Generalized Right Circulant Matrices with Geometric Sequence

A.C.F. Bueno

***Abstract*—In this paper, the determinant, eigenvalues, Euclidean norm, spectral norm and the inverse of generalized right circulant matrices with geometric sequence were obtained.**

***Index Terms*—eigenvalues, Euclidean norm, spectral norm, inverse of a matrix, right circulant matrix, geometric sequence**

***MSC 2010 Codes* – 05C50, 11B50, 15A09, 15A15, 15A60**

I. Introduction

IN [2], the the determinant, eigenvalues, Euclidean norm, spectral norm and the inverse of the matrix

$$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & \dots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & \dots & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & \dots & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & \dots & a & ar \\ ar & ar^2 & \dots & ar^{n-1} & a \end{pmatrix}$$

where $a,\ ar,\ ar^2,\ \dots,\ ar^{n-1}$ are the first n terms of the geometric sequence $\left\{ar^k\right\}_{k=0}^{+\infty}$.

Now, consider the subsequence $\left\{ar^j,\ ar^{j+1},\ ar^{j+2},\ \dots,\ ar^{j+n-1}\right\}$ of $\left\{ar^k\right\}_{k=0}^{+\infty}$ and form the vector $\vec{G} = (ar^j,\ ar^{j+1},\ ar^{j+2},\ \dots,\ ar^{j+n-1})$. The generalized right circulant matrix with geometric sequence is given by

$$RCIRC_n(\vec{G}) = \begin{pmatrix} ar^j & ar^{j+1} & \dots & ar^{j+n-2} & ar^{j+n-1} \\ ar^{j+n-1} & ar^j & \dots & ar^{j+n-3} & ar^{j+n-2} \\ ar^{j+n-2} & ar^{j+n-1} & \dots & ar^{j+n-4} & ar^{j+n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^{j+2} & ar^{j+3} & \dots & ar^j & ar^{j+1} \\ ar^{j+1} & ar^{j+2} & \dots & ar^{j+n-1} & ar^j \end{pmatrix}$$

For the rest of the paper we will use the following conventions for an nxn matrix A: $|A|$ for its determinant, $||A||_E$ for its Euclidean norm and $||A||_2$ for its spectral norm.

II. Main Results

Theorem 2.1:

$$\left|RCIRC_n(\vec{G})\right| = a^n r^{jn}(1-r^n)^{n-1}$$

Proof:

From [2], it has been established that

$$|RCIRC_n(\vec{g})| = a^n(1-r^n)^{n-1}$$

. Note that

$$RCIRC_n(\vec{G}) = r^j RCIRC_n(\vec{g})$$

so

$$\begin{aligned} \left|RCIRC_n(\vec{G})\right| &= \left|r^j RCIRC_n(\vec{g})\right| \\ &= r^{jn}\left|RCIRC_n(\vec{g})\right| \\ &= a^n r^{jn}(1-r^n)^{n-1} \end{aligned}$$

Theorem 2.2:

$$\left\|RCIRC_n(\vec{G})\right\|_E = |ar^j|\sqrt{\frac{n\left(1-r^{2n}\right)}{1-r^2}}$$

Proof:

$$\begin{aligned} \left\|RCIRC_n(\vec{G})\right\|_E &= \sqrt{\sum_{i,j=0}^{n} a_{ij}^2} \\ &= \sqrt{\sum_{k=0}^{n-1} na^2r^{2j+2k}} \\ &= \sqrt{\frac{na^2r^{2j}\left(1-r^{2n}\right)}{1-r^2}} \\ &= |ar^j|\sqrt{\frac{n\left(1-r^{2n}\right)}{1-r^2}} \end{aligned}$$

Theorem 2.3: The eigenvalues of $RCIRC_n(\vec{G})$ are the following: $\lambda_0 = ar^j\left(\frac{1-r^n}{1-r}\right)$ and $\lambda_m = ar^j\left(\frac{1-r^n}{1-r\omega^{-m}}\right)$ where m=1,2, ... , n-1 and $\omega = e^{\frac{2\pi i}{n}}$

Proof: The eigenvalues of any right circulant matrix is given by the Discrete Fourier transform

$$\lambda_m = \sum_{k=0}^{n-1} c_k\omega^{-mk}$$

Hence at $m=0$, we have

$$\lambda_0 = \sum_{k=0}^{n-1} ar^{j+k} = ar^j\left(\frac{1-r^n}{1-r}\right)$$

and for $m \neq 0$

$$\lambda_m = \sum_{k=0}^{n-1} ar^{j+k}\omega^{-mk} = \sum_{k=0}^{n-1} ar^j(r\omega^{-m})^k$$

Aldous Cesar F. Bueno is an instructor in the Department of Mathematics and Physics of the Central Luzon State University, Science City of Muñoz, Nueva Ecija, Phillippines. (e-mail: aldouz_cezar@yahoo.com).

That is, for $m \neq 0$,

$$\lambda_m = ar^j \left(\frac{1 - r^n}{1 - r\omega^{-m}} \right)$$

Theorem 2.4:

$$\left\| RCIRC_n(\vec{G}) \right\|_2 = \max \left\{ \left| ar^j \left(\frac{1 - r^n}{1 - r} \right) \right|, \frac{\left| ar^j (r^n - 1) \right|}{\sqrt{r^2 - 2r\cos\frac{2\pi m}{n} + 1}} \right\}$$

Proof: It suffices to show that

$$|1 - r\omega^{-m}| = \sqrt{r^2 - 2r\cos\frac{2\pi m}{n} + 1}$$

$$\begin{aligned} |1 - r\omega^{-m}| &= \left| 1 - r\left(\cos\frac{2\pi m}{n} - i\sin\frac{2\pi m}{n} \right) \right| \\ &= \left| 1 - r\cos\frac{2\pi m}{n} + ir\sin\frac{2\pi m}{n} \right| \\ &= \sqrt{\left(1 - r\cos\frac{2\pi m}{n} \right)^2 + \left(r\sin\frac{2\pi m}{n} \right)^2} \\ &= \sqrt{1 - 2r\cos\frac{2\pi m}{n} + r^2\cos^2\frac{2\pi m}{n} + r^2\sin^2\frac{2\pi m}{n}} \\ &= \sqrt{r^2 - 2r\cos\frac{2\pi m}{n} + 1} \end{aligned}$$

Theorem 2.5:

$$RCIRC_n^{-1}(\vec{G}) = \frac{1}{ar^j(r^n - 1)} RCIRC_n(-1,\ r,\ 0\ldots,0)$$

Proof: Since

$$RCIRC_n(\vec{G}) = r^j RCIRC_n(\vec{g}),$$

it follows that

$$RCIRC_n^{-1}(\vec{G}) = \frac{1}{r^j} RCIRC_n^{-1}(\vec{g})$$

In [2], it has been established that

$$RCIRC_n^{-1}(\vec{g}) = \frac{1}{a(r^n - 1)} RCIRC_n(-1,\ r,\ 0,\ldots,0)$$

This completes the proof. □

III. Conclusion

In this paper, we have derived the explicit forms of the determinant, eigenvalues, Euclidean norm, spectral norm and the inverse of generalized right circulant matrices with geometric sequence $RCIRC_n(\vec{G})$. All of the explicit forms are functions of the first term ar^j, common ratio r and number of terms n of the subsequence $\left\{ar^j,\ ar^{j+1},\ ar^{j+2},\ \ldots,\ ar^{j+n-1}\right\}$ of $\left\{ar^k\right\}_{k=0}^{+\infty}$.

References

[1] M. Bahsi and S. Solak, "On the Circulant Matrices with Arithmetic Sequence", *International Journal of Contemporary Matematical Sciences*, vol. 5, no. 25, pp. 1213–1222, 2010.

[2] A.C.F. Bueno, "Right Circulant Matrices with Geometric Progression", *International Journal of Applied Mathematical Research*, vol. 1, no 4., pp. 593–603, 2012.