Estimation of Lower Bound for Maximum Lifespan

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***Abstract*—In the absence of age specific mortality data, we estimate lower bound for maximum lifespan in the extended Weibull model. Also, we improved the accuracy of upper bound obtained for maximum lifespan in our earlier work.**

***Index Terms*—Age-dependent stretched exponent (Shape parameter), Characteristic life, Vertex point.**

***MSC 2010 Codes* – 62N05, 62F10, 62P10, 62N02, 26D20**

I. Introduction

Humanity has always been impressed by extreme longevity in the world. In fact, until recent years, it was quite unlikely for anyone to ever live to much more than a hundred [3]. The increase in life expectancy in human populations worldwide is a triumph of biomedical research [12]. Today, however, there are several thousand centenarians lived to the age of 122 [3]. Is it possible that human longevity is on the increase?

Human life expectancy in developed countries has increased steadily for over 150 years, through improvements in public health and lifestyle [12]. Fundamental studies of the aging process have of late attracted the interest of researchers in a variety of disciplines, linking ideas and theories from biochemistry to mathematics [9].

Particularly in modern research findings, it seems to be obvious that the mortality rate does not increase according to the Gompertz model at the highest ages [2], [6] and this deviation from the Gompertz model is a great puzzle to demographers, biologists and gerontologists.

The most frequently used model for mortality rates and life span goes back to Benjamin Gompertz 1825 [1] is widely used in gerontology. The Gompertz model of mortality has been found empirically and while it describes human mortality patterns up to the age of 96 very accurately [7] it has problems to account for the reduced increase of mortality among the oldest individuals [8], [4], [7]. However, so far no model has been suggested which can perfectly approximate the development of mortality over the total life span [11].

It is difficult to find an accurate mathematical model based on old-age mortality pattern [10], [5]. Weon et.al. recently found a useful model derived from the Weibull model with an age-dependent shape parameter to describe the human survival and mortality curves. In this model predicts the maximum longevity to exist around ages $120 - 130$, which indicates that there is an intrinsic limit to human longevity, and that the Weon model allows the best possible description of the demographic trajectories for super centenarians [17].

In our previous works, we found that the estimation of ratio of vertex point to the characteristic life and its maximum lifespan, the reduced $'t'$ intercept and its maximum lifespan.

In the present study, we provide an estimation of lower and upper bounds for maximum lifespan using a new representation for the age-dependent stretched exponent in the absence of age specific mortality data.

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II. Mathematical model: Weon model

Weon et al. had been described human demographic trajectories, a new demographic model namely "Weon model" [13]. The age-dependent shape parameter permits us to model the demographic (survival and mortality) functions, which are expressed as follows,

$$S(t) = e^{(-\frac{t}{\alpha})^{\beta(t)}}. \tag{1}$$

$$\mu(t) = \frac{-d\ln S(t)}{dt}. \tag{2}$$

From (1), we get

$$\mu(t) = \left(\frac{t}{\alpha}\right)^{\beta(t)} \left[\frac{\beta(t)}{t} + \ln\left(\frac{t}{\alpha}\right)\frac{d\beta(t)}{dt}\right]. \tag{3}$$

where $S(t)$ denotes the survival probability at any age t. Here $\beta(t)$ denotes the age-dependent shape parameter as function of age and α denotes the characteristic life. After graphically determining the value α, an mathematical expression determined from (1),

$$\beta(t) = \frac{\ln(-\ln S(t))}{\ln(\frac{t}{\alpha})}. \tag{4}$$

In (4), α shows the characteristic life ($t = \alpha$) when $S = \exp(-1) \approx 36.79\%$. The trends and causes increased characteristic life may be identical with those of average life (when S = 50%). The age-dependence of shape parameter must be associated with the fundamental mechanisms of human aging. Particularly, the shape parameter increases linearly with age for modern survival curve.

The shape parameter $\beta(t)$ using the mathematical expression $\beta(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + ...$, where the associated coefficients were determined by a regression analysis in the plot of $\beta(t)$ versus age. Particularly the quadratic pattern,

$$\beta(t) = \beta_0 + \beta_1 t + \beta_2 t^2. \tag{5}$$

is the mathematically valid before an age limit (the vertex is $\nu = -\frac{\beta_1}{2\beta_2}$). Hypothetically, equation (1) is the survival function of age ($\frac{dS(t)}{dt} < 0$) between $S(0) = 1$ and $S(\omega) = 0$ for a maximum age (ω). For this reason, the slope of the stretched exponent with age can be given by,

$$\frac{d\beta(t)}{dt}\begin{cases} < +\epsilon(t), t < \alpha \\ > -\epsilon(t), t > \alpha. \end{cases} \tag{6}$$

where $\epsilon(t) = \left|-\frac{\beta(t)}{t\ln(\frac{t}{\alpha})}\right|$ is the mathematical constraint of $\beta(t)$. The maximum human lifespan (ω) per each survival curve might be estimated at the mathematical limit of

$$\frac{d\beta(t)}{dt} = -\frac{\beta(t)}{t\ln(t/\alpha)}. \tag{7}$$

Remark: In this paper, we follow the usual notation t_m for maximum lifespan.

III. Estimation of bounds for Maximum lifespan

First, we perceive that the quadratic form [14] $\beta(t)$ can be expressed by

$$\beta(t) = \beta(\nu) - c(\nu - t)^2. \tag{8}$$

Substitution (8) into (7), we get

$$2(t-\nu)t\ln(t/\alpha) = \frac{\beta(\nu)}{c} - (t-\nu)^2. \tag{9}$$

Or, equivalently

$$(t-\nu)^2 + 2(t-\nu)t\ln(t/\alpha) = \frac{\beta(\nu)}{c}. \tag{10}$$

In [14], the curvature and the vertex point give an upper $'t'$ intercept, as can be defined as the "q" point.

$$(q-\nu)^2 = \frac{\beta(\nu)}{c}. \tag{11}$$

From (10) and (11) eliminating $\frac{\beta(\nu)}{c}$ we obtain

$$(t-\nu)^2 + 2(t-\nu)\alpha(t/\alpha)\ln(t/\alpha) = (q-\nu)^2. \tag{12}$$

In [15], it was shown that

$$0.410986 < (t_m/\alpha)\ln(t_m/\alpha) < 0.829297 \tag{13}$$

Since equation (12) is equivalent to (7), in view of (13), (12) gives

$$(t_m-\nu)^2 + 2\alpha(0.410986)(t_m-\nu) < (q-\nu)^2.$$

After some little algebra, we get

$$t_m^2 - 2t_m(\nu - 0.410986\alpha) - 2\alpha\nu(0.410986) - q^2 + 2q\nu < 0. \tag{14}$$

Solving (14) for t_m, we get

$$t_m < (\nu - 0.410986\alpha + (\sqrt{(q-\nu)^2 + 0.168909\alpha^2})). \tag{15}$$

Thus we obtained a upper bound for Maximum lifespan. Notice that, the bound obtained in (15) is more accurate than given in [16]. Next, we estimate a lower bound for Maximum human lifespan (t_m). Similarly using (13), (12) becomes

$$t_m^2 - 2t_m(\nu - 0.829297\alpha) - 2\alpha\nu(0.829297) - q^2 + 2q\nu > 0. \tag{16}$$

Solving (16) for t_m, we get

$$t_m > (\nu - 0.829297\alpha + (\sqrt{(q-\nu)^2 + 0.687734\alpha^2})). \tag{17}$$

For a Comparison with the numerical data [16] we list the numerical values of (15) and (17) in Table 1 and 2 ($Mathematica\ version$ 6.0) for a given ν and α.

TABLE I
(REPRINTED FROM [14])

$\alpha(years)$	$\nu(years)$	q	t_m	estimated $t_m >$	estimated $t_m <$
88.57223	96.34	149.70	122.86	113.676	124.532
88.52680	94.96	154.34	125.19	115.968	128.217
87.83450	94.73	152.27	123.75	114.715	126.557
87.81336	95.93	155.35	125.45	117.096	129.361
87.66873	95.65	155.58	125.59	117.167	129.547
87.34920	94.04	160.56	127.73	119.949	133.729
86.91140	94.57	158.95	126.81	119.137	132.476
86.53163	94.78	158.11	126.21	118.729	131.849
86.30050	94.58	162.12	128.30	121.417	135.398
85.86844	95.55	160.55	127.37	120.755	134.222

TABLE II
(REPRINTED FROM [14])

Estimations for Sweden female survival data sets.

$\alpha(years)$	$\nu(years)$	q	t_m	estimated $t_m >$	estimated $t_m <$
88.07717	92.06	157.23	126.16	116.907	130.410
88.17957	95.29	154.80	125.23	116.445	128.726
88.93178	100.00	157.10	127.34	119.521	131.246
82.71487	94.92	150.53	121.06	114.63	126.103
85.58519	94.70	153.61	123.58	115.963	128.138
86.67277	100.58	158.11	127.16	120.768	132.624
87.57264	96.30	162.70	129.24	122.08	135.836
85.47782	95.36	165.86	129.85	124.449	138.998
88.76823	93.56	153.32	124.54	114.763	127.093
90.37934	95.20	159.00	128.33	118.677	131.881
83.79004	96.27	169.96	131.53	128.068	143.173
88.78873	99.22	142.67	119.77	111.084	119.470
87.53789	90.75	174.44	134.53	128.944	145.868
89.22443	92.76	159.09	127.66	118.138	131.882
91.59045	98.17	161.51	130.31	121.115	134.209
83.82117	95.46	165.53	129.20	124.648	139.091
84.63518	97.72	153.30	123.59	117.062	128.503
87.85314	95.65	155.48	125.55	117.068	129.424
88.49842	94.48	154.52	125.19	115.910	128.306
81.29562	98.59	155.26	123.63	119.244	130.965
86.15168	100.50	159.68	127.78	121.827	134.056
87.42881	94.93	157.52	126.32	118.209	131.169
88.57223	96.34	149.70	122.86	113.676	124.532
89.69486	97.52	149.62	123.38	113.951	124.479
87.43102	103.37	159.62	128.74	122.631	132.185

IV. Conclusion

In this paper, we are able to estimate the lower bound for maximum lifespan thanks to $0.410986 < (t_m/\alpha)\ln(t_m/\alpha) < 0.829297$. Without using the above inequality, we need to solve the equation (12). This, in turn leads to develop a method of solving transcendental equations involving logarithmic functions.

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