A Note on Lind's Right Circulant Matrix with Fibonacci Sequence

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***Abstract*—In this study, we investigate some of the properties of the eigenvalues and the determinant of Lind's right circulant matrix with Fibonacci sequence.**

***Index Terms*—Right circulant matrix, Fibonacci sequence, eigenvalue, determinant.**

***MSC 2010 Codes* – 05C50, 11B50**

I. Introduction

LIND'S right circulant matrix with Fibonacci sequence take the form

$$C_R(\vec{F_r}) = \begin{pmatrix} F_r & F_{r+1} & \cdots & F_{r+n-2} & F_{r+n-1} \\ F_{r+n-1} & F_r & \cdots & F_{r+n-3} & F_{r+n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{r+2} & F_{r+3} & \cdots & F_r & F_{r+1} \\ F_{r+1} & F_{r+2} & \cdots & F_{r+n-1} & F_r \end{pmatrix}$$

where F_n is the n^{th} Fibonacci number given by

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} \quad (1)$$

with $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and $\psi = \frac{1-\sqrt{5}}{2}$.

Lind have shown that the eigenvalues of $C_R(\vec{F_r})$ are given by

$$\lambda(n,r,m) = \frac{F_r - F_{n+r} - (F_{n+r-1} - F_{r-1})\omega^{-mk}}{(1-\phi\omega^{-mk})(1-\psi\omega^{-mk})} \quad (2)$$

where m=0, 1, ..., n-1 and $\omega = e^{2\pi i/n}$. Furthermore, he also established that the determinant of $C_R(\vec{F_r})$ denoted by $D(n,r)$ is given by

$$D(n,r) = \frac{(F_r - F_{n+r})^n - (F_{n+r-1} - F_{r-1})^n}{1 - L_n + (-1)^n} \quad (3)$$

where $L_n = \phi^n + \psi^n$, the n^{th} Lucas number.

In this study, we shall investigate some properties of $\lambda(n,r,m)$ and $D(n,r)$ thorugh the limits. Specifically, we will derive results of the following limits:

$$\lim_{n\to\infty} \frac{\lambda(n+j,r,m)}{\lambda(n,r,m)} \quad (4)$$

$$\lim_{n\to\infty} \frac{D(n+j,r)}{D(n,r)} \quad (5)$$

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II. Preliminary Results

We will need the following lemmas to prove our main results.

Lemma 2.1:

$$\lim_{n\to\infty} e^{2\pi i/n} = 1$$

Proof:

$$\begin{aligned} \lim_{n\to\infty} e^{2\pi i/n} &= e^{\lim_{n\to\infty} 2\pi i/n} \\ &= e^0 \\ &= 1 \end{aligned}$$

Lemma 2.2:

$$\lim_{n\to\infty} \frac{F_r}{F_n} = 0$$

Proof:

$\frac{F_r}{F_n}$ vanishes because F_r is constant and F_n gets larger and larger as $n \to \infty$.

Lemma 2.3:

$$\lim_{n\to\infty} \frac{F_{n+j}}{F_n} = \phi^j$$

Proof:

$$\begin{aligned} \lim_{n\to\infty} \frac{F_{n+j}}{F_n} &= \lim_{n\to\infty} \frac{\phi^{n+j} - \psi^{n+j}}{\phi^n - \psi^n} \\ &= \lim_{n\to\infty} \frac{1 - (\psi/\phi)^{n+j}}{1/\phi^j - (\psi^n/\phi^{n+j})} \\ &= \phi^j \end{aligned}$$

III. Main Results

Theorem 3.1:

$$\lim_{n\to\infty} \frac{\lambda(n+j,r,m)}{\lambda(n,r,m)} = \phi^j$$

Proof:

Let $A(n,r,m) = F_r - F_{n+r} - (F_{n+r-1} - F_{r-1})\omega^{-mk}$ and $B(n,r,m) = (1-\phi\omega^{-mk})(1-\psi\omega^{-mk})$. This leads to

$$\begin{aligned} &\lim_{n\to\infty} \frac{\lambda(n+j,r,m)}{\lambda(n,r,m)} \\ &= \lim_{n\to\infty} \frac{A(n+j,r,m)/B(n+j,r,m)}{A(n,r,m)/B(n,r,m)} \\ &= \left(\lim_{n\to\infty} \frac{A(n+j,r,m)}{A(n,r,m)}\right)\left(\lim_{n\to\infty} \frac{B(n,r,m)}{B(n+j,r,m)}\right) \end{aligned}$$

Next, we solve the limits separately.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{B(n,r,m)}{B(n+j,r,m)} \\
&= \lim_{n\to\infty} \frac{(1-\phi e^{-2\pi imk/n})(1-\psi e^{-2\pi imk/n})}{(1-\phi e^{-2\pi imk/(n+j)})(1-\psi e^{-2\pi imk/(n+j)})} \\
&= \frac{(1-\phi)(1-\psi)}{(1-\phi)(1-\psi)} \text{ (by Lemma 2.1)} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty} \frac{A(n+j,r,m)}{A(n,r,m)} \\
&= \lim_{n\to\infty} \frac{F_r - F_{n+j+r} - (F_{n+j+r-1} - F_{r-1})e^{-2\pi imk/(n+j)}}{F_r - F_{n+r} - (F_{n+r-1} - F_{r-1})e^{-2\pi imk/n}} \\
&= \lim_{n\to\infty} \frac{F_r - F_{n+j+r} - F_{n+j+r-1} + F_{r-1}}{F_r - F_{n+r} - F_{n+r-1} + F_{r-1}} \text{ (by Lemma 2.1)} \\
&= \lim_{n\to\infty} \frac{F_r/F_n - F_{n+j+r}/F_n - F_{n+j+r-1}/F_n + F_{r-1}/F_n}{F_r/F_n - F_{n+r}/F_n - F_{n+r-1}/F_n + F_{r-1}/F_n} \\
&= \frac{-\phi^{j+r} - \phi^{j+r-1}}{-\phi^r - \phi^{r-1}} \\
&= \frac{\phi^{j+r-1}(\phi+1)}{\phi^{r-1}(\phi+1)} \\
&= \phi^j
\end{aligned}$$

Using product of limits, we obtain

$$\left(\lim_{n\to\infty} \frac{A(n+j,r,m)}{A(n,r,m)}\right)\left(\lim_{n\to\infty} \frac{B(n,r,m)}{B(n+j,r,m)}\right) = \phi^j \cdot 1 = \phi^j$$

as desired.

Corollary 3.2:

$$\lim_{n\to\infty} \frac{\lambda(n+1,r,m)}{\lambda(n,r,m)} = \phi$$

Theorem 3.3:

$$\lim_{n\to\infty} \frac{D(n+j,r)}{D(n,r)} = \phi^{jn}$$

Proof:
This follows immediately from Theorem 3.1 since the determinant is just the product of the eigenvalues.

Corollary 3.4:

$$\lim_{n\to\infty} \frac{D(n+1,r)}{D(n,r)} = \phi^n$$

IV. Conclusion

We have established that the eigenvalues and the determinant of Lind's right circulant matrix with Fibonacci sequence is related to the golden ratio.

References

[1] D. Lind, "A Fibonacci circulant", *Fibonacci Quarterly,* vol. 8, pp. 449-455, 1970.