On Two Diophantine Equations $3^x + 19y = z^2$ and $3^x + 91y = z^2$

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Abstract—In this note, we show that the two Diophantine equations $3^x + 19y = z^2$ and $3^x + 91y = z^2$ have exactly two solutions $(x, y, z)$ in non-negative integers. The solutions are $(1, 0, 2), (4, 1, 10)$, and $(1, 0, 2), (2, 1, 10)$, respectively.

Index Terms—Exponential Diophantine equation, integer solutions.

MSC 2010 Codes – 11D61

I. INTRODUCTION

In recent papers, Diophantine equations of type $a^x + b^y = c^z$ have been studied (see for instance [9, 10, 11, 12]). In 2011, Suvarnamani, Singta, and Chotchaisthit [1] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solutions in non-negative integers. In [2], Suvarnamani gave all solutions to the Diophantine equations of type $2^x + p^y = z^2$ for primes $p = 2, 3, 2^{k+1}+1$. In [3], Chotchaisthit studied the Diophantine equation $4^x + p^y = z^2$ where $p$ is a prime number. On the other hand, Peker and Cenberci generalized these results by considering the Diophantine equations $(2^n)^x + p^y = z^2$ and $(4^n)^x + p^y = z^2$ in [4] and in [5], respectively. In 2012, Sroysang published series of papers in relation to the Diophantine equation $a^x + b^y = c^z$ (see [6, 7, 8]). In [13], Rabago gave all solutions to several Diophantine equations of type $p^x + q^y = z^2$.

In most of these papers, the authors used theory of congruences and/or Catalan’s conjecture [14] to find all solutions, or to show the non-existence of solutions to the Diophantine equations of type $p^x + q^y = z^2$.

In the present paper, we study the two Diophantine equations $3^x + 19y = z^2$ and $3^x + 91y = z^2$. Particularly, via elementary methods we show that these two Diophantine equations have exactly two solutions in non-negative integers.

II. PRELIMINARIES

In this section we state some important theorems.

Theorem 2.1: The only solution $(x, z)$ in non-negative integers to the Diophantine equation $3^x + 1 = z^2$ is $(1, 2)$.

Proof: The case when $x = 0$ and $z = 0$ are obvious. So, we only consider the case when $x, z > 0$. Hence, $(z + 1)(z - 1) = 3^x$. It follows that $2 = (z + 1) - (z - 1) = 3^y - 3^x$ where $\alpha + \beta = x$ and $\beta > \alpha$. This implies that $\alpha = 0$ and $3^x - 1 = 2$ or equivalently $3^x = 3^1$. Thus, $x = 1$ and $z = 2$.

Theorem 2.2: [16] If $p$ is an odd prime and $n \geq 2$ is an integer, then the equation $x^2 - 1 = p^n$ has no integer solution.

III. MAIN RESULTS

We now present our results.

Theorem 3.1: The Diophantine equation $3^x + 19^y = z^2$ has exactly two solutions $(x, y, z)$ in non-negative integers, i.e. $(x, y, z) \in \{(1, 0, 2), (4, 1, 10)\}$.

Proof: Let $x, y, z$ be non-negative integers. We first consider the case when one of the unknown $x, y, z$ is zero. By Theorem 2.1 and Theorem 2.2, we see immediately that $(x, y, z) = (1, 0, 2)$. On the other hand, for $x, y, z > 0$ we divide $y$ into two cases.

Case 1. If $y$ is even, say $y = 2n$ for some non-negative integer $n$, then $(z + 19^n)(z - 19^n) = z^2 - 19^{2n} = 3^x$. This implies that $2 \cdot 19^n = (z + 19^n) - (z - 19^n) = 3^y - 3^x$ where $\alpha + \beta = x$ such that $\beta > \alpha$. Hence, $\alpha = 0$ and $2 \cdot 19^n = 3^x - 1$. Adding both sides by $-2$, we obtain $2(19^n - 1) = 3(3^{x-1} - 1)$. That is, $x = 2$ and $19^n = 4$, a contradiction. Thus, $3^x + 19^y = z^2$ is not possible for even positive integer $y$.

Case 2. If $y = 2n + 1$ then we can write $3^x + 19^y = z^2$ as $3(3^{x-1} + 19^{2n}) = 3^x + 3 \cdot 19^{2n} = z^2 - 16 \cdot 19^n = (z + 4 \cdot 19^n)(z - 4 \cdot 19^n)$. Take note that if $3^x + 19^y = z^2$ has a solution in positive integers then $z = 2m$ for some natural number $m$. Hence, $3(3^{x-1} + 19^n) = (2m - 4 \cdot 19^n)(2m - 4 \cdot 19^n) = 4(m + 2 \cdot 19^n)(m - 2 \cdot 19^n)$. We have two possibilities.

Subcase 2.1 If $m = 3 - 2 \cdot 19^n$ then $4(3) = 4(3 - 2 \cdot 19^n + 2 \cdot 19^n) = 3^{x-1} + 19^n$. So, $(3 - 3^{x-1}) = 19^n$ and this gives us $n = 0$. It follows that $3^{x-1} = 11$, a contradiction.

Subcase 2.2 If $m = 3 + 2 \cdot 19^n$ then $4(3) = 4(3 + 2 \cdot 19^n + 2 \cdot 19^n) = 3^{x-1} + 19^n$. Rearranging the equation we obtain $19^n(16 - 19^n) = 3(3^{x-2} - 4)$. Hence, $n = 0$ and $y = 1$. Furthermore, we have $3(3^{x-2} - 4) = 15$ or equivalently $3^2 = 3^2$. Thus, $x = 4$ giving us the unique solution $(x, y, z) = (4, 1, 10)$. This proves the theorem.

Theorem 3.2: The Diophantine equation $3^x + 91^y = z^2$ has exactly two solutions $(x, y, z)$ in non-negative integers, i.e. $(x, y, z) \in \{(1, 0, 2), (2, 1, 10)\}$.

Proof: Let $x, y, z$ be non-negative integers. For the case when one of the unknown is zero we use Theorem 2.1 and Theorem 2.2. Hence, we obtain $(x, y, z) = (1, 0, 2)$.

For $x, y, z > 0$, we let $z = 2m$ and consider the following cases.

Case 1. If $y = 2n$ then $(z + 91^n)(z - 91^n) = z^2 - 91^{2n} = 3^x$.

Hence, $2 \cdot 91^n = (z + 91^n) - (z - 91^n) = 3^y - 3^x$ where $\alpha + \beta = x$ such that $\beta > \alpha$. So, $\alpha = 0$ and $2 \cdot 91^n = 3^x - 1$. It follows that $2(91^{n-1}) = 3(3^{x-1} - 1)$. This implies that $x = 2$ and $91^n = 4$, a contradiction.

Case 2. If $y = 2n + 1$ then we have $3^x + 27 \cdot 91^{2n} = z^2 - 64 \cdot 91^{2n} = (2m - 8 \cdot 91^n)(2m + 8 \cdot 91^n)$. Hence, $3^u(3^{x-u} + 8 \cdot 91^{n-u}) = 3^{x+11} - 3^{x-u}$.
3^{3-u}91^{2n}) = 4(m-4\cdot 91^n)(m+4\cdot 91^n) where u \leq \min\{x, 3\}.

There are two possible cases.

**Subcase 2.1** If \(m = 3^u + 4 \cdot 91^n\) then \(3^{x-u} + 3^{3-u}91^{2n} = 4(m + 4 \cdot 91^n) = 4(3^n + 8 \cdot 91^n)\). This can be written as \(4 \cdot 3^{u+u} - 3^{x-u} = 91^n(3^{u+u} - 32)\). It follows that \(n = 0\) and \(4 \cdot 3^{-3}3^{x-u} = 3^{x-u} - 32\) or equivalently, \(4 \cdot 3^{-3}3^{(x+8)} = 4 \cdot 3^{2u} + 3 \cdot 3^{2u} = 3 \cdot 3^x\). But note that \(x = 2k\) for natural number \(k\). So, \(4(3^{2u} + 8 \cdot 3^u - 7) = 3^x - 1 = (3^k + 1)(3^k - 1).\) Then, either \(k = 0\) or \(k = 1\). If \(k = 0\) then \(3^{2u} + 8 \cdot 3^u = 7\), this is impossible. Now, if \(k = 1\) then \(x = 2\) and \(3^u(3^u+8) = 9 = 3^2\). It is not possible that \(u = 2\) since \(3^u + 8 \neq 1\), so \(u\) must be zero. This implies that \(m = 5\) and \(z = 10\). Here we obtain a solution \((x, y, z) = 3^x + 91^y = 3^2\). In particular we have \((x, y, z) = (2, 1, 10)\).

**Subcase 2.2** If \(m = 3^u - 4 \cdot 91^n\) then we have \(3^{x-u} + 3^{3-u}91^{2n} = 4(3^u + 4 \cdot 91^n - 4 \cdot 91^n) = 4 \cdot 3^n\). It follows that \(3^x + 2 \cdot 91^{2n} = 4 \cdot 3^{2u} - 25 \cdot 91^n = (2 \cdot 3^u + 5 \cdot 91^n)(2 \cdot 3^u - 5 \cdot 91^n).\) Obviously, \(2 \cdot 3^u + 5 \cdot 91^n = 1\) is not possible so, \(2 \cdot 3^u - 5 \cdot 91^n = 1\) and \(2 \cdot 3^u + 5 \cdot 91^n = 3^x + 2 \cdot 91^n\). Hence, \(10 \cdot 91^n = (2 \cdot 3^u + 5 \cdot 91^n) - (2 \cdot 3^u - 5 \cdot 91^n) = 3^x - 1 + 2 \cdot 91^n\). This implies that \(2 \cdot 91^n(5 - 91^n) = 3^x - 1\). Then, \(n = 0\) and \(3^x = 9 = 3^2\) or equivalently, \(x = 2\). Also, we see that \(u = 1\) and \(y = 1\). Thus, we obtain exactly the same solution \((x, y, z)\) from Subcase 2.1 (though \(u = 1\) appears to be an extraneous solution to \(m = 3^u - 4 \cdot 91^n\)). That is, we have \((x, y, z) = (2, 1, 10)\).

In any cases, we see that the same solution \((x, y, z)\) is obtained for \(x, y, z > 0\). Hence, we conclude that \((1, 0, 2)\) and \((2, 1, 10)\) are the only solutions to the Diophantine equation \(3^x + 91^y = z^2\).

**IV. Conclusion**

In the paper, we have shown that \((1, 0, 2)\) and \((4, 1, 10)\) are the exact solutions to \(3^x + 19^y = z^2\) in non-negative integers. Also, we have seen that \((1, 0, 2)\) and \((2, 1, 10)\) are the only solutions to \(3^x + 91^y = z^2\). Here we noticed that \(91 \equiv 19 \equiv 3 \pmod 4\) and obviously, \(19\) is a prime but \(91\) is not. Furthermore, \(19\) is \(91\) when read from right to left. Thus, we may ask, "what are the set of all solutions in non-negative integers to the Diophantine equation \(p^x + q^y = z^2\) where \(p\) and \(q\) are both primes such that \(p\) is equal to \(q\) when \(p\) is read from right to left?".

**References**


