Right Circulant Matrices with Sum of the Terms of Two Geometric Sequences

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***Abstract*—In this paper, we solve for the eigenvalues and the determinant of a right circulant matrix whose entries are the sum of the terms of two geometric sequence. Using this sequence, we also solve the eigenvalues and the determinant of right circulant matrices with Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers.**

***Index Terms*—right circulant matrix, eigenvalue, determinant, geometric sequence, Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers**

***MSC 2010 Codes* – 15A15, 15A18, 15A60**

I. Introduction

BAHSI and Solak [1] provided the explicit forms of the eigenvalues, determinant, Euclidean norm, spectral norm and the inverse of right circulant matrices with terms of an arithmetic sequence as entries. This work was followed by Bueno [2] and did the same thing but he dealt with the terms of a geometric sequence as entries for the right circulant matrices.

In this study, we will use a different sequence. Let $\left\{ar^k\right\}_{k=0}^{+\infty}$ and $\left\{bs^k\right\}_{k=0}^{+\infty}$ be geometric sequences with first terms a and b and common ratios r and s, respectivley. Next we form the sequence $\{h_k\}_{k=0}^{+\infty}$ using the relationship $h_n = ar^n + bs^n$.

Hence, the right circulant matrix that we will dealing with, takes the form

$$C_R(\vec{h}) = \begin{pmatrix} h_0 & h_1 & h_2 & ... & h_{n-2} & h_{n-1} \\ h_{n-1} & h_0 & h_1 & ... & h_{n-3} & h_{n-2} \\ h_{n-2} & h_{n-1} & h_0 & ... & h_{n-4} & h_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_2 & h_3 & h_4 & ... & h_0 & h_1 \\ h_1 & h_2 & h_3 & ... & h_{n-1} & h_0 \end{pmatrix}$$

Our goal is to derive the explicit forms of the eigenvalues and determinant of this matrix.

II. Main Results

We first present the results regarding the eigenvalues of $C_R(\vec{h})$.

Theorem 2.1: The eigenvalues of $C_R(\vec{h})$ are given by

$$\Lambda_m = \frac{h_0 - h_n - [(as + br) - (ar^n s + brs^n)]\omega^{-m}}{(1 - r\omega^{-m})(1 - s\omega^{-m})}$$

where m=0, 1, ... , n-1 and $\omega = e^{2\pi i/n}$.

Proof:

Note that the eigenvalues of a right circulant matrix is just discrete Fourier transform of its first row. That is

$$\Lambda_m = \sum_{k=0}^{n-1} h_k \omega^{-mk}$$

where m=0, 1, ..., n-1 and $\omega = e^{2\pi i/n}$. This results to

$$\begin{aligned} &\sum_{k=0}^{n-1} [ar^k + bs^k]\omega^{-mk} \\ &= \frac{a(1-r^n)}{1 - r\omega^{-m}} + \frac{b(1-s^n)}{1 - s\omega^{-m}} \\ &= \frac{a(1-r^n)(1-s\omega^{-m}) + b(1-s^n)(1-r\omega^{-m})}{(1-r\omega^{-m})(1-s\omega^{-m})} \\ &= \frac{(a+b) - (ar^n + bs^n) - [(as+br) - (ar^n s + brs^n)]\omega^{-m}}{(1-r\omega^{-m})(1-s\omega^{-m})} \\ &= \frac{h_0 - h_n - [(as+br) - (ar^n s + brs^n)]\omega^{-m}}{(1-r\omega^{-m})(1-s\omega^{-m})} \end{aligned}$$

This completes the proof.

As consequences of Theorem 2.1, we have the following corollaries.

Corollary 2.2: If $a = \frac{1}{\sqrt{5}}$, $b = -\frac{1}{\sqrt{5}}$, $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{1-\sqrt{5}}{2}$ then

$$\Lambda_m = \frac{-F_n - (F_{n-1} - 1)\omega^{-m}}{1 - \omega^{-m} - \omega^{-2m}}$$

where F_n is the n^{th} Fibonacci number.

Corollary 2.3: If $a = b = 1$, $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{1-\sqrt{5}}{2}$ then

$$\Lambda_m = \frac{2 - L_n - (L_{n-1} + 1)\omega^{-m}}{1 - \omega^{-m} - \omega^{-2m}}$$

where L_n is the n^{th} Lucas number.

Corollary 2.4: If $a = \frac{1}{2\sqrt{2}}$, $b = -\frac{1}{2\sqrt{2}}$, $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$ then

$$\Lambda_m = \frac{-P_n - (P_{n-1} + 2)\omega^{-m}}{1 - 2\omega^{-m} - \omega^{-2m}}$$

where P_n is the n^{th} Pell number.

Corollary 2.5: If $a = b = 1$, $r = 1+\sqrt{2}$ and $s = 1-\sqrt{2}$ then

$$\Lambda_m = \frac{2 - Q_n - (Q_{n-1} + 2)\omega^{-m}}{1 - 2\omega^{-m} - \omega^{-2m}}$$

where Q_n is the n^{th} Pell-Lucas number.

Corollary 2.6: If $a = \frac{1}{3}$, $b = -\frac{1}{3}$, $r = 2$ and $s = -1$ then

$$\Lambda_m = \frac{-J_n - (2J_{n-1} - 1)\omega^{-m}}{1 - \omega^{-m} - 2\omega^{-2m}}$$

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where J_n is the n^{th} Jacobsthal number.

Corollary 2.7: If $a = b = 1$, $r = 2$ and $s = -1$ then

$$\Lambda_m = \frac{2 - K_n - (2K_{n-1} + 1)\omega^{-m}}{1 - \omega^{-m} - 2\omega^{-2m}}$$

where K_n is the n^{th} Jacobsthal-Lucas number.

Lastly, we present the results on the determinant of $C_R(\vec{h})$.

Theorem 2.8: The determinant of $C_R(\vec{h})$ is

$$|C_R(\vec{h})| = \frac{[h_0 - h_n]^n - [(as + br) - (ar^n s + brs^n)]^n}{1 - (r^n + s^n) + (rs)^n}$$

Proof:

$$|C_R(\vec{h})| = \prod_{m=0}^{n-1} \Lambda_m = \prod_{m=0}^{n-1} \frac{\Gamma}{(1 - r\omega^{-m})(1 - s\omega^{-m})}$$

where

$$\Gamma = (a + b) - (ar^n + bs^n) - [(as + br) - (ar^n s + brs^n)]\omega^{-m}$$

Note that for arbitrary x and y,

$$\prod_{m=0}^{n-1} (x - y\omega^{-m}) = x^n - y^n$$

Using this yields

$$|C_R(\vec{h})| = \frac{[h_0 - h_n]^n - [(as + br) - (ar^n s + brs^n)]^n}{1 - (r^n + s^n) + (rs)^n}$$

as desired.

As special cases of Theorem 2.8, we have the following corollaries.

Corollary 2.9: If $a = \frac{1}{\sqrt{5}}$, $b = -\frac{1}{\sqrt{5}}$, $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{1-\sqrt{5}}{2}$ then

$$|C_R(\vec{h})| = \frac{(-F_n)^n - (F_{n-1} - 1)^n}{1 - L_n + (-1)^n}$$

Corollary 2.10: If $a = b = 1$, $r = \frac{1+\sqrt{5}}{2}$ and $s = \frac{1-\sqrt{5}}{2}$ then

$$|C_R(\vec{h})| = \frac{(2 - L_n)^n - (L_{n-1} + 1)^n}{1 - L_n + (-1)^n}$$

Corollary 2.11: If $a = \frac{1}{2\sqrt{2}}$, $b = -\frac{1}{2\sqrt{2}}$, $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$ then

$$|C_R(\vec{h})| = \frac{(-P_n)^n - (P_{n-1} + 2)^n}{1 - Q_n + (-1)^n}$$

Corollary 2.12: If $a = b = 1$, $r = 1+\sqrt{2}$ and $s = 1-\sqrt{2}$ then

$$|C_R(\vec{h})| = \frac{(2 - Q_n)^n - (Q_{n-1} + 2)^n}{1 - Q_n + (-1)^n}$$

Corollary 2.13: If $a = \frac{1}{3}$, $b = -\frac{1}{3}$, $r = 2$ and $s = -1$ then

$$|C_R(\vec{h})| = \frac{(-J_n)^n - (2J_{n-1} - 1)^n}{1 - K_n + (-2)^n}$$

Corollary 2.14: If $a = b = 1$, $r = 2$ and $s = -1$ then

$$|C_R(\vec{h})| = \frac{(2 - K_n)^n - (2K_{n-1} + 1)^n}{1 - K_n + (-2)^n}$$

III. Conclusion

In summary, we have derived the eigenvalues and the determinant of a right circulant matrix whose entries are sum of the terms of two geometric sequences $\left\{ar^k\right\}_{k=0}^{+\infty}$ and $\left\{bs^k\right\}_{k=0}^{+\infty}$. It turns out that they are all dependent of the first terms a and b, the common ratios r and s, and the number n.

As special cases, we have also derived the eigenvalues and the determinant of the right circulant matrices whose entries are the Fibonacci numbers, the Lucas numbers, the Pell numbers, the Pell-Lucas numbers, the Jacobsthal numbers and the Jacobsthal-Lucas numbers.

References

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