Some Results on Decomposable and Symmetric Decomposable Sets in Decomposable Groups

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***Abstract*—A decomposable set in a group G has interesting properties. In this paper, we investigate such properties. For instance, we prove that if $G = AB$ and $\check{G} = CD$ are isomorphic groups and their decompositions are unique, then the isomorphic image of any decomposable set is decomposable too. Further, we give some elementary results on the decomposator of a subgroup. Symmetric decomposable sets also are introduced. By this concept, we establish some interesting results. As an important consequence, we prove that if $G = AB$ is a decomposable group and S a symmetric decomposable subgroup of G, then S is symmetric decomposable in $K = AS$ if and only if S is idempotent.**

***Index Terms*—Decomposable set, Symmetric decomposable set, Decomposable group, Group theory.**

***MSC 2010 Codes* – 20D10**

I. Introduction

In this section, we study decomposable sets and groups and present some results on them, from [1], which will be needed in the sequel. Any unexplained notions can be founded in [2], [3], [4].

Definition 1.1: Let $G = AB$ be a group and A and B proper subgroups. A subset S of G is called decomposable when $ab \in S$, where $a \in A$ and $b \in B$, implies that $a \in S$.

Lemma 1.2: Let $G = AB$ be a group and S a subgroup of G. Then the following statements are eqivallent:
(i) S is decomposable;
(ii) $S = (A \cap S)(B \cap S)$ where $A \cap B \subseteq S$.

Proposition 1.3: Let $G = AB$ be a group. Then any supersubgroup of A or B is decomposable in G.

Lemma 1.4: Let $G = AB$ be a group. Then
(i) The union and intersection of decomposable subsets of G is decomposable too.
(ii) Suppose that $N \lhd G$ and S/N is a decomposable subset of G, then S is a decomposable subset of G.
(iii) the product of two decomposable subgroup M and N is decomposable provided that $MN = NM$ and $M(B \cap N) = (B \cap N)M$.

Definition 1.5: Let N be a subgroup of $G = AB$ such that $NA = AN$ and $NB = BN$. Then the smallest decomposable subgroup of G which contains N is called to be decompositator of N in G and denoted by $X(N)$.

Remark 1.6: Indeed, $X(N) = \bigcap_{N \subseteq U} U$ where U runs on decomposable subgroups of G.

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II. On Decomposable Sets in a Group

In this section we want to prove some results on decomposable sets in decomposable groups. First of all, we prove the following results.

Theorem 2.1: Let $G = AB$ and $\check{G} = CD$ be decomposable groups with unique decomposition. If G and $\check{G}$ are isomorphic, then the isomorphic image of any decomposable set is decomposable.

Proof: Let $\theta : G \to \check{G}$ be an isomorphism and S a decomposable set of G. By Lemma 1.2, $A \cap B \subseteq S$ and $S = (A \cap S)(B \cap S)$. Then

$$\theta(A) \cap \theta(B) \subseteq \theta(S)$$

And

$$\theta(S) = (\theta(A) \cap \theta(S))(\theta(B) \cap \theta(S))$$

Since θ is an isomorphism. Now, uniqueness of decompositions of G and $\check{G}$ implies that $\theta(A) = C$ and $\theta(B) = D$. This completes the proof. ■

In the following, we present a simple characterization of a decomposable subgroups.

Theorem 2.2: Let $G = K \ltimes H$ and S be an arbitrary subset of G. Then S is decomposable if and only if $S \subseteq (A \cap S)(B \cap S)$.

Proof: It follows from the definition of a semidirect product that $G = KH$ such that $K \cap H = 1$ (see e.g. [4]). Now the proof follows from the Lemma 1.2. ■

Theorem 2.3: Let $G = AB$ and S be a decomposable set in G. If H is a decomposable subgroup of G contain S, then S is a decomposable set in the group $K = (A \cap H)(B \cap H)$.

Proof: It follows from Lemma 1.2 that $A \cap B \subseteq S$ and $S = (A \cap S)(B \cap S)$. Since $S \subseteq H$ and H is a decomposable subgroup of G, so we conclude that

$$H \cap S = (A \cap (H \cap S))(B \cap (H \cap S))$$

And so

$$S = ((A \cap H) \cap S)((B \cap H) \cap S)$$

On the other hand,

$$(A \cap H) \cap (B \cap H) \subseteq S \cap H = S.$$

This concludes the desired result. ■

Now we want to pose the following question:

Question 2.4: Let $G = AB$ and M and N be decomposable sets in G. Suppose further, $S \subset G$ such that $M \subsetneq S \subsetneq N$. Is S decomposable in G?

Theorem 2.5: Let $G = K \ltimes H$ and S be an arbitrary subset of G such that $SK = KS$. Then KS is decomposable.

Proof: It is well-known and easy to see that $G = KH$ such that $K \cap H = 1$. Therefore it is obvious that $M(K \cap H) = (K \cap H)M$. Further, it can be easily seen that K and H are decomposable subgroup of G. Now, the proof is finished by Lemma 1.4 (iii). ■

For a subgroup N of a group G, $X(N)$ and $D_G(N)$ denote the set of all decomposable subgroup of G which contain N and decomposator of N in G, respectively.

Theorem 2.6: Let $G = AB$ and N be a subgroup of G. If $X(N) = A$, then $G = UB$ for every $U \in D_G(N)$.

Proof: One can easily conclude that $N \subseteq A \subseteq U$ for every $U \in D_G(N)$. Therefore, $G = UB$ for every $U \in D_G(N)$. ■

In the following, we have some elementary results on decomposator of a subgroup.

Proposition 2.7: Let G be a group and M and N subgroup of G such that $N \subseteq M$. Then $X(M)$ contains $X(N)$.

Proof: It immediately obtains by Definition 1.5 and Remark 1.6. ■

Corollary 2.8: Let G be a group and M and N subgroup of G such that $N \subseteq M$. Then

$$X(N) = \bigcup_{N \subseteq W \subseteq M} W$$

Where W's are decomposable subgroups of G.

Proof: By Proposition 2.7, we have

$$X(N) = X(N) \cap X(M) = \bigcup_{N \subseteq W \subseteq M} W$$

As desired. ■

Now we want to define a relation on the set of all decomposable subgroups a group G. For convenience, we denote the set of all decomposable subgroups of a group G, by $DS(G)$.

$$M, N \in DS(G) \ \ , \ \ M \lesssim_D N \Leftrightarrow X(M) = X(N).$$

Proposition 2.9: $\lesssim_D$ defines an equivalence relation on $DS(G)$.

Proof: It is clear. ■

Now, the equivalence class of $N \in DS(G)$ is defined as follow:

$$[N] = \{M \in DS(G) : \ X(M) = X(N)\}.$$

Proposition 2.10: Let $N \in DS(G)$. Then $[N] = \{N\}$ if the set of all decomposable subgroup of G which contain N construct an increasing sequence with respect to inclusion.

Proof: Straightforward. ■

It is natural to ask the following question:

Question 2.11: Is true the converse of the Proposition 2.10?

III. Symmetric Decomposable Sets

In this section, we introduce the concept of a symmetric decomposable subset of a group and obtain some results. First, we present the following definition.

Definition 3.1: Let $G = AB$ be a group and A and B proper subgroups of G. A subset S of G is called to be symmetric decomposable if we have

$$S = (A \cap S)S = S(S \cap B).$$

Example 3.2: Let $G = AB$ be an abelian group and A decomposable set. Then A is symmetric decomposable.

In a decomposable abelian group, If a subgroup is both decomposable and symmetric decomposable, it must be idempotent. This is reflected in the next theorem.

Theorem 3.3: Let $G = AB$ be an abelian group and the subgroup S both decomposable and symmetric decomposable. Then S is idempotent.

Proof: Since G is abelian, decomposable and symmetric decomposable, we have

$$S = (A \cap S)(B \cap S) \Rightarrow S^2 = S(B \cap S)(A \cap S) = (A \cap S)S = S$$

This completes the proof. ■

The existence of complement of a subgroup in a decomposable group can be established as follow.

Theorem 3.4: Let $G = AB$ be a decomposable group. If G has a symmetric decomposable subgroup, then A is a complement of B and vice versa.

Proof: From symmetric decomposability of S, one can conclude that $A \cap B = 1$ and this is the desired result. ■

Corollary 3.5: Let $G = AB$ be a decomposable group. If A or B are symmetric decomposable, then A is a complement of B and vice versa.

Corollary 3.6: Let $G = AB$ be a decomposable group. If $S \neq 1$ is a symmetric decomposable subgroup of G, then S is not decomposable.

Proof: By contrary, if S is decomposable, then Lemma 1.2 implies that

$$S = (A \cap S)(B \cap S) \tag{1}$$

By hypothesis, since S is symmetric decomposable subgroup of G, so we have

$$A \cap S = 1$$

And

$$B \cap S = 1.$$

■

Theorem 3.7: Let $G = AB$ be a decomposable group. If S is a symmetric decomposable set of G and $\check{B}$ a supersubgroup of B, then S is a symmetric decomposable set of the group $\check{G} = A\check{B}$.

Proof: Since S is a symmetric decomposable set of G, so by Definition 3.1, it is enough to show that $S = S(\check{B} \cap S)$ and this is obvious from the hypothesis on $\check{B}$. ■

We conclude the section with an interesting characterization.

Theorem 3.8: Let $G = AB$ be a decomposable group and S a symmetric decomposable subgroup of G. Then S is symmetric decomposable in $K = AS$ if and only if S is idempotent.

Proof: $(\Rightarrow)$ Straightforward.

$(\Leftarrow)$ Let S be idempotent. Then

$$S = S^2 = (S \cap S)S \tag{2}$$

On the other hand, by symmetric decomposability of S in G, we have

$$S = (A \cap S)S \tag{3}$$

Now, (2) together with (3) imply that S is symmetric decomposable in $K = AS$. ■

Remark 3.9: It is mentioned that under assumptions of the previous theorem, one can prove that S is symmetric decomposable in $L = SB$ if and only if S is idempotent.

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