagic constants are (i) $c_1 = \frac{7n-5}{2}$ and $c_2 = 3n$, if n is odd and (ii) $c_1 = \frac{7n-4}{2}$ and $c_2 = 3n$, if n is even. This establishes that switching a pendant vertex in path graph P_n, $n \geq 6$ has SEBM total labeling.

Remark 2.1 Switching a pendant vertex in path graph $P_n(n = 3, 4$ or $5)$ admits SEM total labeling.

Theerem 2.2 Switching a vertex in cycle graph $C_n, n \geq 6$ admits SEBM total labeling.

Proof Let $C_n : v_1, v_2, ..., v_n$ be the cycle of order $n \geq 6$. Without loss of generality let us assume that the graph G_v is obtained from C_n by switching the vertex v_n. Let $E(G_v) = E_1 \cup E_2$ where $E_1 = \{v_iv_{i+1} : 1 \leq i \leq n-2\}$ and $E_2 = \{v_nv_i : 2 \leq i \leq n-2\}$ with $|E(G_v)| = 2n-5$. Define a bijective function $g : V(G_v) \cup E(G_v) \to \{1, 2, ..., 3n-5\}$ in the following way.

Case (i): $n \equiv 0\ (mod\ 2)$

For $1 \leq i \leq n-2$; $g(v_iv_{i+1}) = 2(n-1) - i$.

$$g(v_i) = \begin{cases} \frac{n+i+1}{2}, & \text{if } i \ is\ odd, 1 \leq i \leq n-1, \\ 1 + \frac{i}{2}, & \text{if } i\ is\ even, 2 \leq i \leq n-2. \end{cases}$$

$$g(v_nv_i) = \begin{cases} 3n - 3 - \frac{i+1}{2}, & \text{if } i\ is\ odd, 3 \leq i \leq n-1, \\ 2n - 2 + \frac{n-i}{2}, & \text{if } i\ is\ even, 2 \leq i \leq n-2. \end{cases}$$

$g(v_n) = 1$.

Case (ii):$n \equiv 1\ (mod\ 2)$

For $1 \leq i \leq n-2$; $g(v_iv_{i+1}) = 3n - 4 - i$.

$$g(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i\ is\ odd, 1 \leq i \leq n, \\ \frac{n+i+1}{2} - 1, & \text{if } i\ is\ even, 2 \leq i \leq n-1. \end{cases}$$

$$g(v_nv_i) = \begin{cases} 2n - 1 - \frac{i+1}{2}, & \text{if } i\ is\ odd, 3 \leq i \leq n-2, \\ n - 1 + \frac{n-i+1}{2}, & \text{if } i\ is\ even, 2 \leq i \leq n-1. \end{cases}$$

$g(v_n) = n$.

Verification of bimagic property for the case when n is even.

(i) For edges $v_iv_{i+1} \in E_1$,

$$g(v_i) + f(v_{i+1}) + f(v_iv_{i+1}) = \begin{cases} (\frac{n+i+1}{2}) + (1 + \frac{i+1}{2}) + (2(n-1) - i) = \frac{5n}{2}, & \text{if } i\ is\ odd, \\ (1 + \frac{i}{2}) + (\frac{n+i+2}{2}) + (2(n-1) - i) = \frac{5n}{2}, & \text{if } i\ is\ even. \end{cases}$$

(ii) For edges $v_nv_i \in E_2$,

$$g(v_n) + f(v_i) + f(v_nv_i) = \begin{cases} (1 + \frac{n}{2}) + (\frac{i+1}{2}) + (3n - 3 - \frac{i+1}{2}) = \frac{7n-4}{2}, & \text{if } i\ is\ odd, \\ 1 + (1 + \frac{i}{2}) + (2n - 2 + \frac{n-i}{2}) = \frac{5n}{2}, & \text{if } i\ is\ even. \end{cases}$$

Verification of bimagic property for the case when n is odd.

(i) For edges $v_iv_{i+1} \in E_1$,

$$g(v_i) + g(v_{i+1}) + g(v_iv_{i+1}) = \begin{cases} (\frac{i+1}{2}) + (\frac{n+i+2}{2} - 1) + (3n - 4 - i) = \frac{7(n-1)}{2} & \text{if } i\ is\ odd, \\ (\frac{n+i+1}{2} - 1) + (\frac{i+2}{2}) + (3n - 4 - i) = \frac{7(n-1)}{2}, & \text{if } i\ is\ even. \end{cases}$$

(ii) For edges $v_nv_i \in E_2$,

$$g(v_n) + g(v_i) + g(v_nv_i) = \begin{cases} (n) + (\frac{i+1}{2}) + (2n - 1 - \frac{i+1}{2}) = 3n - 1 & \text{if } i\ is\ odd, \\ \{(n) + (\frac{n+i+1}{2} - 1) + (n - 1 + \frac{n-i+1}{2}) = 3n - 1\}, & \text{if } i\ is\ even. \end{cases}$$

Note that, when n is odd, the bimagic constants are $c_1 = \frac{7(n-1)}{2}$ and $c_2 = 3n - 1$. Also note that the bimagic constants are $c_1 = \frac{5n}{2}$ and $c_2 = \frac{7n-4}{2}$. Therefore, the resultant graph has SEBM total labeling.

Remark 2.2 Switching a vertex in cycle graph $C_n(n = 4$ or $5)$ admits SEM total labeling.

Theorem 2.3 Switching a pendant vertex in star graph $K_{1,n}, n \geq 3$ admits SEBM total labeling.

Proof Let $v_1, v_2, ..., v_{n+1}$ be the vertices of $K_{1,n}$ with v_1 as the vertex at the center. Without loss of generality le us assume that the graph G_v is obtained from $K_{1,n}$ by switching the vertex v_2. Let $E(G_v) = E_1 \cup E_2$ where $E_1 = \{v_1v_i : 3 \leq i \leq n+1\}$ and $E_2 = \{v_2v_i : 3 \leq i \leq n+1\}$ with $|E(G_v)| = 2n-2$. A bijective mapping $g : V(G_v) \cup E(G_v) \to \{1, 2, ..., 3n-1\}$ is given below.

For $3 \leq i \leq n+1$; $g(v_i) = n + 4 - i$, $g(v_1v_i) = 2n + i - 2$, $g(v_2v_i) = n - 1 + i$.

and let $g(v_1) = 1$, $g(v_2) = 2$.

In the following, it is determined that the above assignment results in the required labeling.

(i) For any edge $v_1v_i \in E_1$,

$g(v_1) + g(v_i) + g(v_1v_i) = 1 + (n + 4 - i) + (2n + i - 2) = 3(n+1) = c_1$.

(ii) For any edge $v_2v_i \in E_2$,

$g(v_2) + g(v_i) + g(v_2v_i) = 2 + (n + 4 - i) + (n + i - 1) = 2n + 5 = c_2$.

This proves that switching a pendant vertex in star graph has SEBM total labeling.

Theorem 2.4 Switching a pendant vertex in crown graph $C_n \odot K_1, n \geq 3$ admits SEBM total labeling.

Proof Let $v_1, v_2, ..., v_n$ be the pendant vertices and $u_1, u_2, ..., u_n$, be the rim vertices of $C_n \odot K_1$. Let us assume that the graph G_v is obtained by switching a pendant vertex v_1. We denote that $E(G_v) = E_1 \cup E_2 \cup E_3 \cup E_4$ where $E_1 = \{u_1u_n, u_iu_{i+1} : 1 \leq i \leq n-1\}$, $E_2 = \{u_iv_i : 2 \leq i \leq n\}$, $E_3 = \{v_1u_2, v_1u_i : 2 \leq i \leq n\}$, $E_4 = \{v_1v_i : 2 \leq i \leq n\}$ with $|E(G_v)| = 4n-3$. A bijection $g : V(G_v) \cup E(G_v) \to \{1, 2, ..., 3(2n-1)\}$ is given below.

Case (i): $n \equiv 1 \ (mod\ 2)$

For i = 1 to n-1 ; $g(u_iu_{i+1}) = 4n - 2i + 1$.

$$g(u_i) = \begin{cases} n+i, & \text{if } i \ is \ odd, 1 \leq i \leq n, \\ i, & \text{if } i \ is \ even, 2 \leq i \leq n-1. \end{cases}$$

$$g(v_1u_i) = \begin{cases} 5n-i, & \text{if } i \ is \ odd, 3 \leq i \leq n, \\ 6n-i, & \text{if } i \ is \ even, 4 \leq i \leq n-1. \end{cases}$$

$$g(v_i) = \begin{cases} i, & \text{if } i \ is \ odd, 3 \leq i \leq n, \\ n+i, & \text{if } i \ is \ even, 4 \leq i \leq n-1. \end{cases}$$

$$g(v_1v_i) = \begin{cases} 6n-i, & \text{if } i \ is \ odd, 3 \leq i \leq n, \\ 5n-i, & \text{if } i \ is \ even, 4 \leq i \leq n-1. \end{cases}$$

For $2 \leq i \leq n$; $g(u_iv_i) = 4n-2i+2$, $g(v_1) = 1$, $f(u_1u_n) = 2n+1$, $f(v_1u_2) = 5n-1$.

We verify that the bimagic property in the following.

(i) For edges $u_iu_{i+1} \in E_1$,

$$g(u_i) + g(u_{i+1}) + g(u_iu_{i+1}) = \begin{cases} (n+i) + (i+1) + (4n-2i+1) = 5n+2, & \text{if } i \ is \ odd, \\ i + (n+i+1) + (4n-2i+1) = 5n+2, & \text{if } i \ is \ even. \end{cases}$$

For edge $u_1u_n \in E_1$,

$$g(u_1) + g(u_n) + g(u_1u_n) = (n+1) + (2n) + (2n+1) = 5n+2.$$

(ii) For edges $u_iv_i \in E_2$,

$$g(u_i) + g(v_i) + g(u_iv_i) = \begin{cases} i + (n+i) + (4n-2i+2) = 5n+2, & \text{if } i \ is \ odd, \\ (n+i) + i + (4n-2i+2) = 5n+2, & \text{if } i \ is \ even. \end{cases}$$

(iii) For edges $v_1u_i \in E_3$,

$$g(v_1) + g(u_i) + g(v_1u_i) = \begin{cases} 1 + (n+i) + (5n-i) = 6n+1, & \text{if } i \ is \ odd, \\ 1 + (i) + (6n-i) = 6n+1, & \text{if } i \ is \ even. \end{cases}$$

For edge $v_1u_i \in E_3$,

$g(v_1) + g(u_2) + g(v_1u_2) = 1 + (2) + (5n-1) = 5n+2$.

(iv) For edges $v_1v_i \in E_4$,

$$g(v_1) + g(v_i) + g(v_1v_i) = \begin{cases} 1 + (i) + (6n-i) = 6n+1, & \text{if } i \ is \ odd, \\ 1 + (n+i) + (5n-i) = 6n+1, & \text{if } i \ is \ even. \end{cases}$$

Case (ii): $n \equiv 0 \ (mod\ 2)$

For $1 \leq i \leq n-1$; $g(u_iu_{i+1}) = 6n - 1 - 2i$.

$$g(u_i) = \begin{cases} i+1, & \text{if } i \ is \ odd, 1 \leq i \leq n-1, \\ n+i, & \text{if } i \ is \ even, 2 \leq i \leq n. \end{cases}$$

$$g(v_i) = \begin{cases} n+i, & \text{if } i \ is \ odd, 3 \leq i \leq n-1, \\ i+1, & \text{if } i \ is \ even, 2 \leq i \leq n. \end{cases}$$

$$g(v_1v_i) = \begin{cases} 3n+2-i, & \text{if } i \ is \ odd, 3 \leq i \leq n-1, \\ 4n+1-i, & \text{if } i \ is \ even, 2 \leq i \leq n. \end{cases}$$

$$g(v_1v_i) = \begin{cases} 4n+1-i, & \text{if } i \ is \ odd, 3 \leq i \leq n-1, \\ 3n+2-i, & \text{if } i \ is \ even, 2 \leq i \leq n. \end{cases}$$

For $2 \leq i \leq n$ $g(u_iv_i) = 6n - 2i$, $g(v_1) = 1, g(u_1u_n) = 2n+1$.

We verify that the bimagic property below.

(i) For edges $u_iu_{i+1} \in E_1$,

$$g(u_i) + g(u_{i+1}) + g(u_iu_{i+1}) = \begin{cases} (i+1) + (n+i+1) + (6n-1-2i) = 7n+1, & \text{if } i \ is \ odd, \\ (n+i) + (i+1) + (6n-1-2i) = 7n+1, & \text{if } i \ is \ even. \end{cases}$$

For edge $u_1u_n \in E_1$,

$g(u_1) + g(u_n) + g(u_1u_n) = 2 + 2n + 2n + 1 = 4n + 3$.

(ii) For edges $u_iv_i \in E_2$,

$$g(u_i) + g(v_i) + g(u_iv_i) = \begin{cases} (i+1) + (n+i) + (6n-2i) = 7n+1, & \text{if } i \ is \ odd, \\ (n+i) + (i+1) + (6n-2i) = 7n+1, & \text{if } i \ is \ even. \end{cases}$$

(iii) For edges $v_1u_i \in E_3$,

$$g(v_1) + g(u_i) + g(v_1u_i) = \begin{cases} 1 + (i+1) + (4n+1-i) = 4n+3, & \text{if } i \ is \ odd, \\ 1 + (n+i) + (3n+2-i) = 4n+3, & \text{if } i \ is \ even. \end{cases}$$

(iv) For edges $v_1v_i \in E_4$,

$$g(v_1) + g(v_i) + g(v_1v_i) = \begin{cases} 1 + (n+i) + (3n+2-i) = 4n+3, & \text{if } i \ is \ odd, \\ 1 + (i+1) + (4n+1-i) = 4n+3, & \text{if } i \ is \ even. \end{cases}$$

From the above cases we have, when n is even, $c_1 = 7n+1$ and $c_2 = 4n+3$ and when n is odd, $c_1 = 5n+2$ and $c_2 = 6n+1$. Therefore, the resultant graph has SEBM total labeling.

Theorem 2.5 Switching the apex vertex in odd order helm graph $H_n, n \geq 3$ admits SEBM total labeling.

proof Let $v_i : 1 \leq i \leq n$ be the pendant vertices, $u_i : 1 \leq i \leq n$ be the rim vertices and w be the apex vertex of H_n. We assume that the graph G_v is obtained from H_n by switching an apex vertex w. We denote $E(G_v) = E_1 \cup E_2 \cup E_3$ where $E_1 = \{u_1u_n, u_iu_{i+1} : 1 \leq i \leq n-1\}$, $E_2 = \{u_iv_i : 2 \leq i \leq n\}$, $E_3 = \{wv_i : 1 \leq i \leq n\}$ with $|E(G_v)| = 3n$. Define a bijection $g : V(G_v) \cup E(G_v) \to \{1, 2, ..., 5n+1\}$ as follows.

For $1 \leq i \leq n-1$; $g(u_iu_{i+1}) = 2n+1+i$.

$$g(u_i) = \begin{cases} 2n - \frac{i+1}{2}, & \text{if } i \text{ } is \text{ } odd, 1 \leq i \leq n, \\ 2n+2-\frac{i}{2}, & \text{if } i \text{ } is \text{ } even, 2 \leq i \leq n-1. \end{cases}$$

$$g(u_iv_i) = \begin{cases} 4n+2-\frac{i+1}{2}, & \text{if } i \text{ } is \text{ } odd, 1 \leq i \leq n, \\ 4n-\frac{n+1}{2}+2-\frac{i}{2}, & \text{if } i \text{ } is \text{ } even, 2 \leq i \leq n-1. \end{cases}$$

For $1 \leq i \leq n$; $g(wv_i) = 5n+2-i$.
For $1 \leq i \leq n-1$; $g(v_i) = i+1$. $g(u_1u_n) = 3n+1$.
We claim that the bimagic contants are $c_1 = \frac{11n+9}{2}$ and $c_2 = 5n+4$.
(i) For edges $u_iu_{i+1} \in E_1$,

$$g(u_i)+g(u_{i+1})+g(u_iu_{i+1}) = \begin{cases} \begin{array}{c}\{2n+3-\frac{n+1}{2}-\frac{i+1}{2}+2n+2 \\ -\frac{i+1}{2}+2n+1+i = \frac{11n+9}{2}\}\end{array}, & \text{if } i \text{ } is \text{ } odd, \\ \begin{array}{c}\{2n+2-\frac{i}{2}+2n+3-\frac{n+1}{2} \\ -\frac{i+2}{2}+2n+1+i = \frac{11n+9}{2}\}\end{array}, & \text{if } i \text{ } is \text{ } even. \end{cases}$$

For the edge $u_1u_n \in E_1$,
$g(u_1)+g(u_n)+g(u_1u_n) = 2n-\frac{n+1}{2}+2+n+2+3n+1 = \frac{11n+9}{2}$.
(ii) For edges $u_iv_i \in E_2$,

$$g(u_i)+g(v_i)+g(u_iv_i) = \begin{cases} \begin{array}{c}\{2n+3-\frac{n+1}{2}-\frac{i+1}{2}+i \\ +1+4n+2-\frac{i+1}{2} = \frac{11n+9}{2}\}\end{array}, & \text{if } i \text{ } is \text{ } odd, \\ \begin{array}{c}\{2n+2-\frac{i}{2}+i+1+4n \\ -\frac{n+1}{2}+2-\frac{i}{2} = \frac{11n+9}{2}\}\end{array}, & \text{if } i \text{ } is \text{ } even. \end{cases}$$

For edges $wv_i \in E_3$,
$g(w)+g(v_i)+g(wv_i) = 1+i+1+5n+2-i = 5n+4 = c_2$.
Therefore, the resultant graph has SEBM labeling.

Theorem 2.6 Let $G_1(V_1, E_1)$ be a SEM graph of order p_1 and size q_1 and let its corresponding bijective function be f_1. If there exists a vertex $x_1 \in V_1$ such that $f_1(x_1) = p_1$ and adjacent to vertices having consecutive integers as their SEM label, then the graph G obtained from G_1 by duplicating the vertex x_1 admits SEBM total labeling.

proof Let $G_1(V_1, E_1)$ be a SEM graph with $|V_1| = p_1$, $|E_1| = q_1$ and $f_1 : V_1 \cup E_1 \rightarrow \{1,2,3,...,p_1+q_1\}$ such that $f_1(u)+f_1(uv)+f_1(v) = c$ is a constant, for all $uv \in E_1$. Let x_1 in G_1 be such that $f_1(x_1) = p_1$ and adjacent to vertices $\{y_1, y_2, ..., y_{d(x_1)}\}$ with $f_1(y_1) = n_1+1$, $f_1(y_2) = n_1+2, ..., f_1(y_{d(x)}) = n_1+d(x_1)$ for some integer n_1. Let $G(V, E)$ be the graph obtained by duplicating x_1 in G. Let the newly added vertex corresponding to x_1 be x. We observe that $V = V_1 \cup V_2$ where $V_2 = \{x\}$ and $E = E_1 \cup E_2$ where $E_2 = xy_i : 1 \leq i \leq n$. We define a bijective mapping $f : V \cup E \rightarrow \{1,2,...,p_1+q_1+d(x)+1\}$ as follows:
For $u \in E$

$$f(u) = \begin{cases} f_1(u), & \text{if } u \in V_1 \\ p_1+1, & \text{if } u \in V_2 \end{cases}$$

and for all $uv \in E$
$f(uv) = f_1(uv)+1$, if $uv \in E_1$
$f(xy_i) = p_1+q_1+1+d(x_1)-i+1$, for $i = 1,2,...,d(x_1)$
Verification:
For each edge $uv \in E$
(i) if $uv \in E_1$ then,
$f(u)+f(uv)+f(v) = f_1(u)+(f_1(uv)+1)+f_1(v) = c+1 = c_1$
(ii) if $uv \in E_2$ with $u = x$ and $v = y_i$ then,
$f(u)+f(uv)+f(v) = (p_1+1)+(p_1+q_1+1+d(x_1)+1-i)+(n_1+i) = 2p_1+q_1+3+d(x_1)+n_1 = c_2$.
Hence the graph G' has SEBM total labeling.

Theorem 2.7 If G is a SEM graph then there exist at most a vertex in G such that duplication of the vertex by an edge admits edge bimagic total labeling.

proof Let $G(V, E)$ be a SEM graph of order p and size q with a bijective function $f : V \cup E \rightarrow \{1,2,...,p+q\}$ such that $f(u)+f(uv)+f(v) = c$ is a constant, for all $uv \in E$. Let G' be the graph obtained by duplicating the vertex x with $f(x) = p$ in G by an edge yz. Observe that the vertex set $V' = V \cup V_1$ where $V_1 = \{y, z\}$ and edge set $E' = E \cup E_1$ where $E_1 = \{xy, xz, yz\}$. Define a bijection $g : V' \cup E' \rightarrow \{1,2,...,p+q+5\}$ as follows:
For vertices in V', $g(u) = f(u)$ if $u \in V$, $g(y) = p+1$ and $g(z) = p+2$.
The edge labels are defined as follows:
For edges in E',
if $uv \in E$, then $g(uv) = f(uv)+2$, $g(yz) = p+q+3$, $g(xz) = p+q+4$ and $g(xy) = p+q+5$.
Verification:
For edges in E',
(i) if $uv \in E$, then we have
$g(u)+g(uv)+g(v) = f(u)+(f(uv)+2)+f(v) = c+2 = c_1$
(ii) if edges in E_1, then we have
$g(x)+g(xy)+g(y) = (p)+(p+q+5)+(p+1) = 3p+q+6 = c_2$,
$g(x)+g(xz)+g(z) = (p)+(p+q+4)+(p+2) = 3p+q+6 = c_2$,
and $g(y)+g(yz)+g(z) = (p+1)+(p+q+3)+(p+2) = 3p+q+6 = c_2$.
Hence the graph G' admits SEBM total labeling.

Theorem 2.8 If G is a SEM graph then there exist at least $n+1$ graphs from the class $G \hat{e} K_{1,n}$ that admit SEBM total labeling.

proof Let $G(V, E)$ be a SEM graph with $|V| = p$ and $|E| = q$ respectively. Then there exists a bijection $f : V \cup E \rightarrow \{1,2,...,p+q\}$ such that the magic sum $f(u)+f(uv)+f(v)$ is a constant c, for all $uv \in E$ and let $f(x) = p$ for some $x \in V$. Let the graph obtained by edge joining the vertex $x \in V$ and one of the vertices, say z in $K_{1,n}$ be G'. That is $G' = G \hat{e} K_{1,n}$ with vertex set $V' = V \cup V_1$ where $V_1 = V(K_{1,n}) = \{y_1, y_2, ..., y_{n+1}\}$, where y_{n+1} is the vertex with degree n and edge set $E' = E \cup E_1 \cup \{xz\}$ where $E_1 = E(K_{1,n})$.
Case(i): z is a pendant vertex of $K_{1,n}$
Without loss of generality, let us assume that $z = y_1$. We define the bijective function $g : V' \cup E' \rightarrow \{1,2,...,p+q+2(n+1)\}$ as follows:
For $u \in V'$,

if $u \in V$, then $g(u) = f(u)$
if the vertices are in V_1, then $g(y_{n+1}) = p + 1$ and $g(y_i) = p + 1 + i; 1 \leq i \leq n$.
For all $uv \in E^{'}$,
if $uv \in E$, then $g(uv) = f(uv) + n + 1$ and if the edges belong to E_1,
then $g(xy_1) = p + q + 2(n + 1)$, $g(y_{n+1}y_i) = p + q + 2(n + 1) - i; 1 \leq i \leq n$.
Case(ii): z is the central vertex of $K_{1,n}$, that is $z = y_{n+1}$.
We define the bijective function $g : V^{'} \cup E^{'} \to \{1, 2, ..., p + q + 2(n + 1)\}$ in the following way:
For $u \in V^{'}$,
if $u \in V$, then $g(u) = f(u)$
if the vertex in V_1 then,
$g(y_i) = p + i; 1 \leq i \leq n + 1$.
For the edge belong to E_1
we define $g(y_{n+1}y_i) = p + q + 2(n + 1) - i; 1 \leq i \leq n$ and $g(xy_{n+1}) = p + q + 2(n + 1)$.
In the following, the SEBM labeling of g is verified.
Verification for case(i)
For each edge in $E^{'}$
(i) if the edge $uv \in E$ then,
$g(u) + g(uv) + g(v) = f(u) + (f(uv) + n + 1) + f(v) = c + n + 1 = c_1$
(ii) if the edges in E_1, then we obtain
$g(y_{n+1}) + g(y_{n+1}y_i) + g(y_i) = (p + 1) + (p + q + 2(n + 1) - i) + (p + 1 + i) = 3p + q + 2(n + 2) = c_2$;
and for the edge $xz \in E^{'}$
$g(x) + g(xy_1) + g(y_1) = p + (p + q + 2(n + 1)) + (p + 2) = 3p + q + 2(n + 2) = c_2$.
Verification for case(ii)
In this case, constant c_2 alone is verified, as c_1 is as in case(i).
For each edge in $E^{'}$
(i) if the edges in E_1, then we obtain
$g(y_{n+1}) + g(y_{n+1}y_i) + g(y_i) = (p + n + 1) + (p + q + 2(n + 1) - i) + (p + i) = 3p + 3n + q + 3 = 3(p + n + 1) + q = c_2$
and for the edge $xz \in E^{'}$ $g(x) + g(xz) + g(z) = p + (p + q + 2(n+1)) + (p+n+1) = 3p+q+3n+3 = 3(p+n+1)+p = c_2$
Hence the resultant graph admits SEBM total labeling.

Theorem 2.9 If G is a SEM graph then there exist at least one graph from the class $G\hat{e}C_n : n \equiv 1\ (mod\ 2)$ that admit SEBM total labeling.

proof Let $G(V, E)$ be a super edge magic graph with $|V| = p$ and $|E| = q$ and therefore there exists a bijection $f : V \cup E \to \{1, 2, ..., p + q\}$ such that the magic sum $f(u) + f(uv) + f(v)$ is a constant c, for all $uv \in E$ and let $f(x) = p$ for $x \in V$. Let the graph obtained by edge joining the vertex $x \in V$ and one of the vertices in C_n be $G^{'}$. That is $G^{'} = G\ \hat{e}\ C_n$ with $V^{'} = V \cup V_1$ where $V_1 = V(C_n) = \{y_i : 1 \leq i \leq n\}$ and $E^{'} = E \cup E_1 \cup E_2$ where $E_1 = E(C_n) = \{y_1y_n, y_iy_{i+1} : 1 \leq i \leq n - 1\}$ and $E_2 = \{xy_1\}$. We define a bijection $g : V^{'} \cup E^{'} \to \{1, 2, ..., p + q + 2n + 1\}$ as follows:
For $u \in V^{'}$,
if the vertices in V, then $g(u) = f(u)$.
For the vertices in V_1,
let $g(y_i) = p + \frac{i}{2}$ when $i \equiv 0\ (mod\ 2); 2 \leq i \leq n - 1$
and let $g(y_i) = p + \frac{i+1}{2} + \frac{n-1}{2}$ when $i \equiv 1\ (mod\ 2); 1 \leq i \leq n$.
For all $uv \in E^{'}$
if the edges in E_1, then
$g(y_1y_n) = p+q+n+1$ and $g(y_iy_{i+1}) = p+q+2n+1-i; 1 \leq i \leq n - 1$ and for $xy_1 \in E_2$ $g(xy_1) = p + q + 2n + 1$.
In the following, the SEBM labeling of g is verified.
Verification:
For the edges $uv \in E^{'}$,
(i) if $uv \in E$ then we obtain
$g(u)+g(uv)+g(v) = f(u)+(f(uv)+n)+f(v) = c+n = c_1$
(ii) if the edges in E_1 then we have
$g(y_1) + g(y_1y_n) + g(y_n) = (p + \frac{n-1}{2} + 1) + (p + q + n + 1) + (p + \frac{n-1}{2} + \frac{n+1}{2}) = 3p + q + \frac{5n+3}{2} = c_2$
and when $i \equiv 0\ (mod\ 2)$ we obtain
$g(y_i) + g(y_{i+1}) + g(y_iy_{i+1}) = (p + \frac{i}{2}) + (p + \frac{i+2}{2} + \frac{n-1}{2}) + (p + q + 2n + 1 - i) = 3p + q + \frac{5n+3}{2} = c_2$
when $i \equiv 1\ (mod\ 2)$ we have
$g(y_i) + g(y_{i+1}) + g(y_iy_{i+1}) = (p + \frac{i+1}{2} + \frac{n-1}{2}) + (p + q + 2n + 1 - i) + (p + \frac{i+1}{2}) = 3p + q + \frac{5n+3}{2} = c_2$.
(iii) if the edge in E_2 then we obtain
$g(x)+g(xy_1)+g(y_1) = p+(p+q+2n+1)+(p+\frac{n-1}{2}+1) = 3p + q + \frac{5n+3}{2} = c_2$.
Therefore, the result satisfies SEBM total labeling.

III. Conclusion

In our present study, we have examined that the existence of edge bimagic graphs are shown by using some operations on graphs. In future, we would like to apply new techniques in this labeling and investigate its applications.

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