The b-coloring and Hajós Sum of Two Graphs

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***Abstract*—A b-coloring of a graph is a variant of proper coloring in which each color class has a vertex which has at least one neighbor in all the other color classes. The minimum number k for which a graph admits a b-coloring is called the b-chromatic number. We investigate b-chromatic number for the graphs obtained as a Hajós sum of two graphs.**

***Index Terms*—graph coloring, b-coloring, Hajós sum.**

***MSC 2010 Codes* – 05C15; 05C76.**

I. Introduction

We begin with finite, connected and undirected graphs without loops and multiple edges. For any undefined term we refer to Clark and Holton [1]. A *proper k-coloring* of a graph G is a function $c : V(G) \to \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The color class $c(i)$ is the subset of vertices of G with color i. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k-coloring. A vertex that has at least one neighbor in each of the other color classes is called a b-vertex. A b-coloring of a graph G is a proper coloring of G in which each color class has a b-vertex. The concept of b-coloring was introduced in 1999 by Irving and Manlove [2]. The b-chromatic number, $\phi(G)$, of G is the largest integer k such that G has a b-coloring using k colors.

The b-chromatic number for path related graphs is studied by Vaidya and Rakhimol [3]. The same authors have also investigated the b-chromatic numbers of the degree splitting graphs of path, shell and gear graph in [4]. The b-chromatic numbers of one point union of cycles, path union of cycles and t-ply is discussed in [5] while the b-coloring of shell, gear and generalized web graph is obtained in [6]. The discussions on b-chromatic number of some wheel and cycle related graphs are reported in Vaidya and Shukla [7, 8]. The present work is focused on b-coloring of graphs which are obtained from Hajós sum of two graphs. □

II. Some Preliminary Results

In this section, some definitions and preliminary results are given which will serve as prerequisites.

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Definition 2.1 [2] The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m - 1$. □

Proposition 2.2 [1] For any graph G, $\chi(G) \geq 3$ if and only if G has an odd cycle. □

Proposition 2.3 [9] $\chi(G) \leq \varphi(G) \leq m(G)$. □

Proposition 2.4 [10] If P_n, W_n and K_n are respectively path, wheel and complete graph with n vertices, then

1) $\varphi(P_n) = 2$, if $1 < n < 5$ and $\varphi(P_n) = 3$, if $n \geq 5$.
2) $\chi(W_n) = 3$, if n is even and $\chi(W_n) = 4$, if n is odd.
3) $\chi(K_n) = \varphi(K_n) = n$.
4) $\varphi(W_n) = 3$, if $n = 4$ and $\varphi(W_n) = 4$, if $n \neq 4$. □

III. Main Results

Definition 3.1 [11, 12] Let G_1 and G_2 be two graphs. Let uu' be an edge in G_1 and vv' be an edge in G_2. The Hajós sum of the graphs G_1 and G_2, denoted by $H(G_1, G_2)$, is the graph obtained by identifying u and v by a new vertex w, deleting the edges uu' and vv' and adding an edge $u'v'$.
In the graph $H(G_1, G_2)$, $d(u')$ and $d(v')$ remains unchanged while $d(w) = d(u) + d(v) - 2$.

Example 3.2 K_5 and C_4 are shown in *Figure 1* while their Hajós sum is shown in *Figure 2*.

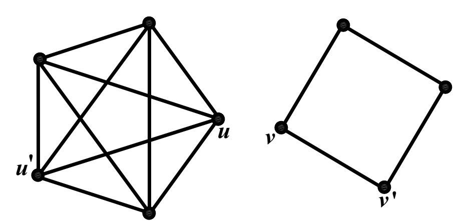

Figure 1

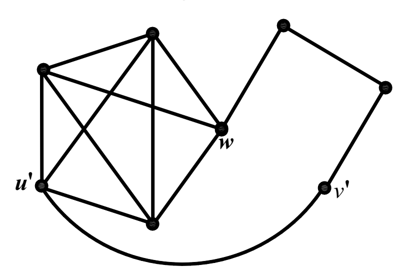

Figure 2

Theorem 3.3

$$\varphi(H(K_2, K_m)) = m - 1 \; ; m \neq 2$$
$$\varphi(H(K_n, K_m)) = \max\{m, n\}$$

Proof: Consider the complete graphs K_n and K_m. Denote the vertices of K_n as $u_1, u_2, \ldots, u_n$ and the vertices of K_m as $v_1, v_2, \ldots, v_m$. Without loss of generality we assume that $n < m$.

Case 1: The graph $H(K_2, K_m)$ is obtained by identifying the vertices u_1 and v_p as w, deleting the edges u_1u_2 and v_pv_q and adding the edge u_2v_q. Obviously, $d(u_2) = 1$ and $d(w) = m-2$. As v_j is any arbitrary vertex in K_m, $d(v_j) = m-1$; $1 \leq j \leq m$; $j \neq p$ in $H(K_2, K_m)$. Thus there is no b- vertex for all the m color classes of K_m. Hence $\varphi(H(K_2, K_m)) \leq m-1$. By assigning the color j $(1 \leq j \leq m)$ to the vertices v_j such that $d(v_j) = m-1$ and assign a proper coloring to the vertices w and u_2 using the colors from $\{1, 2, \ldots, m-1\}$, there will be at least $m-1$ vertices of degree $m-1$. Hence $\varphi(H(K_2, K_m)) = m-1$.

Case 2: Consider the edges u_ku_l in K_n and v_pv_q in K_m. The graph $H(K_n, K_m)$ is obtained by identifying u_k and v_p as w, deleting the edges u_ku_l and v_pv_q and adding an edge u_lv_q. As there are m vertices of degree at least $m-1$, $\varphi(H(K_n, K_m) \leq m$. As $n < m$, there is at least one color which is assigned to the vertices of K_m and not to that of K_n. By assigning the colors $c(v_j) = j$; $1 \leq j \leq m$, $c(u_l) = c(w) = p$; $1 \leq p \leq m$ and assign the color q $(1 \leq q \leq m)$ to any vertex $u_i; i \neq l; 1 \leq i \leq n$, adjacent to w, we get a b-coloring for $H(K_n, K_m)$ and $\varphi(H(K_n, K_m)) = \varphi(K_m) = m = max\{m, n\}$. □

Theorem 3.4

$$\varphi(H(K_2, C_m)) = \begin{cases} 2, & m = 3 \\ 3, & m \neq 3. \end{cases}$$
$$\varphi(H(K_n, C_m)) = n.$$

Proof: Consider the complete graph K_n and cycle C_m. Denote the vertices of K_n as $u_1, u_2, \ldots, u_n$ and the vertices of C_m as $v_1, v_2, \ldots, v_m$.

Case 1: The Hajós sum of K_2 and C_m, $H(K_2, C_m)$, gives a path P_{m+1} and hence,
$$\varphi(H(K_2, C_m)) = \varphi(P_{m+1}) = \begin{cases} 2, & m = 3 \\ 3, & m \neq 3. \end{cases}$$

Case 2: Consider the edges u_ku_l in K_n and v_pv_q in C_m. According to the definition $H(K_n, C_m)$ is obtained by identifying u_k and v_p as w, deleting the edges u_ku_l and v_pv_q and adding an edge u_lv_q. Now $d(u_i) = n-1$ in $H(K_n, C_m)$. As there are n vertices of degree $n-1$, $\varphi(H(K_n, C_m)) \leq n$. By assigning the colors $c(u_i) = i; 1 \leq i \leq n, c(v_q) = k; 1 \leq k \leq m$ and assign the color l to the vertex v_j which is a part of the cycle and is adjacent to w and for the remaining vertices assign a proper coloring from $\{1, 2, \ldots, n\}$, we get the b-vertices for all the n color classes. Thus $\varphi(H(K_n, C_m)) = n$. □

Theorem 3.5

$$\varphi(H(K_3, W_m)) = \begin{cases} 3, & \text{When the apex vertex of } W_m \text{ is identified with arbitrary vertex of } K_3 \\ 4, & \text{When either of the rim vertices of } W_m \text{ is identified with arbitrary vertex of } K_3 \end{cases}$$

and for all $n > 3$, $\varphi(H(K_n, W_m)) = n$.

Proof: Consider the complete graph K_n and wheel $W_m = C_m + K_1$. Denote the vertices of K_n as $u_1, u_2, \ldots, u_n$ and the vertices of W_m as $v, v_1, v_2, \ldots, v_m$.

Case 1: Consider the edge u_1u_2 in K_3 and the edge vv_p in W_m where v is the apex vertex. $H(K_3, W_m)$ is obtained by identifying u_1 and v_p as w, deleting the edges u_1u_2 and vv_p and adding an edge u_2v. $\varphi(H(K_3, W_m)) \leq 4$ as there are at least four vertices of degree at least three. Then the b- coloring using four colors is not possible due to the adjacency of vertices. Thus $\varphi(H(K_3, W_m)) \leq 3$. But $\varphi(H(K_3, W_m)) \geq 3$ as the graph contains an odd cycle. Therefore $\varphi(H(K_3, W_m)) = 3$.

Case 2: Consider the edge u_1u_2 in K_3 and the rim edge v_pv_q in W_m. $H(K_3, W_m)$ is obtained by identifying u_1 and v_p as w, deleting the edges u_1u_2 and v_pv_q and adding an edge u_2v_q. $\varphi(H(K_3, W_m)) \leq 4$ as there are at least four vertices having degree at least 3. By assigning the colors $c(v) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(w) = 4$ and assign the color 3 to the vertex which is a part of K_3 and is adjacent to w and for the remaining vertices assign any proper coloring from $\{1, 2, \ldots, m\}$, we get the b-vertices for the four color classes. Thus $\varphi(H(K_3, W_m)) = 4$.

Case 3: Consider the edges u_ku_l in K_n and v_pv_q in W_m. $H(K_n, W_m)$ is obtained by identifying u_k and v_p as w, deleting the edges u_ku_l and v_pv_q and adding an edge u_lv_q. Even though we deleted the edges u_ku_l and v_pv_q, $d(v_q) = n$ and $d(u_l) = n-1$. $\varphi(H(K_n, W_m)) \leq n$ as there are n vertices of degree at least $n-1$. By assigning the colors $c(u_i) = i; 1 \leq i \leq n$, $c(v_q) = k$ and assign the color l to any vertex $v_j (j \neq q)$ which is adjacent to w, we get the b-vertices for all the n color classes. Thus $\varphi(H(K_n, W_m)) = \varphi(K_n) = n$.□

Theorem 3.6
$$\varphi(H(K_3, P_m)) = \begin{cases} 2, & \text{if } m = 2 \\ 3, & \text{if } m \neq 2. \end{cases}$$ and,

for all $n > 3$,

$$\varphi(H(K_n, P_m)) = \begin{cases} n-1, & \text{When either of the end vertices of } P_m \text{ is identified with arbitrary vertex of } K_n \\ n, & \text{When any internal vertex of } P_m \text{ is identified with arbitrary vertex of } K_n \end{cases}$$

Proof: Consider the complete graph K_n and the path P_m. Denote the vertices of K_n as $u_1, u_2, \dots, u_n$ and the vertices of P_m as $v_1, v_2, \dots, v_m$. Consider the edges $u_k u_l$ in K_n and $v_p v_q$ in P_m.

Case 1: The Hajós sum of K_3 and P_m is a path P_{m+2}, by identifying either an end vertex of path P_m with any vertex of K_3 or any internal vertex of path P_m with any vertex of K_3. Hence $\varphi(H(K_3, P_m)) = \varphi(P_{m+2}) = \begin{cases} 2, & m = 2 \\ 3, & m \neq 2 \end{cases}$

Case 2: Let u_k be any vertex in K_n and v_p be any end vertex of P_m. The graph $H(K_n, P_m)$ is obtained by identifying the vertices u_k and the end vertex v_p as w, deleting the edges $u_k u_l$ and $v_p v_q$ and adding the edge $u_l v_q$. As there are $n-1$ vertices of degree $n-1$ in $H(K_n, P_m)$, $\varphi(H(K_n, P_m)) = n - 1$.

Case 3: Let u_k be any vertex in K_n and v_p be any internal vertex of P_m. The graph $H(K_n, P_m)$ is obtained by identifying the vertices u_k and the vertex v_p as w, deleting the edges $u_k u_l$ and $v_p v_q$ and adding the edge $u_l v_q$. $\varphi(H(K_n, P_m)) \leq n$ as there are n vertices of degree $n-1$. By assigning the colors $c(u_i) = i; 1 \leq i \leq n$, $c(v_q) = k$ and assign the color l to the vertex which is a part of path and is adjacent to w, we get b- vertices for all the n color classes. Thus $\varphi(H(K_n, P_m)) = n.\square$

IV. Conclusion

The Hajós sum of two graphs is useful to construct a connected graph from two graphs. This concept is useful when either of the network is disrupted and certain node(s) is(are) not functioning. Then that node(s) is(are) to be identified(fused) with the node of a network which is functioning properly and thus new network is constructed.

On the other hand the study of b-coloring is important due to its vital applications in many real life problems involving scheduling, channel assignment, routing of networks etc. We explore the concept of b-coloring in the context of Hajós sum of two graphs.

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