Steiner Domination Number of Some Graphs

S. K. Vaidya and S. H. Karkar

***Abstract*—A tree T contained in graph G is a steiner tree with respect to $W \subseteq V(G)$ if T is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all the vertices of G which lie on some steiner tree with respect to W. The set W is a steiner set for G if $S(W) = V(G)$. The minimum cardinality among the steiner sets of G is the steiner number of G, denoted as $s(G)$. The set W is called steiner dominating set if W is both a steiner set and a dominating set. The minimum cardinality among such sets is a steiner domination number, denoted as $\gamma_s(G)$. We investigate steiner domination number of some graphs.**

***Index Terms*—Steiner number, Domination number, Steiner domination number.**

***MSC 2010 Codes* – 05C12, 05C69, 05C76**

I. INTRODUCTION

WE begin with finite connected and undirected graph G without loops and multiple edges. We will provide brief summary of notation and existing results.

Definition 1.1 The $distance$ $d(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest $u - v$ path in G.

Definition 1.2 The $steiner\ distance\ sd(W)$ of a subset W of vertices of a connected graph G is the minimum number of edges in a connected subgraph of G that contains W. If H is a subgraph of minimum size that contains a set W, then H is necessarily a tree, called a $steiner\ tree$ for W or a $steiner$ $W - tree$.

Definition 1.3 The set of all vertices of G that lie on some steiner W-tree is denoted by $S(W)$. If $S(W) = V(G)$, then W is called a $steiner\ set$ for G. A steiner set of minimum cardinality is a *minimum steiner set* and this cardinality is the *steiner number* $s(G)$.

The concept of steiner number was introduced by Chartrand and Zhang [1]. In the same paper authors have proved many results on this newly defined concept. The steiner number of a graph is further studied in Santhakumaran and John [2].

Definition 1.4 A set $S \subseteq V(G)$ of vertices in a graph $G = (V(G), E(G))$ is called a $dominating\ set$ if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S. A dominating set S is a $minimal\ dominating\ set$ if no proper subset $S^{'} \subset S$ is a dominating set. The $domination$ $number$ $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in graph G.

Definition 1.5 Let G be a connected graph with vertex set $V(G)$. A set of vertices W in G is called a $steiner$ $dominating\ set$ if W is both a steiner set and a dominating set. The minimum cardinality of a steiner dominating set of G is called its *steiner domination number*, denoted by $\gamma_s(G)$.

The concept of steiner domination number was introduced by John *et al*. [3]. It is very interesting to investigate steiner domination number of graph or graph families as it is known only for handful number of graphs. We investigate steiner domination number of graphs obtained by splitting of path and splitting of cycle. We also investigate the steiner domination number for friendship graph, middle graphs of path and cycle.

For the standard graph theoretic terminology and notation we follow Chatrand and Lesniak [4] while the terms related to the theory of domination are used in the sense of Haynes *et al*. [5].

Definition 1.6 A vertex v is an $extreme\ vertex$ of a graph G if the subgraph induced by neighbours of v is complete.

Definition 1.7 [6] A systematic visit of each vertex of a tree is called a $tree\ traversal$.

II. MAIN RESULTS

Proposition 2.1 [1] Each extreme vertex of a connected graph G belongs to every steiner dominating set of G.

Definition 2.2 For a graph G the $splitting\ graph$ $S^{'}(G)$ of a graph G is obtained by adding a new vertex $v^{'}$ corresponding to each vertex v of G such that $N(v) = N(v^{'})$.

Theorem 2.3 $\gamma_s(S^{'}(P_n)) = \begin{cases} 6 & \text{if } n = 4, 5 \\ n & \text{otherwise.} \end{cases}$

Proof: The cases when $n = 4$ and $n = 5$ are to be dealt separately. For the graph $S^{'}(P_4)$ given in the Figure 1, $W = \{v_1, v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_4\}$ is a steiner dominating set of minimum cardinality for $S^{'}(P_4)$ since W will not remain a steiner dominating set when any element is removed from W. Hence, $\gamma_s(S^{'}(P_4)) = 6$.

For the graph $S^{'}(P_5)$ in the Figure 2, $W = \{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, v_5\}$ is a steiner dominating set of minimum cardinality for $S^{'}(P_5)$ since W will not remain a steiner dominating set when any element is removed from W. Hence, $\gamma_s(S^{'}(P_5)) = 6$. The respective steiner W - trees are shown by solid lines while solid vertices are of steiner set W in Figure 1 and Figure 2.

Let $v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}$ be the vertices corresponding to $v_1, v_2, v_3, \dots v_n$ of P_n which are added to obtain $S^{'}(P_n)$. Then $V(S^{'}(P_n)) = \{v_1, v_2, v_3, \dots v_n, v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ and $|V(S^{'}(P_n))| = 2n$. According to proposition 2.1, we must include $v_1^{'}$ and $v_n^{'}$ in Steiner dominating set W as they are the only extreme vertices of $S^{'}(P_n)$. We can observe that if $v_1^{'}, v_2^{'} \in W$ then in various Steiner W-tree traversal from $v_1^{'}$ to $v_2^{'}$, the vertices v_1 and v_2 are contained in $S(W)$. That is, if $v_1^{'}, v_2^{'} \in W$ then $v_1^{'}, v_2^{'}, v_1, v_2 \in S(W)$.

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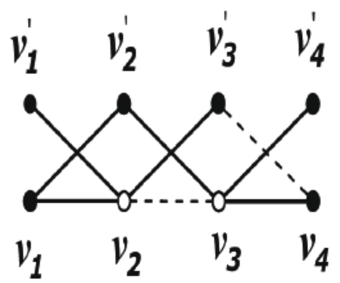

Fig. 1. $S^{'}(P_4)$ and its Steiner tree

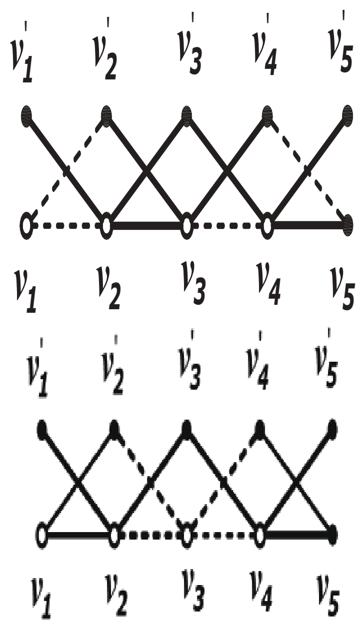

Fig. 2. Two possible Steiner trees in $S^{'}(P_5)$

Therefore, if W = $\{v_1^{'}, v_2^{'}\}$ then $S(W)$ = $V(S^{'}(P_2))$. Hence, W = $\{v_1^{'}, v_2^{'}\}$ is a Steiner set for $S^{'}(P_2)$. Similarly if $v_1^{'}, v_2^{'}, v_3^{'} \in W$, then in various Steiner W-tree traversal from $v_1^{'}$ to $v_3^{'}$ via $v_2^{'}$, contain the vertices v_1,v_2 and v_3. Therefore the vertices v_1,v_2 and v_3 are in $S(W)$. This implies, if $v_1^{'}, v_2^{'}, v_3^{'} \in W$ then $v_1^{'}, v_2^{'}, v_3^{'}, v_1, v_2, v_3 \in S(W)$. Therefore, if W = $\{v_1^{'}, v_2^{'}, v_3^{'}\}$ then $S(W)$ = $V(S^{'}(P_3))$. Hence, W = $\{v_1^{'}, v_2^{'}, v_3^{'}\}$ is a Steiner set for $S^{'}(P_3)$. Also, if $v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'} \in W$ or $v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'} \in W$. Also there is no Steiner tree of minimum size that contains all the vertices of $V(S^{'}(P_4))$ or $V(S^{'}(P_5))$ when $v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'} \in W$ or $v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'} \in W$. But if $v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, v_6^{'} \in W$ then in various Steiner W-tree traversal from $v_1^{'}$ to $v_6^{'}$ via $v_2^{'}$,$v_3^{'}$,$v_4^{'}$,$v_5^{'}$, the vertices v_1,v_2,v_3,v_4,v_5 and v_6 are contained in $S(W)$. That is, if $v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, v_6^{'} \in W$ then $v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, v_6^{'}, v_1, v_2, v_3, v_4, v_5, v_6 \in S(W)$. Therefore, if W = $\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, v_6^{'}\}$ then $S(W)$ = $V(S^{'}(P_6))$. Hence, W = $\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, v_6^{'}\}$ is a Steiner set for $S^{'}(P_6)$. Similarly if W = $\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, v_6^{'}, v_7^{'}\}$ then $S(W)$ = $V(S^{'}(P_7))$. Hence, W = $\{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, v_6^{'}, v_7^{'}\}$ is a Steiner set for $S^{'}(P_7)$ and so on. In general, if $v_1^{'}, v_2^{'}, \dots v_n^{'} \in W$ (where $n \neq 4, 5$) then there are some Steiner W-trees in which tree traversal from $v_1^{'}$ to $v_n^{'}$ via $v_2^{'}$,$v_3^{'}$,$v_4^{'}$, $\dots v_{n-1}^{'}$, the vertices v_1,v_2,v_3, $\dots v_n$ are in $S(W)$. That is, if $v_1^{'}, v_2^{'}, \dots v_n^{'} \in W$ then $v_1, v_2, v_3, \dots v_n \in S(W)$. Equivalently W = $\{v_1^{'}, v_2^{'}, v_3^{'} \dots v_n^{'}\}$ then $S(W)$ = $V(S^{'}(P_n))$. Thus, W = $\{v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ is a Steiner set for $S^{'}(P_n)$. Also, W is a dominating set as each element of $V(S^{'}(P_n)) - W$ is adjacent to some elements of W. Thus, W is a Steiner dominating set with cardinality n.

Claim : W is the only Steiner dominating set of minimum cardinality n.

Let $W^{'} \subset \{v_1, v_2, v_3, \dots v_n, v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$, where $\mid W^{'} \mid \leqslant n$. As $v_1^{'}$ and $v_n^{'}$ being extreme vertices, they must be included in $W^{'}$. In order to prove our claim we consider following two cases.

Case (i): Let $\mid W^{'} \mid$ = n and if $W^{'} = W$ then we have already proved that W is a Steiner dominating set with cardinality n. Therefore we consider that $W^{'} \neq W$. We know that every $v_i^{'}$ for $2 \leq i \leq n-1$ is adjacent with only two vertices v_{i-1} and v_{i+1}. Let $W^{'}$ be a set obtained from W by replacing any element $v_i^{'}$ (except $v_1^{'}$ and $v_n^{'}$) of W by any v_j for $1 \leqslant j \leqslant n$.

Subcase (i): If we replace $v_i^{'} \in W$ by any v_j, where v_j is not adjacent to $v_i^{'}$. We can observe that there is no element which is adjacent to $v_i^{'}$. Thus, if we replace any $v_i^{'} \in W$ by its nonadjacent vertex v_j then no vertex in $W^{'}$ dominates $v_i^{'}$. That is, $W^{'}$ will not be a dominating set.

Subcase (ii): If we replace any $v_i^{'} \in W$ by a vertex v_j which is adjacent to $v_i^{'}$ then $W^{'}$ will not be a Steiner set as $v_i^{'} \notin S(W^{'})$ and $v_i^{'} \notin W^{'}$.

That is, in general if we replace $v_i^{'}$ by any element v_j for $1 \leqslant j \leqslant n$ then $W^{'}$ will not be a Steiner dominating set. Even if we replace k elements ($k > 1$) of $\{v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ by k elements from $\{v_1, v_2, v_3, \dots.v_n\}$ then the resultant set $W^{'}$ will not be a Steiner dominating set.

Thus, $W = \{v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ is the only Steiner dominating set of cardinality n.

Case (ii): Let $|W^{'}| < n$. If we remove arbitrary element $v_i^{'} \in W$ then the resultant set $W^{'}$ is not a Steiner set as $v_i^{'} \notin S(W^{'})$. Thus, $W^{'}$ will not be Steiner dominating set. Therefore, there is no Steiner dominating set with cardinality less than n.

Thus, from above cases $W = \{v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ is the only Steiner dominating set of minimum cardinality n. Hence, we proved that $\gamma_s(S^{'}(P_n)) = \begin{cases} 6 & \text{if } n = 4, 5 \\ n & \text{otherwise.} \end{cases}$

Theorem 2.4 $\gamma_s(S^{'}(C_n)) = n$.

Proof: Let $v_1^{'}$,$v_2^{'}$,$v_3^{'}$,$\dots v_n^{'}$ be the vertices corresponding to $v_1, v_2, v_3, \dots v_n$ of C_n which are added to obtain $S^{'}(C_n)$. Then $V(S^{'}(C_n)) = \{v_1, v_2, v_3, \dots v_n, v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ and $|V(S^{'}(C_n))| = 2n$. According to proposition 2.1, we must include $v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}$ in Steiner dominating set W as they are the extreme vertices of $S^{'}(C_n)$. If $W = \{v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ then we can observe that there are some Steiner W-trees which contain all the vertices of $V(S^{'}(C_n))$. That is, if $v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'} \in W$ then $v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}, v_1, v_2, v_3, \dots v_n \in S(W)$. If W = $\{v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ then $S(W)$ = $V(S^{'}(C_n))$. Therefore, W = $\{v_1^{'}, v_2^{'}, v_3^{'}, \dots v_n^{'}\}$ is a Steiner set for $S^{'}(C_n)$. Moreover W is a dominating set as each element of $V(S^{'}(C_n)) - W$ is adjacent to some elements of W. Consequently, W is a Steiner dominating set with cardinality n.

Claim : W is the only Steiner dominating set of minimum cardinality n.

As each element of W is an extreme vertex, $W - \{v_i^{'}\}$ for $1 \leqslant i \leqslant n$ will not be a Steiner dominating set. Even if we replace k elements ($k > 1$) of $\{v_1^{'}, v_2^{'}, v_3^{'}, \ldots v_n^{'}\}$ by k elements from $\{v_1, v_2, v_3, ...v_n\}$ then the resultant set $W^{'}$ will not be a Steiner dominating set. In other words, there does not exist any Steiner dominating set other than W with $|W| \leq n$. Therefore, W is the only Steiner dominating set of cardinality n. Thus, W is a Steiner dominating set of minimum cardinality for $S^{'}(C_n)$. Hence, we proved that $\gamma_s(S^{'}(C_n)) = n$.

Definition 2.5 A $friendship\ graph$ F_n is a one point union of n copies of cycle C_3.

Theorem 2.6 $\gamma_s(F_n) = 2n$.

Proof: For F_n, $|V(F_n)| = 2n + 1$. Let $V(F_n) = \{v_1, v_2, v_3, \ldots v_n, v_{n+1}, \ldots v_{2n}, v\}$, where v is the apex vertex. According to proposition 2.1, we must include $v_1, v_2, v_3, \ldots v_{2n}$ in Steiner dominating set W as they are the extreme vertices of F_n. In tree traversal from v_1 to v_{2n} via $v_2, v_3, \ldots v_{2n-1}$, the vertex v is inevitably included. Thus, if $W = \{v_1, v_2, v_3, \ldots v_{2n}\}$ then $S(W) = V(F_n)$. Therefore, $W = \{v_1, v_2, v_3, \ldots v_{2n}\}$ is a Steiner set. Moreover W is also a dominating set as $V(F_n) - W = \{v\}$ and v is adjacent to all the elements of W where $|W| = 2n$.

As each element of W being an extreme vertex if we remove any element from W then the resultant set $W - \{v_i\}$ for $1 \leqslant i \leqslant 2n$ will not be a Steiner dominating set. Similarly if we replace any element of W by v, then each vertex of W being an extreme vertex the resultant set will not be a Steiner dominating set. Thus, there does not exist any Steiner dominating set other than W with $|W| \leqslant 2n$. Thus, W is a Steiner dominating set with minimum cardinality $2n$. Hence, $\gamma_s(F_n) = 2n$.

Definition 2.7 The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Theorem 2.8 $\gamma_s(M(P_n)) = n$.

Proof: Here, $|V(M(P_n))| = 2n - 1$. Let $v_1, v_2, v_3, \ldots v_n$ be the vertices and $e_1, e_2, e_3, \ldots e_{n-1}$ be the edges of path P_n. Then $V(M(P_n)) = \{v_1, v_2, v_3, \ldots v_n, e_1, e_2, e_3, \ldots e_{n-1}\}$. According to proposition 2.1, we must include $v_1, v_2, v_3, \ldots v_n$ in Steiner dominating set W as they are the extreme vertices of $M(P_n)$. In tree traversal from v_1 to v_n via $v_2, v_3, \ldots v_{n-1}$, the vertices $e_1, e_2, e_3, \ldots e_{n-1}$ are inevitably included. That is, if $v_1, v_2, v_3, \ldots v_n \in W$ then $v_1, v_2, v_3, \ldots v_n, e_1, e_2, e_3, \ldots e_{n-1} \in S(W)$. Hence, $S(W) = V(M(P_n))$. Therefore $W = \{v_1, v_2, v_3, \ldots v_n\}$ is a Steiner set. Moreover W is also a dominating set as each element of $V(M(P_n)) - W = \{e_1, e_2, e_3, \ldots e_{n-1}\}$ is adjacent to some elements of W. Thus, W is a Steiner dominating set of $M(P_n)$ with cardinality n

As each element of W being an extreme vertex if we remove any element from W, the resultant set $W - \{v_i\}$ for $1 \leqslant i \leqslant n$ will no longer remain a Steiner dominating set. Thus, there does not exist any Steiner dominating set other than W with $|W| \leqslant n$. But we have proved that W is a Steiner dominating set with cardinality n. That is, the minimum cardinality of Steiner dominating set for $M(P_n)$ is n. Hence, we have the result.

Theorem 2.9 $\gamma_s(M(C_n)) = n$.

Proof: For $M(C_n)$, $|V(M(C_n))| = 2n$. Let $v_1, v_2, v_3, \ldots v_n$ be the vertices and $e_1, e_2, e_3, \ldots e_n$ be the edges of cycle C_n. Then $V(M(C_n)) = \{v_1, v_2, v_3, \ldots v_n, e_1, e_2, e_3, \ldots e_n\}$. According to proposition 2.1, we must include $v_1, v_2, v_3, \ldots v_n$ in Steiner dominating set W as they are the extreme vertices of $M(C_n)$. In various Steiner W-tree traversal from v_1 to v_n via $v_2, v_3, \ldots v_{n-1}$, the vertices $e_1, e_2, e_3, \ldots e_n$ are included in $S(W)$. So, if $v_1, v_2, v_3, \ldots v_n \in W$ then $v_1, v_2, v_3, \ldots v_n, e_1, e_2, e_3, \ldots e_n \in S(W)$. Hence, $S(W) = V(M(C_n))$. Therefore $W = \{v_1, v_2, v_3, \ldots v_n\}$ is a Steiner set. Moreover W is also a dominating set as each element of $V(M(C_n)) - W = \{e_1, e_2, e_3, \ldots e_n\}$ is adjacent to some elements of W. Thus, W is a Steiner dominating set of $M(C_n)$ with cardinality n.

As each element of W being an extreme vertex if we remove any element from W, the resultant set $W - \{v_i\}$ for $1 \leqslant i \leqslant n$ will not be a Steiner dominating set, implying that there does not exist any Steiner dominating set other than W with $|W| \leqslant n$. Thus, the minimum cardinality of Steiner dominating set for $M(C_n)$ is n. Hence, $\gamma_s(M(C_n)) = n$.

III. Conclusions

It is very interesting to investigate Steiner domination number of any graph or graph families as the Steiner domination number for very few graphs are known. We take up this task and investigate Steiner domination number for the larger graphs obtained by the means of some graph operations on standard graphs.

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