Fixed Point Results for Some New Type of Contraction Conditions in Dislocated Quasi-Metric Space

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***Abstract*—In this paper, we have established some new common fixed point results for a single and a pair of continuous self-mapping in the context of dislocated quasi-metric space using some new type of rational contractive conditions. Our established results extend and modify some well-known existing fixed point results of the literature. Appropriate examples are given for the usability of our constructed results and corollaries.**

***Index Terms*—Complete dislocated quasi-metric space, contraction mapping, Cauchy sequence, self-mapping, fixed point.**

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I. INTRODUCTION

FIXED point theory is one of the most important field in the development of non-linear analysis. Also fixed point theory is widely Applicable in many branches of Science such as Chemistry, Biology, Economics, Computer Science and Engineering etc.

The Banach contraction theorem [1] is one of the most important result in functional analysis. It is popular tool for proving the existence of solution of problems in various branches of mathematics. A comprehensive literature and generalization of the Banach contraction theorem can be found in [2] and [3]. In 2000, Hitzler and Seda [4] introduced the concept of dislocated metric space and generalized the Banach contraction principle in such a space. Dislocated metric space play an important role in Topology, Logical Programming and Electronics Engineering etc. Zeyada et al. [5] initiated the concept of complete dislocated quasi-metric space and generalized the result of Hitzler and Seda [4] in dislocated quasi metric space. After the work of Zeyada et al. [5] a lot of papers have been published containing fixed point results for a single and a pair of self-mappings with different type of contractive conditions in dislocated quasi-metric space (see [6], [7], [8], [9], [10], [11]).

II. PRELIMINARIES

Throughout this paper $\mathbb{R}^+$ will represent the set of non-negative real numbers.

We begin with the following definition as a recall from [4].

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Definition 2.1. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a distance function satisfying the conditions,

d_1) $d(x,x) = 0$;

d_2) $d(x,y) = d(y,x) = 0$ implies that $x = y$;

d_3) $d(x,y) = d(y,x)$;

d_4) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

If d satisfy the conditions from d_1 to d_4 then it is called metric on X, if d satisfy conditions d_2 to d_4 then it is called dislocated metric (d-metric) on X, and if d satisfy conditions d_2 and d_4 only then it is called dislocated quasi-metric (dq-metric) on X.

Clearly every metric space is a dislocated metric space but the converse is not necessarily true, as clear from the following example.

Example 2.2. Let X={0,1} define the distance function $d : X \times X \to \mathbb{R}^+$ as

$$d(x,y) = \max\{x,y\}.$$

Clearly the define function is dislocated metric space but not a metric space.

Also every metric space is dislocated quasi-metric space but the converse is not true and every dislocated metric space is dislocated quasi metric space but the converse is not true, as clear from the following example.

Example 2.3. Let $X = [0,1]$ we define the function $d : X \times X \to \mathbb{R}^+$ as,

$$d(x,y) = |x-y| + |x| \quad \text{ for all } x,y \in X.$$

Evidently the defined function is dq-metric space but not a metric space nor dislocated metric space.

In our main work we will use the following definitions which can be found in [5].

Definition 2.4. A sequence $\{x_n\}$ in dq-metric space (X,d) is called Cauchy sequence if for $\epsilon > 0$ there exists a positive integer $n_0 \in N$ such that for $m,n \geq n_0$, we have $d(x_m,x_n) < \epsilon$.

Definition 2.5. A sequence $\{x_n\}$ is called dq-convergent in (X,d) if for $n \in N$ we have

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

In this case x is called the dq-limit of the sequence $\{x_n\}$.

Definition 2.6. A dq-metric space (X,d) is said to be complete if every Cauchy sequence in X converge to a point of X.

Definition 2.7. Let (X, d) be a dq-metric space, a mapping $T : X \to X$ is called contraction if there exists $0 \le \alpha < 1$ such that

$$d(Tx, Ty) \le \alpha \cdot d(x, y) \quad \text{for all} \quad x, y \in X.$$

Lemma 2.8[5]. Limit in dq-metric space is unique.

Theorem 2.9.[5]. Let (X, d) be a complete dq-metric space $T : X \to X$ be a contraction, then T has a unique fixed point.

In [9], Kohli et al. proved the following result in dislocated quasi metric space.

Theorem 2.10. Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying the following condition

$$d(Tx, Ty) \le \alpha \cdot d(x, y) + \beta \cdot d(y, Ty) + \gamma \cdot \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}$$

$\forall\ x, y \in X$, $\alpha, \beta, \gamma \ge 0$, and $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point.

Remark 2.11. It is obvious that the the following statement hold for real numbers a, b and c:

If $a < b$ and $c > 0$, then $ac < bc$. □

III. Main Results

Theorem 3.1. Let (X, d) be a complete dq-metric space and $S, T : X \to X$ be two continuous self mappings satisfying the condition

$$d(Sx, Ty) \le \alpha \cdot d(x, y) + \beta \cdot [d(x, Sx) + d(y, Ty)] + \gamma \cdot \frac{d(x, Sx) \cdot d(y, Ty)}{1 + d(x, y)} + \mu \cdot \frac{d(x, Sx) \cdot d(x, Ty)}{1 + d(x, y)}$$

$\forall\ x, y \in X$ where $\alpha, \beta, \gamma, \mu \ge 0$ with $\alpha + 2\beta + \gamma + 2\mu < 1$. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ for $n = 0, 1, 2, ...$ by the rule

$$x_0, x_1 = Sx_0, x_3 = Sx_2, ..., x_{2n+1} = Sx_{2n}$$

and

$$x_2 = Tx_1, x_4 = Tx_3, ..., x_{2n} = Tx_{2n-1}.$$

Now we have to show that $\{x_n\}$ is a Cauchy sequence in X for this consider

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

using given condition in the theorem we have

$$\le \alpha \cdot d(x_{2n}, x_{2n+1}) + \beta \cdot [d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})] + \gamma \cdot \frac{d(x_{2n}, Sx_{2n}).d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \mu \cdot \frac{d(x_{2n}, Sx_{2n}).d(x_{2n}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}$$

using definition of the sequence and by Remark 2.11 we have

$$< \alpha \cdot d(x_{2n}, x_{2n+1}) + \beta \cdot [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma \cdot d(x_{2n+1}, x_{2n+2}) + \mu \cdot d(x_{2n}, x_{2n+2})$$

$$\le \alpha \cdot d(x_{2n}, x_{2n+1}) + \beta \cdot [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma \cdot d(x_{2n+1}, x_{2n+2}) + \mu \cdot d(x_{2n}, x_{2n+1}) + \mu . d(x_{2n+1}, x_{2n+2})$$

$$[1 - (\beta + \gamma + \mu)] \cdot d(x_{2n+1}, x_{2n+2}) \le (\alpha + \beta + \mu) \cdot d(x_{2n}, x_{2n+1})$$

$$d(x_{2n+1}, x_{2n+2}) \le \frac{(\alpha + \beta + \mu)}{1 - (\beta + \gamma + \mu)} \cdot d(x_{2n}, x_{2n+1}). \quad (1)$$

Let

$$h = \frac{(\alpha + \beta + \mu)}{1 - (\beta + \gamma + \mu)} < 1$$

using in (1) we have

$$d(x_{2n+1}, x_{2n+2}) \le h \cdot d(x_{2n}, x_{2n+1})$$

$$d(x_{2n}, x_{2n+1}) \le h \cdot d(x_{2n-1}, x_{2n}).$$

So

$$d(x_n, x_{n+1}) \le h \cdot d(x_{n-1}, x_n)$$

$$d(x_{n-1}, x_n) \le h \cdot d(x_{n-2}, x_{n-1})$$

$$d(x_n, x_{n+1}) \le h^2 \cdot d(x_{n-2}, x_{n-1}).$$

Similarly proceeding we get

$$d(x_n, x_{n+1}) \le h^n \cdot d(x_0, x_1).$$

Since $h < 1$ taking $n \to \infty$, implies that $h^n \to 0$.

$$d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete dq-metric space. Therefore there must exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Also the sub-sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to u. Since S and T are continuous so

$$T \lim_{n\to\infty} x_{2n+1} = Tu \quad \Rightarrow \quad Tu = u.$$

Similarly we can show that $Su = u$. Therefore u is the common fixed point of S and T.

Uniqueness: Let u and v are two distinct common fixed points of S and T. Consider

$$d(u, v) = d(Su, Tv) \le \alpha \cdot d(u, v) + \beta \cdot [d(u, Su) + d(v, Tv)] + \gamma \cdot \frac{d(u, Su).d(v, Tv)}{1 + d(u, v)} + \mu \cdot \frac{d(u, Su).d(u, Tv)}{1 + d(u, v)}$$

by use of given condition in the theorem and considering u, v are fixed points of S and T we have

$$d(u, u) = d(v, v) = 0$$

using these in the above we get

$$d(u, v) \le \alpha \cdot d(u, v).$$

The above inequality is possible if $d(u, v) = 0$ similarly we can show that $d(v, u) = 0$ hence $u = v$.

Therefore S and T have a unique common fixed point.

We deduce the following Corollary from Theorem III.

Corollary 3.2. Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot [d(x,Tx) + d(y,Ty)] + \gamma \cdot \frac{d(x,Tx).d(y,Ty)}{1+d(x,y)} + \mu \cdot \frac{d(x,Tx).d(x,Ty)}{1+d(x,y)}$$

$\forall\ x,y \in X$ where $\alpha,\beta,\gamma,\mu \geq 0$ with $\alpha+2\beta+\gamma+2\mu < 1$. Then T has a unique fixed point in X.

Proof. Putting $S = T$ in Theorem III, we can get the required result easily.

Remarks 3.3.

- Putting $\beta = \gamma = \mu = 0$ in Corollary 3.2, we obtain the result of Zeyada et al. [5].
- Putting $\gamma = \mu = 0$ in Corollary 3.2, we get the result of Aage and Salunke [7].
- Putting $\gamma = \mu = 0$ in Theorem 3.1, we obtain the result of Aage and Salunke [7].

Example 3.4. Let $X = [0,1]$ and d be a complete dq-metric on X define by,

$$d(x,y) = |x| \text{ for all } \ x,y \in X.$$

Let $Sx = 0$ and $Tx = \frac{x}{5}\ \forall\ x \in X$ with $\alpha = 1/4, \beta = 1/6, \gamma = 1/8, \mu = 1/12$ satisfy all the conditions of the Theorem 3.1. Therefore $x = 0$ is the unique common fixed point of S and T.

Theorem 3.5. Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot d(y,Ty) + \gamma \cdot \frac{[1+d(x,Tx)] \cdot d(y,Ty)}{1+d(x,y)} + \mu \cdot \frac{d(x,Tx) \cdot d(y,Ty)}{1+d(y,Ty) \cdot d(x,y)} \cdot d(x,Ty)$$

$\forall\ x,y \in X$ where $\alpha,\beta,\gamma,\mu \geq 0$ with $\alpha+\beta+\gamma+2\mu < 1$. Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ by the rule,

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n \text{ for all } n \in N.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X consider,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

By the use of given condition in the theorem we have

$$d(x_n,x_{n+1}) \leq \alpha \cdot d(x_{n-1},x_n) + \beta \cdot d(x_n,Tx_n) + \gamma \cdot \frac{[1+d(x_{n-1},Tx_{n-1})] \cdot d(x_n,Tx_n)}{1+d(x_{n-1},x_n)} + \mu \cdot (\frac{d(x_n,Tx_n) \cdot d(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},x_n).d(x_n,Tx_n)}).d(x_{n-1},Tx_n).$$

Now by the use of definition of the sequence and Remark 2.11 we have the following inequality

$$d(x_n,x_{n+1}) \leq \alpha \cdot d(x_{n-1},x_n) + \beta \cdot d(x_n,x_{n+1}) + \gamma \cdot d(x_n,x_{n+1}) + \mu \cdot d(x_{n-1},x_{n+1})$$

$$(1-(\beta+\gamma+\mu)) \cdot d(x_n,x_{n+1}) \leq (\alpha+\mu) \cdot d(x_{n-1},x_n)$$

$$d(x_n,x_{n+1}) \leq (\frac{\alpha+\mu}{1-(\beta+\gamma+\mu)}) \cdot d(x_{n-1},x_n). \quad (2)$$

Let

$$h = \frac{\alpha+\mu}{1-(\beta+\gamma+\mu)} < 1 \text{ because } \alpha+\beta+\gamma+2\mu < 1$$

thus (2) become

$$d(x_n,x_{n+1}) \leq h \cdot d(x_{n-1},x_n)$$

$$d(x_{n-1},x_n) \leq h \cdot d(x_{n-2},x_{n-1}).$$

Hence

$$d(x_n,x_{n+1}) \leq h^2.d(x_{n-2},x_{n-1}).$$

Proceeding in the similar way we get

$$d(x_n,x_{n+1}) \leq h^n \cdot d(x_0,x_1).$$

Since $h < 1$ taking limit $n \to \infty$, $h^n \to 0$

$$d(x_n,x_{n+1}) \to 0.$$

Thus $\{x_n\}$ is a Cauchy sequence in complete dq-metric space X. So there must exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Since T is continuous so

$$T \lim_{n\to\infty} x_n = Tu \quad \Rightarrow \quad Tu = u.$$

Hence u is the fixed point of T.

Uniqueness: Let $u \neq v$ be two distinct fixed points of T in X, then consider

$$d(u,v) = d(Tu,Tv) \leq \alpha \cdot d(u,v) + \beta \cdot d(v,Tv) + \gamma \cdot \frac{[1+d(u,Tu)]\c{d}otd(v,Tv)}{1+d(u,v)} + \mu \cdot (\frac{d(u,Tu) \cdot d(v,Tv)}{1+d(v,Tv) \cdot d(u,v)}) \cdot d(u,Tv).$$

By use of given condition in the theorem and considering u,v are the fixed points of T we have

$$d(u,u) = d(v,v) = 0.$$

Using this in the above inequality we get

$$d(u,v) \leq \alpha \cdot d(u,v).$$

Since $\alpha < 1$ so the above inequality is possible only if $d(u,v) = 0$ similarly we can show that $d(v,u) = 0 \Rightarrow u = v$. Hence T has a unique fixed point in X.

We deduce the following corollary from Theorem 3.5.

Corollary 3.6.

Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying the condition,

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot d(y,Ty) + \gamma \cdot \frac{[1+d(x,Tx)].d(y,Ty)}{1+d(x,y)}$$

$\forall\ x, y \in X$ where $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point.

Proof. Putting $\mu = 0$ in Theorem 3.5, we can get the required result easily.

Remark 3.7.

- Corollary 3.6 is the result of Kohli et al. [9].

Example 3.8. Let $X = [0, 1]$ we define dq-metric on X by,

$$d(x, y) = |x| \quad \text{for all} \quad x, y \in X.$$

Where $Tx = \frac{x}{2}\ \forall\ x \in X$, with $\alpha = 1/5, \beta = 1/6, \gamma = 1/10, \mu = 1/12$, satisfy all the conditions of Theorem 3.5. Hence $x = 0$ is the unique fixed point of T in X.

IV. Conclusion

In this paper, the established new and modified fixed point results generalize the results of Aege and Salunke [7], Kohli et al. [9] and Zeyada et al. [5]. Also examples are given in the support of these new type of fixed point results.

References

[1] Ervin Kreyszig, Introductory Functional Analysis with Applications, John Willey and Sons, New York, 1989.

[2] B.K. Dass and S. Gupta, "An extension of Banach's contraction principle through rational expression", *Indian Journal of Pure and Applied Mathematics*, vol. 6, pp. 1455-1458, 1975.

[3] B.E. Rohades, "A comparison of various definition of contractive mappings", *Transfer Amer. Soc.*, vol. 226, pp. 257-290, 1977.

[4] P. Hitzler, Generalized metrics and Topology in Logic Programming Semantics, Ph.D Thesis, National Univeristy of Ireland, University College, Cork, 2001.

[5] F.M. Zeyada, G.H. Hassan and M.A. Ahmad, "A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space", *Arabian Journal Sci. Engg.,* vol. 31, pp. 111-114, 2005.

[6] C.T. Aage and J.N. Salunke, "The results of fixed points in dislocated and dislocated quasi-metric space", *Applied Mathematical Sciences*, vol. 2, pp. 2941-2948, 2008.

[7] C.T. Aage and J.N. Salunke, "Some results of fixed point theorem in dislocated quasi-metric space", *Bulletin of Marathadawa Mathematical Society*, vol. 9, pp. 1-5, 2008.

[8] A. Isufati, "Fixed point theorem in dislocated quasi-metric spaces", *Applied Mathematical Sciences*, vol. 4, pp. 217-223, 2010.

[9] M. Kohli, R. Sharivastava and M. Sharma, "Some results on fixed point theorems in dislocated quasi-metric space", *Int. Journal of Theoritical and Applied Sciences* , vol. 2, pp. 27-28, 2010.

[10] A. Muraliraj and R.J. Hussin, "New results on fixed point theorems in dislocated quasi-metric space", *Int. Journal of Mathematical Sciences and Applications*, vol. 3, pp. 321-326, 2013.

[11] M. Sarwar, M.U. Rahman and G. Ali, "Some fixed point results in dislocated quasi-metric (dq-metric) spaces", *Journal of Inequalities and Applications*, vol. 278, pp. 1-11, 2014.