On Two Diophantine Equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$

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***Abstract*—In this note, we show that the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ have exactly two solutions (x, y, z) in non-negative integers. The solutions are $\{(1, 0, 2), (4, 1, 10)\}$, and $\{(1, 0, 2), (2, 1, 10)\}$, respectively.**

***Index Terms*—Exponential Diophantine equation, integer solutions.**

***MSC 2010 Codes* – 11D61**

I. Introduction

In recent papers, Diophantine equations of type $a^x + b^y = c^z$ have been studied (*see* for instance $[9, 10, 11, 12]$). In 2011, Suvarnamani, Singta, and Chotchaisthit [1] showed that the two Diophantine equations $4^x+7^y = z^2$ and $4^x+11^y = z^2$ have no solutions in non-negative integers. In [2], Suvarnamani gave all solutions to the Diophantine equations of type $2^x+p^y = z^2$ for primes $p = 2, 3, 2^{k+1}+1$. In [3], Chotchaisthit studied the Diophantine equation $4^x + p^y = z^2$ where p is a prime number. On the other hand, Peker and Cenberci generalized these results by considering the Diophantine equations $(2^n)^x + p^y = z^2$ and $(4^n)^x + p^y = z^2$ in [4] and in [5], respectively. In 2012, Sroysang published series of papers in relation to the Diophantine equation $a^x+b^y = c^z$ (*see* $[6, 7, 8]$). In [13], Rabago gave all solutions to several Diophantine equations of type $p^x + q^y = z^2$.

In most of these papers, the authors used theory of congruences and/or Catalan's conjecture [14] to find all solutions, or to show the non-existence of solutions to the Diophantine equations of type $p^x + q^y = z^2$.

In the present paper, we study the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$. Particularly, via elementary methods we show that these two Diophantine equations have exactly two solutions in non-negative integers.

II. Preliminaries

In this section we state some important theorems.

Theorem 2.1: The only solution (x, z) in non-negative integers to the Diophantine equation $3^x + 1 = z^2$ is $(1, 2)$.

Proof: The case when $x = 0$ and $z = 0$ are obvious. So, we only consider the case when $x, z > 0$. Hence, $(z+1)(z-1) = 3^x$. It follows that $2 = (z + 1) - (z - 1) = 3^\beta - 3^\alpha$ where $\alpha + \beta = x$ and $\beta > \alpha$. This implies that $\alpha = 0$ and $3^x - 1 = 2$ or equivalently $3^x = 3^1$. Thus, $x = 1$ and $z = 2$. ■

Theorem 2.2: [16] If p is an odd prime and $n \geq 2$ is an integer, then the equation $x^2 - 1 = p^n$ has no integer solution.

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III. Main Results

We now present our results.

Theorem 3.1: The Diophantine equation $3^x + 19^y = z^2$ has exactly two solutions (x, y, z) in non-negative integers, *i.e.* $(x, y, z) \in \{(1, 0, 2), (4, 1, 10)\}$.

Proof: Let x, y, z be non-negative integers. We first consider the case when one of the unknown x, y, z is zero. By Theorem 2.1 and Theorem 2.2, we see immediately that $(x, y, z) = (1, 0, 2)$. On the other hand, for $x, y, z > 0$ we divide y into two cases.

Case 1. If y is even, say $y = 2n$ for some non-negative integer n, then $(z + 19^n)(z - 19^n) = z^2 - 19^{2n} = 3^x$. This implies that $2 \cdot 19^n = (z+19^n) - (z-19^n) = 3^\beta - 3^\alpha$ where $\alpha+\beta = x$ such that $\beta > \alpha$. Hence, $\alpha = 0$ and $2 \cdot 19^n = 3^x - 1$. Adding both sides by -2, we obtain $2(19^n-1) = 3(3^{x-1}-1)$. That is, $x = 2$ and $19^n = 4$, a contradiction. Thus, $3^x + 19^y = z^2$ is not possible for even positive integer y.

Case 2. If $y = 2n + 1$ then we can write $3^x + 19^y = z^2$ as $3(3^{x-1} + 19^{2n}) = 3^x + 3 \cdot 19^{2n} = z^2 - 16 \cdot 19^{2n} = (z + 4 \cdot 19^n)(z-4\cdot 19^n)$. Take note that if $3^x+19^y = z^2$ has a solution in positive integers then $z = 2m$ for some natural number m. Hence, $3(3^{x-1} + 19^{2n}) = (2m + 4 \cdot 19^n)(2m - 4 \cdot 19^n) = 4(m + 2 \cdot 19^n)(m - 2 \cdot 19^n)$. We have two possibilities.

Subcase 2.1 If $m = 3 - 2 \cdot 19^n$ then $4(3) = 4(3 - 2 \cdot 19^n + 2 \cdot 19^n) = 3^{x-1} + 19^n$. So, $3(4 - 3^{x-1}) = 19^{2n}$ and this gives us $n = 0$. It follows that $3^{x-1} = 11$, a contradiction.

Subcase 2.2 If $m = 3 + 2 \cdot 19^n$ then $4(3 + 4 \cdot 19^n) = 3^{x-1} + 19^{2n}$. Rearranging the equation we obtain $19^n(16 - 19^n) = 3(3^{x-2} - 4)$. Hence, $n = 0$ and $y = 1$. Furthermore, we have $3(3^{x-2} - 4) = 15$ or equivalently $3^2 = 3^{x-2}$. Thus, $x = 4$ giving us the unique solution $(x, y, z) = (4, 1, 10)$. This proves the theorem. ■

Theorem 3.2: The Diophantine equation $3^x + 91^y = z^2$ has exactly two solutions (x, y, z) in non-negative integers, *i.e.* $(x, y, z) \in \{(1, 0, 2), (2, 1, 10)\}$.

Proof: Let x, y, z be non-negative integers. For the case when one of the unknown is zero we use Theorem 2.1 and Theorem 2.2. Hence, we obtain $(x, y, z) = (1, 0, 2)$.

For $x, y, z > 0$, we let $z = 2m$ and consider the following cases.

Case 1. If $y = 2n$ then $(z+91^n)(z-91^n) = z^2-91^{2n} = 3^x$. Hence, $2 \cdot 91^n = (z + 91^n) - (z - 91^n) = 3^\beta - 3^\alpha$ where $\alpha + \beta = x$ such that $\beta > \alpha$. So, $\alpha = 0$ and $2 \cdot 91^n = 3^x - 1$. It follows that $2(91^n - 1) = 3(3^{x-1} - 1)$. This implies that $x = 2$ and $91^n = 4$, a contradiction.

Case 2. If $y = 2n + 1$ then we have $3^x + 27 \cdot 91^{2n} = z^2 - 64 \cdot 91^{2n} = (2m - 8 \cdot 91^n)(2m + 8 \cdot 91^n)$. Hence, $3^u(3^{x-u} +$

$3^{3-u}\cdot 91^{2n}) = 4(m-4\cdot 91^n)(m+4\cdot 91^n)$ where $u \leq \min\{x, 3\}$. There are two possible cases.

Subcase 2.1 If $m = 3^u + 4 \cdot 91^n$ then $3^{x-u} + 3^{3-u} \cdot 91^{2n} = 4(m + 4 \cdot 91^n) = 4(3^u + 8 \cdot 91^n)$. This can be written as $4 \cdot 3^u - 3^{x-u} = 91^n(3^{3-u} \cdot 91^n - 32)$. It follows that $n = 0$ and $4 \cdot 3^u - 3^{x-u} = 3^{3-u} - 32$ or equivalently, $4 \cdot 3^u(3^u + 8) = 4 \cdot 3^{2u} + 32 \cdot 3^u = 3^3 + 3^x$. But note that $x = 2k$ for natural number k. So, $4(3^{2u} + 8 \cdot 3^u - 7) = 3^x - 1 = (3^k + 1)(3^k - 1)$. Then, either $k = 0$ or $k = 1$. If $k = 0$ then $3^{2u} + 8 \cdot 3^u = 7$, this is impossible. Now, if $k = 1$ then $x = 2$ and $3^u(3^u + 8) = 9 = 3^2$. It is not possible that $u = 2$ since $3^u + 8 \neq 1$, so u must be zero. This implies that $m = 5$ and $z = 10$. Here we obtain a solution (x, y, z) to $3^x + 91^y = z^2$. In particular we have $(x, y, z) = (2, 1, 10)$.

Subcase 2.2 If $m = 3^u - 4 \cdot 91^n$ then we have $3^{x-u} + 3^{3-u} \cdot 91^{2n} = 4(3^u + 4 \cdot 91^n - 4 \cdot 91^n) = 4 \cdot 3^u$. It follows that $3^x + 2 \cdot 91^{2n} = 4 \cdot 3^{2u} - 25 \cdot 91^{2n} = (2 \cdot 3^u + 5 \cdot 91^n)(2 \cdot 3^u - 5 \cdot 91^n)$. Obviously, $2 \cdot 3^u + 5 \cdot 91^n = 1$ is not possbile so, $2 \cdot 3^u - 5 \cdot 91^n = 1$ and $2 \cdot 3^u + 5 \cdot 91^n = 3^x + 2 \cdot 91^{2n}$. Hence, $10 \cdot 91^n = (2 \cdot 3^u + 5 \cdot 91^n) - (2 \cdot 3^u - 5 \cdot 91^n) = 3^x - 1 + 2 \cdot 91^{2n}$. This implies that $2 \cdot 91^n(5 - 91^n) = 3^x - 1$. Then, $n = 0$ and $3^x = 9 = 3^2$ or equivalently, $x = 2$. Also, we see that $u = 1$ and $y = 1$. Thus, we obtain exactly the same solution (x, y, z) from Subcase 2.1 (though $u = 1$ appears to be an extraneous solution to $m = 3^u - 4 \cdot 91^n$). That is, we have $(x, y, z) = (2, 1, 10)$.

In any cases, we see that the same solution (x, y, z) is obtained for $x, y, z > 0$. Hence, we conclude that $(1, 0, 2)$ and $(2, 1, 10)$ are the only solutions to the Diophantine equation $3^x + 91^y = z^2$. ■

IV. Conclusion

In the paper, we have shown that $(1, 0, 2)$ and $(4, 1, 10)$ are the exact solutions to $3^x + 19^y = z^2$ in non-negative integers. Also, we have seen that $(1, 0, 2)$ and $(2, 1, 10)$ are the only solutions to $3^x + 91^y = z^2$. Here we noticed that $91 \equiv 19 \equiv 3 \pmod 4$ and obviously, 19 is a prime but 91 is not. Furthermore, 19 is 91 when read from right to left. Thus, we may ask, "what are the set of all solutions in non-negative integers to the Diophantine equation $p^x + q^y = z^2$ where p and q are both primes such that p is equal to q when p is read from right to left?".

References

[1] A. Suvarnamani, A. Singta and S. Chotchaisthit, "On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$", *Sci. Technol. RMUTT J.,* **1**, pp. 25–28, 2011.

[2] A. Suvarnamani, Solutions of the Diophantine equation $2^x + p^y = z^2$, *Int. J. of Mathematical Sciences and Applications,* **1**, no. 3, pp. 1415–1419, 2011.

[3] S. Chotchaisthit, "On the Diophantine equation $4^x + p^y = z^2$ where p is a prime number", *American Jr. of Mathematics and Sciences,* **1**, no. 1, pp. 191–193, 2012.

[4] B. Peker and S. I. Cenberci, "On the Diophantine equations of $(2^n)^x + p^y = z^2$", *American Jr. of Mathematics and Sciences,* **1**, no. 1, 2012.

[5] B. Peker and S. I. Cenberci, "On the Solutions of the equation $(4^n)^x + p^y = z^2$", *Far East Journal of Mathematical Sciences,* **71**, no. 1, pp. 21–27, 2012.

[6] B. Sroysang, "More on the Diophantine Equation $8^x + 19^y = z^2$", *International Journal of Pure and Applied Mathematics,* **81**, no. 4, pp. 601–604, 2012.

[7] B. Sroysang, "On the Diophantine Equation $3^x + 5^y = z^2$", *International Journal of Pure and Applied Mathematics,* **81**, no. 4, pp. 605–608, 2012.

[8] B. Sroysang, "On the Diophantine Equation $31^x + 32^y = z^2$", *International Journal of Pure and Applied Mathematics,* **81**, no. 4, pp. 609–612, 2012.

[9] B. Sroysang, "On the Diophantine Equation $7^x + 8^y = z^2$", *International Journal of Pure and Applied Mathematics,* **84**, no. 1, pp. 111–114, 2013.

[10] J. F. T. Rabago, "On the Diophantine Equation $2^x + 17^y = z^2$", *General Mathematics,* (2013), submitted.

[11] J. F. T. Rabago, "On two Diophantine Equations of type $p^x + q^y = z^3$", *General Mathematics,* (2013), submitted.

[12] J. F. T. Rabago, "Impossibility of $2^x + 11^y = z^2$ and $2^x + 19^y = z^2$", *Int. J. of Math. and Soft Computing,* submitted.

[13] J. F. T. Rabago, "More on Diophantine Equations of type $p^x + q^y = z^2$", *Int. J. of Math. and Scietific Computing,* vol. 3, no. 1, pp. 15–16, 2013.

[14] P. Mihailescu, "Primary cyclotomic units and a proof of Catalan's conjecture", *J. Reine Angew Math.,* **572**, pp. 167-195, 2004.