On Detour Domination in Graphs

Samir K. Vaidya and Raksha N. Mehta

***Abstract*—The usual distance between two vertices in a graph is the shortest path between them while detour distance is the longest path. We held the discussion of dominating sets in graphs in the context of detour distance. We derive detour domination number of some graphs.**

***Index Terms*—Distance, Detour distance, Detour domination.**

***MSC 2010 Codes* - 05C12, 05C69.**

I. Introduction

We begin with finite, connected and undirected graph without loops and multiple edges. For any graph theoretic terminology we rely upon Chartrand and Lesniak [1] as well as Haynes *et al.* [2]. We provide brief summary of definitions and existing results needed for the present work.

Definition 1.1 The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest $u - v$ path in G.

A brief account of distance in graphs and its related parameters can be found in Buckley and Harary [3].

Definition 1.2 The length of the longest $u - v$ path between two vertices u and v in a connected graph G is called detour distance $D(u, v)$.

The concept of detour distance was introduced by Chartrand *et al.* [4, 5].

Remark 1.3 $D(u, v) = d(u, v)$, for any tree T.

Definition 1.4 A vertex $u(\neq v)$ is called a *detour neighbor* of v if $\overline{D}(v) = D(u, v)$ where

$$\overline{D}(v) = min\{D(u, v)/u \in V(G) - \{v\}\}.$$

If u is a detour neighbor of v, then it is not necessary that v is also a detour neighbor of u. For example, in Figure 1, v_1 and v_2 are detour neighbors of u_1, u_2 and u_3, but u_1, u_2 and u_3 are not the detour neighbors of v_1 and v_2. Moreover v_1 and v_2 are detour neighbors of each other. The set of all neighbors of v is denoted by $N_D(v)$. In Figure 1, $N_D(u_1) = N_D(u_2) = N_D(u_3) = \{v_1, v_2\}$.

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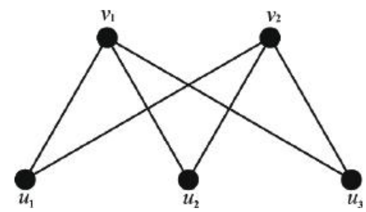


Figure 1

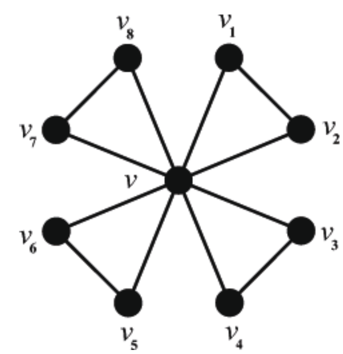


Figure 2

Definition 1.5 A vertex u is said to *detour dominate* a vertex v if $u = v$ or u is a detour neighbor of v. In graph G_1 of Figure 1, u_1, u_2 and u_3 detour dominate v_1 and v_2 while v_1 and v_2 detour dominate each other whereas in Figure 2, v detour dominates all the vertices of the graph G_2. A set S of vertices of G is called a *detour dominating set* if every vertex of G is detour dominated by some vertex of S. A detour dominating set of G with minimum cardinality is a *minimum detour dominating set* and this cardinallity is the *detour domination number* denoted as $\gamma_D(G)$.

The concept of detour domination was introduced by Chartrand *et al.* [6] and it was further explored by Chartrand and Zhang [7].

We formalize the following concepts.

Definition 1.6 A vertex v is said to be a *detour isolate* if it is not a detour neighbor of any vertex of the graph G. In other words, $N_D(V) \cap \{v\} = \varnothing$ where $v \in V$, then v is a detour isolate. In Figure 1, u_1, u_2 and u_3 are detour isolates as they are not detour neighbors of any vertices of the graph G_1, whereas the graph G_2 in Figure 2 has no detour isolates.

It is obvious that a detour isolate is always an element of any detour dominating set.

Definition 1.7 A detour dominating set S is a minimal detour dominating set if no proper subset $S' \subset S$ is a detour dominating set. For the graph of Figure 1, $\gamma_D(G) = 3$ as $S = \{u_1, u_2, u_3\}$ is the only detour dominating set while in Figure 2, $S_1 = \{v_1, v_3, v_5, v_7\}$, $S_2 = \{v_2, v_4, v_6, v_8\}$ and $S_3 = \{v\}$ are minimal detour dominating sets and $\gamma_D(G) = 1$ as $\mid S_3 \mid = 1$ is minimum.

II. Main Results

We prove the following result analogous to Ore [8].

Theorem 2.1 A detour dominating set S is a minimal detour dominating set if and only if for each vertex $u \in S$ either of the following two conditions holds.
(i) u is a detour isolate in S.
(ii) For all $u \in S$ and $v \in S - \{u\}$ $\exists$ at least a vertex $u' \in N_D(u)$ such that $u' \notin N_D(u) \cap N_D(v)$.

Proof: Suppose S is a minimal detour dominating set and neither of the conditions (i) and (ii) hold. That is, if u is not a detour isolate in S, then there exist some vertex $v' \in S$ such that $u \in N_D(v')$. Then $S - \{u\}$ still remains a detour dominating set. Also for all $u \in S$ and $v \in S - \{u\}$ $\exists$ at least a vertex $u' \in N_D(u)$ such that $u' \in N_D(u) \cap N_D(v)$. That is, u' is detour dominated by some vertex $v \in S - \{u\}$. Hence, $S - \{u\}$ is a detour dominating set which is a contradiction. Therefore either of the conditions must hold.

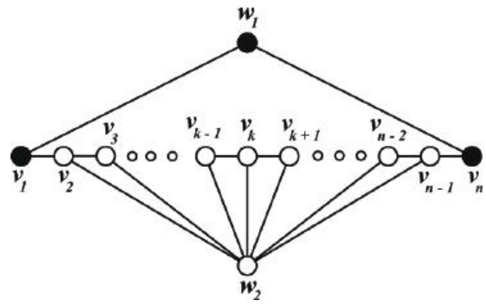
Conversely suppose that (i) or (ii) holds. But S is not a minimal detour dominating set. Hence, $S - \{u\}$ is still a detour dominating set. Then $\exists$ some vertex u such that $N_D[u] \subseteq \cup N_D(v)$, for some $v \in S - \{u\}$. Consequently $\exists$ at least a vertex $u' \in N_D(u) \cap N_D(v)$. Hence, (ii) does not hold.

Also if $S - \{u\}$ is a detour dominating set, then in that case u is a detour neighbor of at least a vertex of $S - \{u\}$. That is, (i) does not hold. Thus neither condition (i) nor (ii) holds which contradicts our assumption. Therefore, S is a minimal detour dominating set.

Definition 2.2 Let G be a graph with $V(G) = S_1 \cup S_2 \cup \ldots S_t \cup T$ where S_i is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$ where $i = 1, 2, \ldots, t$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2 \ldots, w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$.

Theorem 2.3 For $n \geq 6$, $\gamma_D[DS(P_n)] = 3$.

Proof: Let $v_1, v_2, v_3 \ldots v_n$ be the vertices of P_n, we add two vertices w_1 and w_2 in order to obtain $DS(P_n)$ as P_n contains vertices with two different types of degrees. Hence, $\mid V[DS(P_n)] \mid = n + 2$ and $\mid E[DS(P_n)] \mid = 2n - 1$. The graph $DS(P_n)$ is shown in the following Figure 3.



$DS(P_n)$
Figure 3

Claim 1: w_1 detour dominates w_2, v_{n-1} and v_k, $(k = 2, 3 \ldots, n-1)$.

On traversing right through $w_1 v_n v_{n-1} w_2 v_{n-2} \ldots v_1$ then the detour distance from w_1 to v_1 is $n + 1$ and on traversing left through $w_1 v_1 v_2 w_2 v_3 \ldots v_n$ then the detour distance from w_1 to v_n is also $n + 1$ while on traversing left through $w_1 v_1 v_1 v_2 \ldots v_{n-1} w_2$ or traversing right through $w_1 v_n v_{n-1} \ldots v_2 w_2$, in either case, the detour distance from w_1 to w_2 is n. Now, on traversing right through $w_1 v_n v_{n-1} \ldots v_{k+1} w_2 v_2 \ldots v_k$, $(k = 2, 3 \ldots, n-2)$ then the detour distance from w_1 to v_k is n. The detour distance from w_1 to v_{n-1} traversing left through $w_1 v_1 v_2 w_2 v_3 \ldots v_{n-1}$ is n. Hence, w_1 detour dominates v_k, v_{n-1} and w_2 which are at a minimum detour distance n.

Claim 2: w_2 detour dominates w_1.

On traversing left through $w_2 v_{k+1} \ldots\ v_n w_1 v_1 \ldots v_k$, $(k = 1, 2, 3 \ldots, n-1)$ then the detour distance from w_2 to v_k is $n + 1$. The detour distance from w_2 to v_n traversing right through $w_2 v_{n-1} \ldots v_1 w_1 v_n$ is $n + 1$. Now, on traversing left through $w_2 v_2 \ldots v_n w_1$ then the detour distance from w_2 to w_1 is n. Hence, w_2 detour dominates w_1 which is at minimum detour distance n.

Claim 3: v_1 detour dominates v_{n-1}.

On traversing through the path $v_1 w_1 v_n v_{n-1} \ldots v_{k+1} w_2 v_2 \ldots v_k$, $(k = 2, 3 \ldots, n-2)$ then the detour distance from v_1 to v_k is $n + 1$ and on traversing through the path $v_1 v_2 w_2 v_3 \ldots v_{n-1}$ then the detour distance from v_1 to v_{n-1} is $n - 1$ while on traversing through the path $v_1 v_2 w_2 v_3 \ldots v_n$ then the detour distance from v_1 to v_n is n. The detour distance from v_1 to w_j, $j = 1$ or 2 is $n + 1$ on traversing through the path $v_1 w_1 v_n v_{n-1} \ldots v_2 w_2$ or $v_1 v_2 w_2 v_3 \ldots v_n w_1$. Hence, v_1 detour dominates v_{n-1} which is at minimum detour distance $n - 1$.

Claim 4: v_2 detour dominates v_{n-1}.

On traversing through the path $v_2 v_1 w_1 v_n v_{n-1} \ldots v_{k+1} w_2 v_3 \ldots v_k$, $(k = 3 \ldots, n-2)$ then the detour distance from v_2 to v_k is $n + 1$. On traversing through the path $v_2w_2v_3 \dots v_nw_1v_1$ then the detour distance from v_2 to v_1 is $n+1$ while on traversing through the path $v_2w_2v_3 \dots v_n$ then the detour distance from v_2 to v_n is $n-1$. On traversing through the path $v_2w_2v_3 \dots v_{n-1}$ then the detour distance from v_2 to v_{n-1} is $n-2$. The detour distance from v_2 to w_1 or v_2 to w_2 traversing through the path $v_2w_2v_3 \dots v_nw_1$ or $v_2v_1w_1v_n \dots v_3w_2$ is n and $n+1$ respectively. Hence, v_2 detour dominates v_{n-1} which is at a minimum detour distance $n-2$.

Claim 5: v_k detour dominates w_1.

The detour distance from v_k to v_j, $(k = 3, 4, \dots, n-2$ and $j = 1, 2, \dots, n,\ k \neq j)$, traversing through any path covers all the vertices of the graph which is of length $n+1$. On traversing through the path $v_kv_{k-1} \dots v_2w_2v_{k+1} \dots v_nw_1$ then the detour distance from v_k to w_1 is n while the detour distance from v_k to w_2 traversing through any path will cover all the vertices of the graph is $n+1$. Hence, v_k detour dominates w_1.

Claim 6: v_{n-1} detour dominates v_2.

The detour distance from v_{n-1} to v_k, $(k = 3 \dots, n)$ covers all the vertices of the graph which is of length $n+1$. The detour distance is symmetric. So, the detour distance from v_{n-1} to v_2 is $n-2$ as discussed in Claim 4, the detour distance from v_{n-1} to v_1 is $n-1$ as discussed in Claim 3, the detour distance from v_{n-1} to w_1 is n as discussed in Claim 1 and the detour distance from v_{n-1} to w_2 is $n+1$ as discussed in Claim 2. Hence, v_{n-1} detour dominates v_2.

Claim 7: v_n detour dominates v_2.

The detour distance from v_n to v_k, $(k = 3 \dots, n-1)$, on traversing through any path covers all the vertices of the graph which is of length $n+1$. The detour distance is symmetric. So, the detour distance from v_n to v_1 is n as discussed in Claim 3, the detour distance from v_n to v_2 is $n-1$ as discussed in Claim 4, the detour distance from v_n to w_1 is $n+1$ as discussed in Claim 1, the detour distance from v_n to w_2 is $n+1$ as discussed in Claim 2. Hence, v_n detour dominates v_2.

Hence, from the above arguments we conclude that the vertices v_1 and v_n are not detour dominated by any other vertices except themselves. That is, they are detour isolates. Consequently, v_1 and v_n must be in every detour dominating set. Thus, by Theorem 2.1, detour dominating set $S = \{w_1, v_1, v_n\}$ is a minimal detour dominating set. Here, v_1 and v_n cannot be removed from S being detour isolates and $N_D[w_1] = \{v_2, v_3, \dots, v_{n-1}, w_1, w_2\}$. That is, $\mid N_D[w_1] \mid = n$. From the above arguments it is clear w_1 is the only vertex that detour dominates remaining n vertices of the graph. Hence, S is the only detour dominating set with minimum cardinality. Therefore, $\gamma_D[DS(P_n)] = 3$, for $n \geq 6$.

Definition 2.4 The *wheel graph* W_n with n vertices is defined to be the join of K_1 and C_n. The vertex corresponding to K_1 is known as apex while the vertices corresponding to C_n are known as rim vertices.

Definition 2.5 The *helm graph* H_n is a graph obtained from wheel graph W_n by attaching a pendant edge to each rim vertex. It contains three types of vertices, the vertex of degree n called apex, n pendant vertices and n vertices of degree four.

Theorem 2.6 For $n \geq 3$, $\gamma_D(H_n) = n+1$.

Proof: For helm graph H_n, let $u_1, u_2, \dots, u_n$ be the pendant vertices, $v_1, v_2, \dots, v_n$ be the vertices of degree four and v be the apex. So, $V(H_n) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, v\}$. The vertices u_i and v_i, where $(i = 1, 2, \dots, n)$ are detour neighbors of each other since there is unique path between these vertices. So, u_i and v_i detour dominate each other. Consequently, either u_i or v_i must be in every detour dominating set. The vertex v is detour isolate as it is not the detour neighbor of u_i or v_i and thus v must be in every detour dominating set. Thus, by Theorem 2.1, $S_1 = \{u_1, u_2, \dots, u_n, v\}$ or $S_2 = \{v_1, v_2, \dots, v_n, v\}$ are minimal detour dominating sets with $\mid S_1 \mid = \mid S_2 \mid = n+1$. Thus minimum $n+1$ vertices are essential for a minimum detour dominating set. Hence, $\gamma_D(H_n) = n+1$, for $n \geq 3$.

III. Conclusion

We have introduced the concepts of detour isolate and minimal detour dominating set and derive a necessary and sufficient condition for a set to be a minimal detour dominating set. In addition to this, the detour domination numbers of helm and degree splitting graph of path are also investigated.

Acknowledgement

The authors are highly thankful to the referees for their valuable comments and kind suggestions.

References

[1] G. Chartrand and L. Lesniak, "Graphs & Digraphs", 4/e, *CRC press*, 2005.

[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, "Fundamentals of Domination in Graphs", *Marcel Dekker*, New York, 1998.

[3] F. Buckley and F. Harary, "Distance in Graphs", *Addison - Wesley*, Redwood City, CA, 1990.

[4] G. Chartrand, G. L. Johns and P. Zhang, "On the Detour Number and Geodetic Number of a Graph", *Ars Combinatoria*, vol. 72, pp 3 - 15, 2004.

[5] G. Chartrand, G. L. Johns and P. Zhang, "The Detour Number of a Graph", *Utilitas Mathematica*, vol. 64, pp 97 - 113, 2003.

[6] G. Chartrand, T. W. Haynes, M. A. Henning and P. Zhang, "Detour Domination in Graphs", *Ars Combinatoria*, vol. 71, pp 149 - 160, 2004.

[7] G. Chartrand and P. Zhang, "Distance in Graphs - Taking the Long View", *AKCE J. Graphs. Combin.*, vol. 1, no. 1, pp 1-13, 2004.

[8] O. Ore, "Theory of Graphs", *Amer. Math. Soc. Colloq. Publi.*, 38 (Amer Math. Soc., Providence, RI), 1962.