# The $b$-Chromatic Number of Some Path Related Graphs

S. K. Vaidya and Rakhimol V. Isaac

***Abstract*—A $b$-coloring of a graph $G$ is a proper coloring with additional property that each color class contains a vertex that has a neighbor in all the other color classes. Here we investigate the $b$-chromatic number of some path related graphs.**

***Index Terms*—Coloring, proper coloring, $b$-coloring, $b$-vertex, $b$-chromatic number.**

***MSC 2010 Codes* - 05C15, 05C38, 05C76.**

## I. Introduction

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For any graph theoretic terminology and notations we refer to West [1]. A *proper k-coloring* of a graph $G$ is a function $c : V(G) \rightarrow \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The color class $c_i$ is the subset of vertices of $G$ that are assigned to color $i$. The chromatic number $\chi(G)$ is the minimum integer $k$ for which $G$ admits proper $k$-coloring. The concept of graph coloring is one of the potential areas of research in graph theory. Some variants of graph coloring are also introduced. Some of them are edge coloring, $a$-coloring, $b$-coloring etc. This work is focused on the $b$-coloring of graphs.

A proper $k$-coloring $c$ of a graph $G$ is a *b-coloring* if for every color class $c_i$, there is a vertex with color $i$ which has at least one neighbor in every other color classes. Such vertex is called a *b-vertex*. The *b-chromatic number* of a graph $G$, denoted by $\varphi(G)$, is the largest integer $k$ for which $G$ admits a $b$-coloring for $k$ colors and $G$ is called *b-colorable graph*.

The concept of $b$-coloring was introduced by Irving and Manlove [2]. In the same paper they investigated several results on this newly defined concept and proved that determining the $b$-chromatic number is NP-hard problem. The $b$-coloring of regular graphs is studied by Blidia *et al.* [3] while $b$-coloring of tight graphs is studied by Sales and Sampaio [4] and also by Havet *et al.* [5]. The discussion on the $b$-chromatic number of some power graphs is carried out by Effantin and Kheddouci [6]. The present work is aimed to investigate $b$-chromatic number of some path related graphs.

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## II. Main Results

**Proposition 2.1:**([2]) If $G$ admits a $b$-coloring with $m$ colors, $G$ must have at least $m$ vertices with degree at least $m-1$.

It is obvious that $\chi(G) \leq \varphi(G) \leq \Delta(G)+1$ , where $\Delta(G)$ is the maximum degree of $G$.

**Proposition 2.2:**([7]) If $K_n, P_n$ and $C_n$ are respectively the complete graph, path and cycle on $n$ vertices, then

1) $\varphi(K_n) = n$, for all $n$.
2) $\varphi(P_n) = \varphi(C_n) = 3$, for all $n \geq 5$.
3) $\varphi(P_2) = \varphi(P_3) = \varphi(P_4) = 2$.
4) $\varphi(C_3) = 3$ and $\varphi(C_4) = 2$.

**Definition 2.3:** The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbors of the corresponding vertex $u''$ in $G''$.

**Theorem 2.4:** $\varphi(D_2(P_n)) = \begin{cases} 2, & n = 2,3,4 \\ 3, & n = 5,6,7 \\ 4, & n = 8,9,10 \\ 5, & n \geq 11. \end{cases}$

**Proof:** Let $D_2(P_n)$ be the shadow graph of path $P_n$ with vertices $v_1, v_2, ....v_n$ in first copy of $P_n$ and $v'_1, v'_2, ...., v'_n$ in second copy of $P_n$. The four vertices $v_1, v_n, v'_1$ and $v'_n$ are of degree 2 and the remaining vertices are of degree 4. Also in $D_2(P_n)$ each $v_i$ is adjacent to the vertices $v_{i-1}, v'_{i-1}, v_{i+1}$ and $v'_{i+1}$ where $i = 2, 3, .., n-1$.

The proof is divided into several cases.

**Case 1:** When $n = 2$.

$|V(D_2(P_2))| = 4$ and $V(D_2(P_2)) = \{v_1, v_2, v'_1, v'_2\}$. By Proposition 2.2, $\varphi(D_2(P_2)) = 2$ as the graph $D_2(P_2)$ is isomorphic to $C_4$.

**Case 2:** When $n = 3$.

$|V(D_2(P_3))| = 6$ and $V(D_2(P_3)) = \{v_1, v_2, v_3, v'_1, v'_2, v'_3\}$. Also the graph $D_2(P_3)$ has four vertices of degree 2 and two vertices of degree 4. As $\Delta(D_2(P_3)) = 4$, $\varphi(D_2(P_3)) \leq 5$. If $\varphi(D_2(P_3)) = 5$ then $D_2(P_3)$ must have five vertices of degree 4, which is not possible, as we stated earlier that $D_2(P_3)$ has only two vertices of degree 4. Consequently, $\varphi(D_2(P_3) \neq 5$. If $\varphi(D_2(P_3)) = 4$ then $D_2(P_3)$ must have four vertices of degree 3, which is not possible, as $D_2(P_3)$ has no vertices of degree 3. Consequently, $\varphi(D_2(P_3)) \neq 4$. By Proposition 2.1, $\varphi(D_2(P_3)) \leq 3$ as $D_2(P_3)$ has four vertices of degree 2. Suppose $\varphi(D_2(P_3)) = 3$, then we color the vertices as $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 1$, $c(v'_1) = 1$, $c(v'_2) = 3$, $c(v'_3) = 1$. This gives $b$-vertices for the color classes $c_1$ and $c_2$. But there is no $b$-vertex for the color class $c_3$. Thus due to the adjacency of vertices in $D_2(P_3)$, any proper coloring using three colors is not a $b$-coloring. Clearly $\varphi(D_2(P_3)) \neq 3$.

Thus $\varphi(D_2(P_3)) = 2$. Consequently, we color the vertices as $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 1$, $c(v_1') = 1$, $c(v_2') = 2$, $c(v_3') = 1$. Then $v_1$ and $v_2$ are the $b$-vertices for the color classes $c_1$ and $c_2$ respectively.

**Case 3:** When $n = 4$.

$|V(D_2(P_4))| = 8$ and $V(D_2(P_4)) = \{v_1, v_2, v_3, v_4, v_1', v_2', v_3', v_4'\}$. Also the graph $D_2(P_4)$ has four vertices of degree 2 and four vertices of degree 4. As $\Delta(D_2(P_4)) = 4$, $\varphi(D_2(P_4)) \leq 5$. If $\varphi(D_2(P_4)) = 5$ then $D_2(P_4)$ must have five vertices of degree 4, which is not possible, as we stated earlier that $D_2(P_4)$ has only four vertices of degree 4. Consequently, $\varphi(D_2(P_4)) \neq 5$. If $\varphi(D_2(P_4)) = 4$ then $D_2(P_4)$ must have four vertices of degree 3, which is not possible, as $D_2(P_4)$ has no vertices of degree 3. Consequently, $\varphi(D_2(P_4)) \neq 4$. By Proposition 2.1, $\varphi(D_2(P_4)) \leq 3$ as $D_2(P_4)$ has four vertices of degree 2. Suppose $\varphi(D_2(P_4)) = 3$, then we color the vertices as $c(v_1) = 2$, $c(v_2) = 1$, $c(v_3) = 3$, $c(v_4) = 2$, $c(v_1') = 3$, $c(v_2') = 1$, $c(v_3') = 3$, $c(v_4') = 2$. This gives $b$-vertices for the color classes $c_1$ and $c_3$. But there is no $b$-vertex for the color class $c_2$. Thus due to the adjacency of vertices in $D_2(P_4)$, any proper coloring using three colors is not a $b$-coloring. Clearly $\varphi(D_2(P_4)) \neq 3$. Thus $\varphi(D_2(P_4)) = 2$. Consequently, we color the vertices as $c(v_1) = 2$, $c(v_2) = 1$, $c(v_3) = 2$, $c(v_4) = 1$, $c(v_1') = 2$, $c(v_2') = 1$, $c(v_3') = 2$, $c(v_4') = 1$. Then $v_1$ and $v_2$ are the $b$-vertices for the color classes $c_2$ and $c_1$ respectively.

**Case 4:** When $n = 5$.

$|V(D_2(P_5))| = 10$ and $V(D_2(P_5)) = \{v_1, v_2, v_3, v_4, v_5, v_1', v_2', v_3', v_4', v_5'\}$. Also the graph $D_2(P_5)$ has four vertices of degree 2 and six vertices of degree 4. As $\Delta(D_2(P_5)) = 4$, $\varphi(D_2(P_5)) \leq 5$. Due to the adjacency of vertices in $D_2(P_5)$, at most three $b$-vertices can be generated for any proper coloring. Thus $\varphi(D_2(P_5)) = 3$. Consequently, we color the vertices as $c(v_1) = c(v_1') = 1$, $c(v_2) = c(v_2') = 2$, $c(v_3) = c(v_3') = 3$, $c(v_4) = c(v_4') = 1$, $c(v_5) = c(v_5') = 1$, which is a $b$-coloring with the $b$-vertices $v_4, v_2$ and $v_3$ for the color classes $c_1, c_2$ and $c_3$ respectively.

**Case 5:** When $n = 6$.

$|V(D_2(P_6))| = 12$ and $V(D_2(P_6)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_1', v_2', v_3', v_4', v_5', v_6'\}$. Also the graph $D_2(P_6)$ has four vertices of degree 2 and eight vertices of degree 4. As $\Delta(D_2(P_6)) = 4$, $\varphi(D_2(P_6)) \leq 5$. Due to the adjacency of vertices in $D_2(P_6)$, at most three $b$-vertices can be generated for any proper coloring. Thus $\varphi(D_2(P_6)) = 3$. Consequently, we color the vertices as $c(v_1) = c(v_1') = 1$, $c(v_2) = c(v_2') = 2$, $c(v_3) = c(v_3') = 3$, $c(v_4) = c(v_4') = 1$, $c(v_5) = c(v_5') = 2$, $c(v_6) = c(v_6') = 3$, which is $b$-coloring with the $b$-vertices $v_4, v_2$ and $v_3$ for the color classes $c_1$, $c_2$ and $c_3$ respectively.

**Case 6:** When $n = 7$.

$|V(D_2(P_7))| = 14$ and $V(D_2(P_7)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1', v_2', v_3', v_4', v_5', v_6', v_7'\}$. Also the graph $D_2(P_7)$ has four vertices of degree 2 and ten vertices of degree 4. As $\Delta(D_2(P_7)) = 4$, $\varphi(D_2(P_7)) \leq 5$. Due to the adjacency of vertices in $D_2(P_7)$, at most three $b$-vertices can be generated for any proper coloring. Thus $\varphi(D_2(P_7)) = 3$. Consequently, we color the vertices as $c(v_1) = c(v_1') = 1$, $c(v_2) = c(v_2') = 2$, $c(v_3) = c(v_3') = 3$, $c(v_4) = c(v_4') = 1$, $c(v_5) = c(v_5') = 2$, $c(v_6) = c(v_6') = 3$, $c(v_7) = c(v_7') = 1$, which is $b$-coloring with the $b$-vertices $v_4, v_2$ and $v_3$ for the color classes $c_1$, $c_2$ and $c_3$ respectively.

**Case 7:** When $n = 8$.

$|V(D_2(P_8))| = 16$ and $V(D_2(P_8)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8'\}$. Also the graph $D_2(P_8)$ has four vertices of degree 2 and twelve vertices of degree 4. As $\Delta(D_2(P_8)) = 4$, $\varphi(D_2(P_8)) \leq 5$. Due to the adjacency of vertices in $D_2(P_8)$, at most four $b$-vertices can be generated for any proper coloring. Thus $\varphi(D_2(P_8)) = 4$. Consequently, we color the vertices as $c(v_1) = 1$, $c(v_1') = 4$, $c(v_2) = 2$, $c(v_2') = 2$, $c(v_3) = 3$, $c(v_3') = 3$, $c(v_4) = 1$, $c(v_4') = 4$, $c(v_5) = 3$, $c(v_5') = 2$, $c(v_6) = 1$, $c(v_6') = 1$, $c(v_7) = 4$, $c(v_7') = 4$, $c(v_8) = 3$, $c(v_8') = 2$, which is a $b$-coloring with the $b$-vertices $v_6', v_2, v_3$ and $v_7'$ for the color classes $c_1$, $c_2$, $c_3$ and $c_4$ respectively.

**Case 8:** When $n = 9$.

$|V(D_2(P_9))| = 18$ and $V(D_2(P_9)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8', v_9'\}$. Also the graph $D_2(P_9)$ has four vertices of degree 2 and fourteen vertices of degree 4. As $\Delta(D_2(P_9)) = 4$, $\varphi(D_2(P_9)) \leq 5$. Due to the adjacency of vertices in $D_2(P_9)$, at most four $b$-vertices can be generated for any proper coloring. Thus $\varphi(D_2(P_9)) = 4$. Consequently, we color the vertices as $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(v_4) = 1$, $c(v_5) = 3$, $c(v_6) = 1$, $c(v_7) = 4$, $c(v_8) = 3$, $c(v_9) = 4$, $c(v_1') = 4$, $c(v_2') = 2$, $c(v_3') = 3$, $c(v_4') = 4$, $c(v_5') = 2$, $c(v_6') = 1$, $c(v_7') = 4$, $c(v_8') = 2$, $c(v_9') = 4$, which is a $b$-coloring with the $b$-vertices $v_6', v_2, v_3$ and $v_7'$ for the color classes $c_1$, $c_2$, $c_3$ and $c_4$ respectively.

**Case 9:** When $n = 10$.

$|V(D_2(P_{10}))| = 20$ and $V(D_2(P_{10})) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8', v_9', v_{10}'\}$. Also the graph $D_2(P_{10})$ has four vertices of degree 2 and sixteen vertices of degree 4. As $\Delta(D_2(P_{10})) = 4$, $\varphi(D_2(P_{10})) \leq 5$. Due to the adjacency of vertices in $D_2(P_{10})$, at most four $b$-vertices can be generated for any proper coloring. Thus $\varphi(D_2(P_{10})) = 4$. Consequently, we color the vertices as $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(v_4) = 1$, $c(v_5) = 3$, $c(v_6) = 1$, $c(v_7) = 1$, $c(v_8) = 3$, $c(v_9) = 4$, $c(v_{10}) = 2$, $c(v_1') = 4$, $c(v_2') = 2$, $c(v_3') = 3$, $c(v_4') = 4$, $c(v_5') = 1$, $c(v_6') = 1$, $c(v_7') = 4$, $c(v_8') = 2$, $c(v_9') = 4$, $c(v_{10}') = 2$, which is a $b$-coloring with the $b$-vertices $v_6', v_2, v_3$ and $v_7'$ for the color classes $c_1$, $c_2$, $c_3$ and $c_4$ respectively.

**Case 10:** When $n = 11$.

$|V(D_2(P_{11}))| = 22$ and $V(D_2(P_{11})) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8', v_9', v_{10}', v_{11}'\}$. Also the graph $D_2(P_{11})$ has four vertices of degree 2 and eighteen vertices of degree 4. As $\Delta(D_2(P_{11})) = 4$, $\varphi(D_2(P_{11})) \leq 5$. Due to the adjacency of vertices in $D_2(P_{11})$, at most five $b$-vertices can be generated for any proper coloring. Thus $\varphi(D_2(P_{11})) = 5$. Consequently, we color the vertices as $c(v_1) = 2$, $c(v_2) = 1$, $c(v_3) = 3$, $c(v_4) = 2$, $c(v_5) = 4$, $c(v_6) = 3$, $c(v_7) = 5$, $c(v_8) = 4$, $c(v_9) = 3$, $c(v_{10}) = 5$, $c(v_{11}) = 2$, $c(v_1') = 4$, $c(v_2') = 1$, $c(v_3') = 5$, $c(v_4') = 2$, $c(v_5') = 1$, $c(v_6') = 3$, $c(v_7') = 2$, $c(v_8') = 4$, $c(v_9') = 1$, $c(v_{10}') = 5$, $c(v_{11}') = 4$, which is a $b$-coloring with the $b$-vertices $v_2, v_4, v_6, v_8$ and $v_{10}$ for the color classes $c_1$, $c_2$, $c_3$, $c_4$ and $c_5$ respectively.

**Case 11:** When $n > 11$.

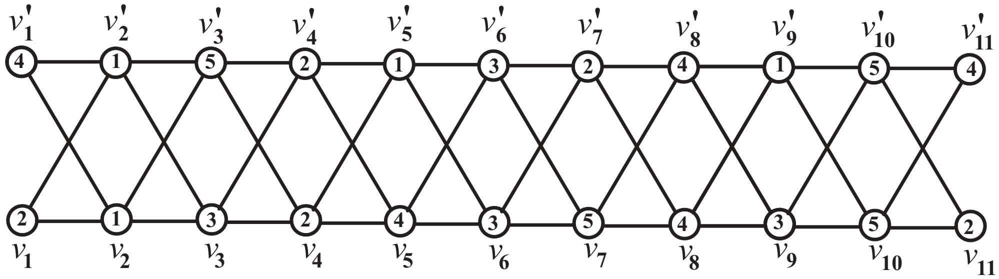


*Figure 1*

$|V(D_2(P_n))|$ =$2n$. We color the vertices $v_1, v_2, ...., v_{11}, v'_1, v'_2, \dots, v'_{11}$ as in $D_2(P_{11})$ and for the remaining vertices assign the colors as

$$\begin{array}{lll} c(v_{2i}) = & c(v'_{2i}) = & 1 \\ c(v_{2i+1}) = & c(v'_{2i+1}) = & 2 \quad ; i = 6, 7, 8, \dots \end{array}$$

The $b$-vertices are same as the $b$-vertices in the case of $D_2(P_{11})$. Thus $\varphi(D_2(P_n)) = 5$, for all $n > 11$. Hence the theorem.

**Illustration 2.5:** The graph $D_2(P_{11})$ and its $b$-coloring is shown in *Figure 1*.

**Definition 2.6:** The splitting graph of a graph $G$, $S'(G)$, is obtained by adding new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of $v$ and $v'$ respectively.

**Theorem 2.7:** $\varphi(S'(P_n)) = \begin{cases} 2, & n = 2, 3, 4 \\ 3, & n = 5 \\ 4, & n = 6, 7 \\ 5, & n \geq 8. \end{cases}$

**Proof:** Let $v_1, v_2, .., v_n$ be the vertices of path $P_n$ and $v'_1, v'_2, ..., v'_n$ be the newly added vertices corresponding to the vertices $v_1, v_2, .., v_n$ to form $S'(P_n)$. In $S'(P_n)$, $v_1$ is adjacent to $v_2$ and $v'_2$, $v_n$ is adjacent to $v_{n-1}$ and $v'_{n-1}$ and each $v_i$ is adjacent to $v_{i-1}, v_{i+1}, v'_{i-1}$ and $v'_{i+1}$ where $i = 2, 3, ..n - 1$.

The proof is divided into following cases.

<u>**Case 1:** When $n = 2$.</u>

$|V(S'(P_2))| = 4$ and $V(S'(P_2)) = \{v_1, v_2, v'_1, v'_2\}$. By Proposition 2.2, $\varphi(S'(P_2)) = 2$ as the graph $S'(P_2)$ is isomorphic to $P_4$.

<u>**Case 2:** When $n = 3$.</u>

$|V(S'(P_3))| = 6$ and $V(S'(P_3)) = \{v_1, v_2, v_3, v'_1, v'_2, v'_3\}$. Also the graph $S'(P_3)$ has two vertices of degree 1, three vertices of degree 2 and one vertex of degree 4. As $\Delta(S'(P_3)) = 4$, $\varphi(S'(P_3)) \leq 5$. If $\varphi(S'(P_3)) = 5$, then $S'(P_3)$ must have five vertices of degree 4 which is not possible, as we stated earlier that $S'(P_3)$ has only one vertex of degree 4. Consequently, $\varphi(S'(P_3)) \neq 5$. If $\varphi(S'(P_3)) = 4$, then $S'(P_3)$ must have four vertices of degree 3 which is not possible, as $S'(P_3)$ has no vertex of degree 3. Consequently, $\varphi(S'(P_3)) \neq 4$. Therefore $\varphi(S'(P_3))$ can be either 3 or 2. But due to the adjacency of vertices in $S'(P_3)$, at most two $b$-vertices can be generated for any proper coloring. Thus $\varphi(S'(P_3)) = 2$. Consequently, we color the vertices as $c(v_1) = c(v'_1) = 1$, $c(v_2) = c(v'_2) = 2$, $c(v_3) = c(v'_3) = 1$, which is a $b$-coloring with $b$-vertices $v_1$ and $v_2$ for the color classes $c_1$ and $c_2$ respectively.

<u>**Case 3:** When $n = 4$.</u>

$|V(S'(P_4))| = 8$ and $V(S'(P_4)) = \{v_1, v_2, v_3, v_4, v'_1, v'_2, v'_3, v'_4\}$. Also the graph $S'(P_4)$ has two vertices of degree 1, four vertices of degree 2 and two vertices of degree 4. As $\Delta(S'(P_4)) = 4$, $\varphi(S'(P_4)) \leq 5$. If $\varphi(S'(P_4)) = 5$ then $S'(P_4)$ must have five vertices of degree 4 which is not possible, as we stated earlier that $S'(P_4)$ has only two vertices of degree 4. Consequently, $\varphi(S'(P_4)) \neq 5$. If $\varphi(S'(P_4)) = 4$ then $S'(P_4)$ must have four vertices of degree 3 which is not possible, as $S'(P_4)$ has no vertex of degree 3. Consequently, $\varphi(S'(P_4)) \neq 4$. Therefore $\varphi(S'(P_4))$ can be either 3 or 2. But due to the adjacency of vertices in $S'(P_4)$, at most two $b$-vertices can be generated for any proper coloring. Thus $\varphi(S'(P_4)) = 2$. Consequently, we color the vertices as $c(v_1) = c(v'_1) = 1$, $c(v_2) = c(v'_2) = 2$, $c(v_3) = c(v'_3) = 3$, $c(v_4) = c(v'_4) = 2$, which is a $b$-coloring with $b$-vertices $v_1$ and $v_2$ for the color classes $c_1$ and $c_2$ respectively.

<u>**Case 4:** When $n = 5$.</u>

$|V(S'(P_5))| = 10$ and $V(S'(P_5)) = \{v_1, v_2, v_3, v_4, v_5, v'_1, v'_2, v'_3, v'_4, v'_5\}$. Also the graph $S'(P_5)$ has two vertices of degree 1, five vertices of degree 2 and three vertices of degree 4. As $\Delta(S'(P_5)) = 4$, $\varphi(S'(P_5)) \leq 5$. If $\varphi(S'(P_5)) = 5$ then $S'(P_5)$ must have five vertices of degree 4 which is not possible, as we stated earlier that $S'(P_5)$ has only three vertices of degree 4. Consequently, $\varphi(S'(P_5)) \neq 5$. If $\varphi(S'(P_5)) = 4$, then $S'(P_5)$ must have four vertices of degree 3 which is not possible, as $S'(P_5)$ has no vertex of degree 3. Consequently, $\varphi(S'(P_5)) \neq 4$. Therefore $\varphi(S'(P_5))$ can be either 3 or 2. Due to the adjacency of vertices in $S'(P_5)$, at most three $b$-vertices can be generated for any proper coloring. Thus $\varphi(S'(P_5)) = 3$. Consequently, we color the vertices as $c(v_1) = c(v'_1) = 1$, $c(v_2) = c(v'_2) = 2$, $c(v_3) = c(v'_3) = 3$, $c(v_4) = c(v'_4) = 1$, $c(v_5) = c(v'_5) = 2$, which is a $b$-coloring with $b$-vertices $v_4, v_2$ and $v_3$ for the color classes $c_1, c_2$ and $c_1$ respectively.

<u>**Case 5:** When $n = 6$.</u>

$|V(S'(P_6))| = 12$ and $V(S'(P_6))$= $\{v_1, v_2, v_3, v_4, v_5, v_6, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6\}$. Also the graph $S'(P_6)$ has two vertices of degree 1, six vertices of degree 2 and four vertices of degree 4. As $\Delta(S'(P_6)) = 4$, $\varphi(S'(P_6)) \leq 5$. If $\varphi(S'(P_6)) = 5$ then $S'(P_6)$ must have five vertices of degree 4 which is not possible, as we stated earlier that $S'(P_6)$ has only four vertices of degree 4. Consequently, $\varphi(S'(P_6)) \neq 5$. Therefore $\varphi(S'(P_6))$ can be either 4, 3 or 2. Due to the adjacency of vertices in $S'(P_6)$, at most four $b$-vertices can be generated for any proper coloring. Thus $\varphi(S'(P_6)) = 4$. Consequently, we color the vertices as $c(v_1) = 3$, $c(v_2) = 1$, $c(v_3) = 2$, $c(v_4) = 2$, $c(v_5) = 4$, $c(v_6) = 2$, $c(v'_1) = 4$, $c(v'_2) = 4$,

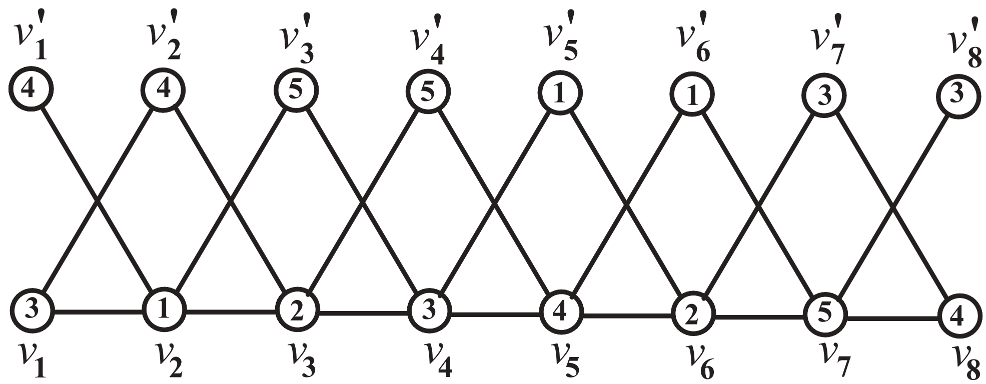


*Figure 2*

$c(v'_3) = 2$, $c(v'_4) = 2$, $c(v'_5) = 1$, $c(v'_6) = 1$, which is a $b$-coloring with $b$-vertices $v_2, v_3, v_4$ and $v_5$ for the color classes $c_1, c_2, c_3$ and $c_4$ respectively.

**Case 6:** When $n = 7$.

$|V(S'(P_7))| = 14$ and $V(S'(P_7)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7\}$. Also the graph $S'(P_7)$ has two vertices of degree 1, seven vertices of degree 2 and five vertices of degree 4. As $\Delta(S'(P_7)) = 4$, $\varphi(S'(P_7)) \le 5$. But due to the adjacency of vertices in $S'(P_7)$, at most four $b$-vertices can be generated for any proper coloring. Thus $\varphi(S'(P_7)) = 4$. Consequently, we color the vertices as $c(v_1) = 4$, $c(v_2) = 1$, $c(v_3) = 2$, $c(v_4) = 3$, $c(v_5) = 1$, $c(v_6) = 4$, $c(v_7) = 2$, $c(v'_1) = 3$, $c(v'_2) = 3$, $c(v'_3) = 4$, $c(v'_4) = 4$, $c(v'_5) = 1$, $c(v'_6) = 4$, $c(v'_7) = 3$, which is a $b$-coloring with $b$-vertices $v_2, v_3, v_4$ and $v_6$ for the color classes $c_1, c_2, c_3$ and $c_4$ respectively.

**Case 7:** When $n = 8$.

$|V(S'(P_8))| = 16$ and $V(S'(P_8)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8\}$. Also the graph $S'(P_8)$ has two vertices of degree 1, eight vertices of degree 2 and six vertices of degree 4. As $\Delta(S'(P_8)) = 4$, $\varphi(S'(P_8)) \le 5$. Due to the adjacency of vertices in $S'(P_8)$, at most five $b$-vertices can be generated for any proper coloring. Thus $\varphi(S'(P_8)) = 5$. Consequently, we color the vertices as $c(v_1) = 3$, $c(v_2) = 1$, $c(v_3) = 2$, $c(v_4) = 3$, $c(v_5) = 4$, $c(v_6) = 2$, $c(v_7) = 5$, $c(v_8) = 4$, $c(v'_1) = 4$, $c(v'_2) = 4$, $c(v'_3) = 4$, $c(v'_4) = 5$, $c(v'_5) = 1$, $c(v'_6) = 1$, $c(v'_7) = 3$, $c(v'_8) = 3$, which is a $b$-coloring with $b$-vertices $v_2, v_3, v_4, v_5$ and $v_7$ for the color classes $c_1, c_2, c_3, c_4$ and $c_5$ respectively.

**Case 8:** When $n > 8$.

$|V(S'(P_n))| = 2n$. We color the vertices $v_1, v_2, ..., v_8, v'_1, v'_2, ..., v'_8$ as in $S'(P_8)$ and for the remaining vertices assign the colors as

$$\begin{array}{llll} c(v_{2i+1}) = & c(v'_{2i+1}) = & 1; & i = 4, 5, 6, ... \\ c(v_{2i}) = & c(v'_{2i}) = & 2; & i = 5, 6, 7, 8, ... \end{array}$$

The $b$-vertices are same as the $b$-vertices in the case of $S'(P_8)$. Thus $\varphi(S'(P_n)) = 5$; for all $n > 8$. Hence the theorem.

**Illustration 2.8:**The graph $S'(P_8)$ and its $b$-coloring is shown in *Figure 2*.

**Definition 2.9:** The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident on it.

**Remark 2.10:** As reported in Vijayalakshmi *et al.* [8], $\varphi(M(P_n)) = n$ which is incorrect as we have the following theorem.

**Theorem 2.11:** $\varphi(M(P_n)) = \begin{cases} 2, & n = 2 \\ 3, & n = 3, 4 \\ 4, & n = 5, 6, 7 \\ 5, & n \ge 8. \end{cases}$

**Proof:** Let $v_1, v_2, .., v_n$ be the vertices and $e_1, e_2, ..., e_{n-1}$ be the edges of path $P_n$. $M(P_n)$ is the middle graph of $P_n$ with vertices $v_1, v_2, ...., v_{n-1}, v_n, e_1, e_2, ...., e_{n-1}$ such that $e_1$ is adjacent to $v_1, v_2$ and $e_2$, $e_{n-1}$ is adjacent to $v_{n-1}, v_n$ and $e_{n-2}$ and $e_i$ is adjacent to $v_i, v_{i+1}, e_{i-1}$ and $e_{i+1}; i = 2, 3, \ldots, n-2$

The proof is divided into following cases.

**Case 1**: When $n = 2$.

$|V(M(P_2))| = 3$ and $V(M(P_2))$ =$\{v_1, e_1, v_2\}$. By Proposition 2.2, $\varphi(M(P_2)) = 2$ as the graph $M(P_2)$ is isomorphic to $P_3$.

**Case 2**: When $n = 3$.

$|V(M(P_3))| = 5$ and $V(M(P_3))$ =$\{v_1, e_1, v_2, e_2, v_3\}$. Also the graph $M(P_3)$ has two vertices of degree 1, one vertex of degree 2 and two vertices of degree 3. Since $M(P_3)$ contains a $K_3$, $\varphi(M(P_3)) \ge 3$. As $\Delta(M(P_3)) = 3$, $\varphi(M(P_3)) \le 4$. If $\varphi(M(P_3)) = 4$ then $M(P_3)$ must have four vertices of degree 3, which is not possible as we stated earlier that $M(P_3)$ has only two vertices of degree 3. Consequently, $\varphi(M(P_3)) \ne 4$. Thus $\varphi(M(P_3)) = 3$ and we color the vertices as $c(e_1) = 1$, $c(e_2) = 3$, $c(v_1) = c(v_2) = c(v_3) = 2$, which is a $b$-coloring with $b$-vertices $e_1, v_2$ and $e_2$ for the color classes $c_1, c_2$ and $c_3$ respectively.

**Case 3**: When $n = 4$.

$|V(M(P_4))| = 7$ and $V(M(P_4))$ =$\{v_1, e_1, v_2, e_2, v_3, e_3, v_4\}$. Also the graph $M(P_4)$ has two vertices of degree 1, two vertices of degree 2, two vertices of degree 3 and one vertex of degree 4. Since $M(P_4)$ contains a $K_3$, $\varphi(M(P_4)) \ge 3$. As $\Delta(M(P_4)) = 4$, $\varphi(M(P_4)) \le 5$. Thus $3 \le \varphi(M(P_4)) \le 5$. If $\varphi(M(P_4)) = 5$ then $M(P_4)$ must have five vertices of degree 4 which is not possible as we stated earlier that $M(P_4)$ has only one vertex of degree 4. Consequently, $\varphi(M(P_4)) \ne 5$. If $\varphi(M(P_4)) = 4$ then $M(P_4)$ must have four vertices of degree

3 which is not possible as $M(P_4)$ has only two vertices of degree 3. Consequently, $\varphi(M(P_4)) \neq 4$. Thus $\varphi(M(P_4)) = 3$. Consequently, we color the vertices as $c(v_1) = 2$, $c(v_2) = 2$, $c(v_3) = 2$, $c(v_4) = 2$, $c(e_1) = 1$, $c(e_2) = 3$, $c(e_3) = 1$, which is a $b$-coloring with $b$-vertices $e_1, e_2$ and $e_3$ for the color classes $c_1, c_2$ and $c_3$ respectively.

**Case 4**: When $n = 5$.

$|V(M(P_5))| = 9$ and $V(M(P_5))$ $=\{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5\}$. Also the graph $M(P_5)$ has two vertices of degree 1, three vertices of degree 2, two vertices of degree 3 and two vertices of degree 4. Since $M(P_5)$ contains a $K_3$, $\varphi(M(P_5)) \geq 3$. As $\Delta(M(P_5) = 4$, $\varphi(M(P_5)) \leq 5$. Thus $3 \leq \varphi(M(P_5)) \leq 5$. If $\varphi(M(P_5)) = 5$ then $M(P_5)$ must have five vertices of degree 4 which is not possible as we stated earlier that $M(P_5)$ has only two vertices of degree 4. Consequently, $\varphi(M(P_5)) \neq 5$. Suppose $\varphi(M(P_5)) = 4$, we color the vertices as $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 1$, $c(v_4) = 1$, $c(v_5) = 4$, $c(e_1) = 1$, $c(e_2) = 4$, $c(e_3) = 2$, $c(e_4) = 3$, which is a $b$-coloring with $b$-vertices $e_1, e_3, e_4$ and $e_2$ for the color classes $c_1, c_2, c_3$ and $c_4$ respectively. Thus $\varphi(M(P_5)) = 4$.

**Case 5**: When $n = 6$.

$|V(M(P_6))| = 11$ and $V(M(P_6)) = \{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6\}$. Also the graph $M(P_6)$ has two vertices of degree 1, four vertices of degree 2, two vertices of degree 3 and three vertices of degree 4. Since $M(P_6)$ contains a $K_3$, $\varphi(M(P_6)) \geq 3$. As $\Delta(M(P_6) = 4$, $\varphi(M(P_6)) \leq 5$. Thus $3 \leq \varphi(M(P_6)) \leq 5$. If $\varphi(M(P_6)) = 5$ then $M(P_6)$ must have five vertices of degree 4 which is not possible as we stated earlier that $M(P_6)$ has only three vertices of degree 4. Consequently, $\varphi(M(P_6)) \neq 5$. Suppose $\varphi(M(P_6)) = 4$, we color the vertices as $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 1$, $c(v_4) = 1$, $c(v_5) = 4$, $c(v_6) = 2$, $c(e_1) = 1, c(e_2) = 4$, $c(e_3) = 2$, $c(e_4) = 3$, $c(e_5) = 1$, which is a $b$-coloring with $b$-vertices $e_1, e_3, e_4$ and $e_2$ for the color classes $c_1, c_2, c_3$ and $c_4$ respectively. Thus $\varphi(M(P_6)) = 4$.

**Case 6**: When $n = 7$.

$|V(M(P_7))| = 13$ and $V(M(P_7))$ $=\{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_7\}$. Also the graph $M(P_7)$ has two vertices of degree 1, five vertices of degree 2, two vertices of degree 3 and four vertices of degree 4. Since $M(P_7)$ contains a $K_3$, $\varphi(M(P_7)) \geq 3$. As $\Delta(M(P_7) = 4$, $\varphi(M(P_7)) \leq 5$. Thus $3 \leq \varphi(M(P_7)) \leq 5$. If $\varphi(M(P_7)) = 5$ then $M(P_7)$ must have five vertices of degree 4 which is not possible as we stated earlier that $M(P_7)$ has only four vertices of degree 4. Consequently, $\varphi(M(P_7)) \neq 5$. Suppose $\varphi(M(P_7)) = 4$, we color the vertices as $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 1$, $c(v_4) = 1$, $c(v_5) = 4$, $c(v_6) = 2$, $c(v_7) = 4$, $c(e_1) = 1$, $c(e_2) = 4$, $c(e_3) = 2$, $c(e_4) = 3$, $c(e_5) = 1$, $c(e_6) = 3$, which is a $b$-coloring with $b$-vertices $e_1, e_3, e_4$ and $e_2$ for the color classes $c_1, c_2, c_3$ and $c_4$ respectively. Thus $\varphi(M(P_7)) = 4$.

**Case 7**: When $n = 8$.

$|V(M(P_8))| = 15$ and $V(M(P_8))$ $=\{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_7, e_7, v_8\}$. Also $M(P_8)$ has two vertices of degree 1, six vertices of degree 2, two vertices of degree 3 and five vertices of degree 4. Since $M(P_8)$ contains a $K_3$, $\varphi(M(P_8)) \geq 3$. As $\Delta(M(P_8) = 4$, $\varphi(M(P_8)) \leq 5$. Thus $3 \leq \varphi(M(P_8)) \leq 5$. As $M(P_8)$ has five vertices of degree 4, we claim that $\varphi(M(P_8)) = 5$. Then we color the vertices as $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(v_4) = 5$, $c(v_5) = 1$, $c(v_6) = 2$, $c(v_7) = 1$, $c(v_8) = 4$, $c(e_1) = 5$, $c(e_2) = 1$, $c(e_3) = 2$, $c(e_4) = 3$, $c(e_5) = 4$, $c(e_6) = 5$, $c(e_7) = 3$, which is a $b$-coloring with $b$-vertices $e_2, e_3, e_4, e_5$ and $e_6$ for the color classes $c_1, c_2, c_3, c_4$ and $c_5$ respectively. Thus $\varphi(M(P_8)) = 5$.

**Case 8**: When $n > 8$.

$|V(M(P_n))| = 2n - 1$ . We color the vertices $v_1, v_2, ..., v_8$, $e_1, e_2, ..., e_8$ as in $M(P_8)$ and for the remaining vertices assign the colors as

$$\begin{array}{lll} c(v_{2i}) = & c(v_{2i+1}) = & 4, \\ c(e_{2i}) = & 2, & \\ c(e_{2i+1}) = & 3; & i = 4, 5, 6, .. \end{array}$$

The $b$-vertices are same as the $b$-vertices in the case of $M(P_8)$. Thus $\varphi(M(P_n)) = 5$, for all $n > 8$. Hence the theorem.

**Illustration 2.12:** The graph $M(P_8)$ and its $b$-coloring is shown in *Figure 3*.

## III. Concluding Remarks

The $b$-chromatic number of $P_n$ is known while we investigate the $b$-chromatic numbers for $D_2(P_n), S'(P_n)$ and $M(P_n)$. The present work throws some light on the $b$-coloring of larger graphs obtained by means of some graph operations on standard graphs.

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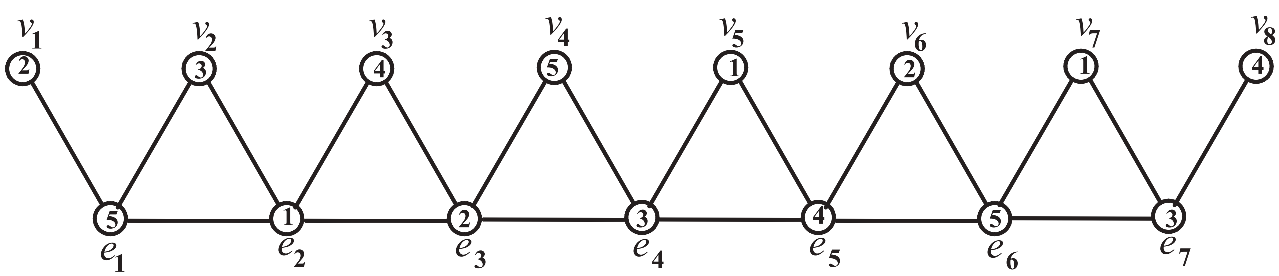


*Figure 3*