Coupled Common Fixed Points for a Pair of Compatible Maps satisfying Geraghty Contraction in Partially Ordered Metric Spaces

G. V. R. Babu* and P. Subhashini**

***Abstract*—In this paper, we prove the existence of coupled coincidence and coupled common fixed points for a pair of compatible maps satisfying Geraghty contraction in partially ordered complete metric spaces. Our results generalize the results of Bhaskar and Lakshmikantham (2006) and Choudhury and Kundu (2012). Examples are provided to illustrate our results.**

***Index Terms*—Partially ordered set, mixed $g-$monotone property, coupled coincidence point, coupled common fixed point.**
***MSC 2010 Codes* – 47H10, 54H25.**

I. Introduction

In 1987, mixed monotone operators were introduced by Guo and Lakshmikantham [1]. In 2004, Ran and Reurings [2] generalized Banach contraction principle by introducing a partial order in metric spaces.

In 2006, Bhaskar and Lakshmikantham [3] established the existence of coupled fixed points for mixed monotone operators in partially ordered metric spaces.

In 1976, Jungck [4] proved common fixed point results in metric spaces for a pair of $commuting\ maps$. In 1986, Jungck [5] introduced the concept of $compatible\ maps$ as a generalization of commuting maps and established the existence of fixed points in metric spaces.

In 2009, Lakshmikantham and Ciric [6] introduced the concept of *commuting maps in the context of coupled fixed points* and proved coupled coincidence and coupled common fixed points in partially ordered metric spaces.

In 2010, Choudhury and Kundu [7] generalized the concept of commuting maps in the context of coupled fixed points by introducing $compatible\ maps$ and established the existence of coupled coincidence points in partially ordered metric spaces.

Many researchers, namely Choudhury, Metiya and Kundu [8]; Alsulami, Alotaibi[9]; Berinde[10]; Harjani, Lopez, Sadarangani[11]; Hemant Kumar, Shatanawi[12]; Karapınar, Luong, Thuan[13]; Luong, Thuan[14]; Luong, Thuan, Hai[15]; Sabetghadam, Masiha[16] worked in this direction.

Throughout this paper, R denotes the set of all real numbers and $R^+ = [0, \infty)$.

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Definition 1.1: [3] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a map. We say that F has the *mixed monotone property* if $F(x, y)$ is non-decreasing in x and is non-increasing in y. *i.e*, for any $x, y \in X$,

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \ \Rightarrow\ F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \ \Rightarrow\ F(x, y_1) \succeq F(x, y_2).\ \Box$$

Definition 1.2: [3] Let X be a non-empty set and

$$F : X \times X \to X$$

be a map. An element $(x, y) \in X \times X$ is said to be a *coupled fixed point of* F if $F(x, y) = x$ and $F(y, x) = y$. $\Box$

Definition 1.3: [6] Let $(X, \preceq)$ be a partially ordered set and

$$F : X \times X \to X \quad \text{and} \quad g : X \to X$$

be two maps. We say that F has the *mixed g-monotone property* if for any $x,\ y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y)$$

$$y_1, y_2 \in X, \quad g(y_1) \preceq g(y_2) \quad \Rightarrow \quad F(x, y_1) \succeq F(x, y_2).\Box$$

Definition 1.4: [6] Let X be a non-empty set and

$$F : X \times X \to X \quad \text{and} \quad g : X \to X$$

be two maps. An element $(x, y) \in X \times X$ is said to be a *coupled coincidence point* of F and g if

$$F(x, y) = g(x) \quad \text{and} \quad F(y, x) = g(y)$$

and (x, y) is said to be a *coupled common fixed point of F and g* if

$$F(x, y) = g(x) = x \quad \text{and} \quad F(y, x) = g(y) = y. \quad \Box$$

Definition 1.5: [6] Let X be a non-empty set and

$$F : X \times X \to X \quad \text{and} \quad g : X \to X$$

be two maps. We say that F and g are *commutative* if $g(F(x, y)) = F(g(x), g(y))$ for all x, y in X. $\Box$

Definition 1.6: [7] Let (X, d) be a metric space. The mappings

$$F : X \times X \to X \quad \text{and} \quad g : X \to X$$

are said to be *compatible* if

$$\lim_{n\to\infty} d(g(F(x_n, y_n)),\ F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)),\ F(g(y_n), g(x_n))) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$$

and

$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$$

for all $x, y \in X$. □

Remark 1.7: Let (X, d) be a metric space. If the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative then F and g are compatible but its converse need not be true.

Example 1.8: Let $X = R$ with the usual metric.

We define $F : X \times X \to X$ by

$$F(x, y) = x^2 - y^2,\ x, y \in X$$

and $g : X \to X$ by

$$g(x) = x^2,\ x \in X.$$

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} x_n = l_1 \ \text{ and } \ \lim_{n\to\infty} x_n = l_2.$$

Let

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n)$$

and

$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n).$$

Then it follows that $l_1 = 0$ and $l_2 = 0$ so that

$$d(g(F(x_n, y_n)),\ F(g(x_n), g(y_n))) \to 0 \ \text{ as } \ n \to \infty$$

and

$$d(g(F(y_n, x_n)),\ F(g(y_n), g(x_n))) \to 0 \ \text{ as } \ n \to \infty.$$

Hence, the mappings F and g are compatible in X.

Now, $F(gx, gy) = x^4 - y^4$ and $g(F(x, y)) = (x^2 - y^2)^2$

Hence $F(gx, gy) \neq g(F(x, y))$ for all $x, y \in X$.

Thus F and g are not commutative. □

The following are the main theorems of Bhaskar and Lakshmikantham [3].

Theorem 1.9: [3] Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Let $F : X \times X \to X$ be a continuous map having the mixed monotone property on X. Assume that there exists k $\in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \tag{1.9.1}$$

for $x, y, u, v \in X$ with $x \succeq u,\ y \preceq v$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$. □

Theorem 1.10: [3] Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists k $\in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \tag{1.10.1}$$

for $x, y, u, v \in X$ with $x \succeq u,\ y \preceq v$.

Further assume that X has the following properties:

(i) if $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to x$, then $x_n \preceq x$ for all $n = 1,\ 2,\ 3, ...$;

(ii) if $\{y_n\}$ is a non-increasing sequence in X with $y_n \to y$, then $y_n \succeq y$ for all $n = 1,\ 2,\ 3, ..$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that

$$x = F(x, y) \ \text{ and } \ y = F(y, x). \ \square$$

We use the following notation introduced by Geraghty [17].

$$S = \{\beta : R^+ \to [0, 1)/\beta(t_n) \to 1 \ \Rightarrow \ t_n \to 0\}.$$

Geraghty [17] established the following theorem.

Theorem 1.11: [17] Let $f : X \to X$ be a selfmap on a complete metric space X and there exists $\beta \in S$ such that

$$d(f(x),\ f(y)) \leq \beta(d(x, y))d(x, y),\ \forall x, y \in X.$$

Then for any choice of initial point x_0, the iteration

$$x_n = f(x_{n-1}), \ \text{ for } \ n = 1,\ 2,\ 3, ...$$

converges to the unique fixed point z of f in X. □

In 2010 Amini-Harandi and Emami [18] extended Theorem 1.11 to partially ordered complete metric spaces.

Theorem 1.12: [18] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be an increasing map and there exists $\beta \in S$ such that

$$d(f(x),\ f(y)) \leq \beta(d(x, y))d(x, y), \ \ \forall x, y \in X \text{ with } x \succeq y.$$

Assume that either f is continuous or X is such that if an increasing sequence $x_n \to x$ in X, then $x_n \preceq x$ for all $n = 1,\ 2,\ 3, ...$ .. Besides if for each $x, y \in X$, there exists $z \in X$ such that which is comparable to x and y. Then f has a unique fixed point. □

Recently Choudhury and Kundu [19] established Theorem 1.12 in the context of coupled fixed points in partially ordered metric spaces.

Theorem 1.13: [19] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous map having the mixed monotone property and there exists $\beta \in S$ such that

$$d(F(x, y), F(u, v)) \leq \beta(\frac{d(x, u) + d(y, v)}{2})(\frac{d(x, u) + d(y, v)}{2}) \tag{1.13.1}$$

for $x, y, u, v \in X$ with $x \succeq u,\ y \preceq v$.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0),$$

then F has a coupled fixed point in X. *i.e.,* there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$. □

Theorem 1.14: [19] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property and there exists $\beta \in S$ such that

$$d(F(x,y), F(u,v)) \leq \beta(\frac{d(x,u)+d(y,v)}{2})(\frac{d(x,u)+d(y,v)}{2}) \quad (1.14.1)$$

for $x, y, u, v \in X$ with $x \succeq u,\ y \preceq v$. Also suppose that X has the following properties:

(i) if $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to x$, then $x_n \preceq x$ for all $n = 1,\ 2,\ 3, ...$;
(ii) if $\{y_n\}$ is a non-increasing sequence in X with $y_n \to y$, then $y_n \succeq y$ for all $n = 1,\ 2,\ 3, \ldots$.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0),$$

then F has a coupled fixed point in X, textit i.e., there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$. □

In this paper we prove the existence of coupled coincidence and coupled common fixed points for a pair of compatible maps satisfying Geraghty contraction in partially ordered complete metric spaces. Our results generalize the results of Bhaskar, Lakshmikantham [3] and Choudhury, Kundu [19].

In the following, we write $\Phi = \{\varphi / \ \varphi : R^+ \to R^+$ satisfying $\varphi(t) \leq t$ for $t > 0$ and $\varphi(t) = 0$ if and only if $t = 0\}$.

II. MAIN RESULTS

Theorem 2.1: Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X,\ g : X \to X$ be two mappings such that F is a continuous map having the mixed $g-$monotone property on X. Assume that there exists $\beta \in S$ and $\varphi \in \Phi$ such that

$$d(F(x,y), F(u,v)) \leq \beta(\tfrac{d(gx,gu)+d(gy,gv)}{2}) \cdot \varphi(\tfrac{d(gx,gu)+d(gy,gv)}{2}) \quad (2.1.1)$$

for $x, y, u, v \in X$ with $gx \succeq gu,\ gy \preceq gv$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous, F and g are compatible maps.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ then F and g have a coupled coincidence point in X. *i.e.,* there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Proof. Let $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0).$$

If $gx_0 = F(x_0, y_0)$ and $gy_0 = F(y_0, x_0)$, then (x_0, y_0) is a coupled coincidence point of F and we are through.

Otherwise, since $F(X \times X) \subseteq g(X)$ we define sequences $\{gx_n\}$ and $\{gy_n\} \in X$ such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for all $n = 0,\ 1,\ 2, \ldots$.

Now, we show that $\{gx_n\}$ is an increasing sequence and $\{gy_n\}$ is a decreasing sequence. *i.e.,*

$$gx_n \preceq x_{n+1} \text{ and } gy_n \succeq y_{n+1}\ \forall n = 0, 1, \ldots \quad (2.1.2)$$

The proof is by mathematical induction.

We have $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$.

Hence $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$ so that (2.1.2) is true for $n = 0$.

Assume that (2.1.2) is true for some n. By using the mixed g-monotone property of F, we have

$$gx_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_n) = gx_{n+1}$$

and

$$gy_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_n) = gy_{n+1}.$$

Hence (2.1.2) is true for $n + 1$. Thus by the mathematical induction, (2.1.2) follows.

Let us denote $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$.

<u>*Case*</u> (i) : Suppose that $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$ for some n. *i.e.,*

$$gx_{n+1} = F(x_n, y_n) = gx_n$$

and

$$gy_{n+1} = F(y_n, x_n) = gy_n.$$

Hence (x_n, y_n) is a coupled coincidence point.

<u>*Case*</u> (ii) : Suppose that $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n.

We have

$$gx_n \preceq gx_{n+1} \text{ and } gy_n \succeq gy_{n+1},\ \forall n = 0, 1, 2, \ldots$$

Now, $d(gx_{n+1}, gx_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))$

$$\leq \beta(\tfrac{d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})}{2})\ \varphi(\tfrac{d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})}{2})$$
$$< d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}). \quad (2.1.3)$$

Again,

$d(gy_n, gy_{n+1})) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)))$

$$\leq \beta(\tfrac{d(gy_{n-1},gy_n)+d(gx_{n-1},gx_n)}{2})\ \varphi(\tfrac{d(gy_{n-1},gy_n)+d(gx_{n-1},gx_n)}{2})$$
$$< d(gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n). \quad (2.1.4)$$

Adding (2.1.3) and (2.1.4), we have

$$d(gx_{n+1}, gx_n) + d(gy_n, gy_{n+1}) < d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n).$$

Hence $\{\delta_n\}$ is non-negative decreasing sequence of reals and hence it converges to a real number r(say), $r \geq 0$.

If possible, we assume that $r > 0$.

From (2.1.3) and (2.1.4), we have

$$d(gx_{n+1}, gx_n) + d(gy_n, gy_{n+1})$$
$$\leq 2\beta(\tfrac{d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)}{2})\ \varphi(\tfrac{d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)}{2})$$
$$\leq \beta(\tfrac{d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)}{2}) (d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n))$$
$$< d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n).$$

On taking limits as $n \to \infty$, we get

$$r \leq \lim_{n\to\infty} \beta(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2})r \leq r.$$

Hence

$$\lim_{n\to\infty} \beta(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}) = 1.$$

Since $\beta \in S$, we have

$$d(gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n) \to 0, \text{as } n \to \infty,$$

which is a contradiction for $r > 0$.

Hence, $r = 0$.

Next, we prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Otherwise, assume that at least one of the sequences $\{gx_n\}$ and $\{gy_n\}$ is not a Cauchy sequence.

Then there exists an $\epsilon > 0$ for which we can find subsequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $n(k) > m(k) > k$ such that

$$d_k = d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)})\} \geq \epsilon. \quad (2.1.5)$$

Further, we choose $n(k)$ to be the smallest positive integer such that $n(k) > m(k)$ satisfying (2.1.5). Hence $d_k \geq \epsilon$ and

$$d(gx_{m(k)}, gx_{n(k)-1}) + d(gy_{m(k)}, gy_{n(k)-1})\} < \epsilon. \quad (2.1.6)$$

From the triangle inequality and (2.1.6), we have

$$\begin{aligned}\epsilon \leq d_k &\leq d(gx_{m(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{n(k)}) + \\ &\quad d(gy_{m(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{n(k)}) \\ &< \epsilon + d(gx_{n(k)-1}, gx_{n(k)}) + d(gy_{n(k)-1}, gy_{n(k)}). \quad (2.1.7)\end{aligned}$$

Now, we prove that $\lim_{k\to\infty} d_k = \epsilon$.

On taking limit superior as $k \to \infty$ in (2.1.7), we get

$$\epsilon \leq \limsup_{k\to\infty} d_k \leq \epsilon.$$

Hence,

$$\limsup_{k\to\infty} d_k = \epsilon. \quad (2.1.8)$$

Again, on taking limit inferior as $k \to \infty$ in (2.1.5) and from (2.1.8), we get $\epsilon \leq \liminf_{k\to\infty} d_k \leq \limsup_{k\to\infty} d_k = \epsilon$. Hence

$$\liminf_{k\to\infty} d_k = \epsilon. \quad (2.1.9)$$

Thus, from (2.1.8) and (2.1.9), $\lim_{k\to\infty} d_k$ exists and

$$\lim_{k\to\infty} d_k = \epsilon. \quad (2.1.10)$$

Again, for all $k \geq 0$, we have

$$\begin{aligned}d_k &= d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_n(k) \\ &\leq d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{n(k)+1}) \\ &\quad + d(gx_{n(k)+1}, gx_{n(k)}) + d(gy_{m(k)}, gy_{m(k)+1}) \\ &\quad + d(gy_{m(k)+1}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{n(k)}). \quad (2.1.11)\end{aligned}$$

Now, since

$$gx_{m(k)} \preceq gx_{n(k)} \text{ and } gy_{m(k)} \succeq gy_{n(k)},$$

we have

$$\begin{aligned}d(gx_{m(k)+1}, gx_{n(k)+1}) &= d\big(F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)})\big) \\ &\leq \beta\Big(\tfrac{d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)})}{2}\Big) \\ &\qquad \varphi\Big(\tfrac{d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)})}{2}\Big) \\ &\leq \beta\big(\tfrac{d_k}{2}\big)\tfrac{d_k}{2}. \quad (2.1.12)\end{aligned}$$

Similarly, we can prove that

$$d(gy_{m(k)+1}, gy_{n(k)+1}) \leq \beta\big(\frac{d_k}{2}\big)\frac{d_k}{2}. \quad (2.1.13)$$

Adding (2.1.12) and (2.1.13), we get

$$d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1}) \leq \beta\big(\frac{d_k}{2}\big)d_k.$$

Now, from (2.1.11)

$$d_k \leq \delta_{m(k)} + \delta_{n(k)} + \beta\big(\frac{d_k}{2}\big)d_k < \delta_{m(k)} + \delta_{n(k)} + d_k.$$

On taking limits as $k \to \infty$ and by using (2.1.10), we get

$$\epsilon \leq \lim_{k\to\infty} \beta\big(\frac{d_k}{2}\big)\epsilon \leq \epsilon.$$

Hence,

$$\lim_{k\to\infty} \beta\big(\frac{d_k}{2}\big) = 1.$$

Therefore, $d_k \to 0$ as $k \to \infty$, a contradiction to (2.1.10).

Hence, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X.

Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \text{ and } \lim_{n\to\infty} gy_n = y.$$

We have

$$x = \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} F(x_n, y_n)$$

and

$$y = \lim_{n\to\infty} gy_{n+1} = \lim_{n\to\infty} F(y_n, x_n).$$

Now, since F and g are compatible, we have

$$\lim_{n\to\infty} d\big(gF(x_n, y_n), F(gx_n, gy_n)\big) = 0 \quad (2.1.14)$$

and

$$\lim_{n\to\infty} d\big(gF(y_n, x_n), F(gy_n, gx_n)\big) = 0. \quad (2.1.15)$$

Now, we prove that $gx = F(x, y)$ and $gy = F(y, x)$.

For all $n \geq 0$, we have

$$\begin{aligned}d(gx, F(gx_n, gy_n)) &\leq d(gx, gF(x_n, y_n)) \\ &\quad + d(gF(x_n, y_n), F(gx_n, gy_n)).\end{aligned}$$

On taking limits as $n \to \infty$ and from (2.1.14) we get

$$d(gx, F(x, y)) = 0,$$

since F and g are continuous.

Similarly it is easy to see that

$$d(gy, F(y, x)) = 0.$$

Hence $gx = F(x, y)$ and $gy = F(y, x)$.

Thus (x, y) is a coupled coincidence point of F and g.

This completes the proof of the theorem. □

Theorem 2.2: Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X,\ g : X \to X$ be two mappings such that F has the mixed $g-$monotone property on X. Assume that there exists $\beta \in S$ and $\varphi \in \Phi$ such that

$$\begin{aligned}d(F(x, y), F(u, v)) &\leq \beta\big(\tfrac{d(gx, gu) + d(gy, gv)}{2}\big) \\ &\qquad \varphi\big(\tfrac{d(gx, gu) + d(gy, gv)}{2}\big)\end{aligned}$$

for $x, y, u, v \in X$ with $gx \succeq gu,\ gy \preceq gv$. (2.2.1)

Suppose that $F(X\times X)\subseteq g(X)$, g is continuous, g is non-decreasing and F and g are compatible maps. Further, assume that:

(a) if $\{x_n\}$ is a non-decreasing sequence in X with $x_n\to x$, then $x_n\preceq x$ for all $n=1,\ 2,\ 3,...$;

(b) if $\{y_n\}$ is a non-increasing sequence in X with $y_n\to y$, then $y_n\succeq y$ for all $n=1,\ 2,\ 3,...$.

If there exist $x_0,y_0\in X$ such that $gx_0\preceq F(x_0,y_0)$ and $gy_0\succeq F(y_0,x_0)$, then F and g have a coupled coincidence point in X. *i.e.,* there exist $x,y\in X$ such that $gx=F(x,y)$ and $gy=F(y,x)$.

Proof: From Theorem 2.1, we have

$$\lim_{n\to\infty} gx_n=x \text{ and } \lim_{n\to\infty} gy_n=y.$$

Hence, from the hypotheses (a) and (b), we have

$$gx_n\preceq x \text{ and } gy_n\succeq y,\ \forall n=1,\ 2,\ 3,\dots$$

Since g is non-decreasing, we have

$$ggx_n\preceq gx \text{ and } ggy_n\succeq gy, \forall n=1,\ 2,\ 3,\dots$$

Since g is continuous and F and g are compatible mappings, we have

$$\lim_{n\to\infty} ggx_n=gx=\lim_{n\to\infty} gF(x_n,y_n)=\lim_{n\to\infty} F(gx_n,gy_n)$$

and

$$\lim_{n\to\infty} ggy_n=gy=\lim_{n\to\infty} gF(y_n,x_n)=\lim_{n\to\infty} F(gy_n,gx_n).$$

Now, we have

$d(gx,F(x,y))\le d(gx,ggx_{n+1})+d(ggx_{n+1},F(x,y))$.

On taking limits as $n\to\infty$, we get

$$\begin{aligned} d(gx,F(x,y))&\le \lim_{n\to\infty} d(gx,ggx_{n+1})+\\ &\quad\lim_{n\to\infty} d(gF(x_n,y_n),F(x,y))\\ &=\lim_{n\to\infty} d(F(gx_n,gy_n),F(x,y)). \qquad (2.2.2)\end{aligned}$$

Now from (2.2.1)

$$\begin{aligned} &d(F(x,y),F(gx_n,gy_n))\\ &\le \beta\big(\tfrac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\big)\ \varphi\big(\tfrac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\big)\\ &< \tfrac{d(gx,ggx_n)+d(gy,ggy_n)}{2}.\end{aligned}$$

On taking limits $n\to\infty$, we get

$$\lim_{n\to\infty} d(F(x,y),F(gx_n,gy_n)=0.$$

Hence from (2.2.2) we have $gx=F(x,y)$.

Similarly, we can prove that $gy=F(y,x)$.

Thus (x,y) is a coupled coincidence point of F and g.

This completes the proof. □

We relax the conditions continuity and non-decreasing property of g, compatibility of F and g and completeness of X in Theorem 2.2 by assuming $g(X)$ is a complete subspace of X, and prove the following theorem.

Theorem 2.3: Let $(X,\preceq,d)$ be a partially ordered metric space. Let $F:X\times X\to X$, $g:X\to X$ and F has the mixed g-monotone property and $F(X\times X)\subseteq g(X)$, $g(X)$ is a complete subspace of X. Assume that there exists $\beta\in S$ and $\varphi\in\Phi$ such that

$$\begin{aligned} d(F(x,y),F(u,v))&\le \beta(\tfrac{d(gx,gu)+d(gy,gv)}{2})\\ &\qquad\varphi(\tfrac{d(gx,gu)+d(gy,gv)}{2})\end{aligned}$$

for $x,y,u,v\in X$ with $gx\succeq gu,\ gy\preceq gv$. (2.3.1)

Further assume that X has the following property :

(a) if $\{x_n\}$ is a non-decreasing sequence in X with $x_n\to x$, then $x_n\preceq x$ for all $n=1,\ 2,\ 3,...$;

(b) if $\{y_n\}$ is a non-increasing sequence in X with $y_n\to y$, then $y_n\succeq y$ for all $n=1,\ 2,\ 3,...$.

If there exist $x_0,y_0\in X$ such that $g(x_0)\preceq F(x_0,y_0)$ and $g(y_0)\succeq F(y_0,x_0)$, then there exist $x,y\in X$ such that $g(x)=F(x,y)$ and $g(y)=F(y,x)$. *i.e.,* F and g have a coupled coincidence point.

Proof: On proceeding as in proof of Theorem 2.1, the sequences $\{gx_n\}$ and $\{gy_n\}$ defined by

$$gx_{n+1}=F(x_n,y_n) \text{ and } gy_{n+1}=F(y_n,x_n)$$

are Cauchy in $g(X)$.

Since $g(X)$ is complete, there exist $x,y\in X$ such that

$$\lim_{n\to\infty} g(x_n)=gx$$

and

$$\lim_{n\to\infty} g(y_n)=gy.$$

Hence, by our assumptions (a) and (b), we have $gx_n\preceq gx$ and $gy_n\succeq gy$ for all $n\ge 0$.

Now, we prove that $gx=F(x,y)$ and $gy=F(y,x)$.

We consider $d(gx_{n+1},F(x,y))=d(F(x_n,y_n),F(x,y))$

$$\begin{aligned}&\le \beta\big(\tfrac{d(gx,gx_n)+d(gy,gy_n)}{2}\big)\ \varphi\big(\tfrac{d(gx,gx_n)+d(gy,gy_n)}{2}\big)\\ &< \tfrac{d(gx,gx_n)+d(gy,gy_n)}{2}. \qquad (2.3.2)\end{aligned}$$

On taking limits as $n\to\infty$, in (2.3.2), we get $d(gx,F(x,y)=0$.

Similarly, we get $d(gy,F(y,x))=0$.

Hence, $F(x,y)=gx$ and $F(y,x)=gy$.

Thus, (x,y) is a coupled coincidence point.

This completes the proof. □

Remark 2.4: Theorem 2.3 (Theorem 2.1 and Theorem 2.2) does not guarantee the uniqueness of the coupled fixed point. Now we present an example in support of this statement. □

Example 2.5: Let $X=\{0,\frac{1}{2},2\}$ with the usual metric.

We consider the following relation on X.

For any $x,y\in X$, $x\preceq y\Leftrightarrow\ x=y$ or $(0,\frac{1}{2})$.

Clearly (X,d) is a complete metric space and $(X,\preceq)$ is a partially ordered set.

We have

$$\preceq:=\{(0,0),(\frac{1}{2},\frac{1}{2}),(2,2),(0,\frac{1}{2})\}.$$

Set

$$\begin{aligned} A&=\{(0,0),(0,\tfrac{1}{2}),(0,2),(2,0),(2,2),(2,\tfrac{1}{2})\}\\ B&=\{(\tfrac{1}{2},0),(\tfrac{1}{2},2)\}\\ C&=\{(\tfrac{1}{2},\tfrac{1}{2})\}\end{aligned}$$

We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A \\ \frac{1}{2} & \text{if } (x,y) \in B \\ 2 & \text{if } (x,y) \in C \end{cases}$$

and $g : X \to X$ by

$$g0 = 0,\ g\frac{1}{2} = 2,\ g2 = \frac{1}{2}.$$

Clearly, F has the mixed $g-$monotone property and $g(X)$ is a complete subspace of X and properties (a), (b) of Theorem 2.3 hold. By choosing $x_0 = 0,\ y_0 = 2$ we have

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0).$$

The inequality (2.3.1) clearly holds, since for every $(x,y),(u,v) \in X \times X$ whenever $gx \succeq gu$ and $gy \preceq gv$, we have $d(F(x,y), F(u,v)) = 0$.

So F satisfies all the hypotheses of Theorem 2.3 and F, g have two coincidence points $(0,0)$, $(\frac{1}{2}, \frac{1}{2})$ even though X is complete.

Further, let us choose $((x,y),(u,v)) = ((\frac{1}{2}, 0),(0,0))$.

In this case, $x \succeq u$ and $y \preceq v$ and

$$d(F(x,y), F(u,v)) = \frac{1}{2} \leq \frac{k}{2}\frac{1}{2} = \frac{k}{2}[d(x,u) + d(y,v)],$$

which is absurd for any $0 \leq k < 1$.

Hence the inequality (1.10.1) fails to hold. Further $d(F(x,y), F(u,v)) = \frac{1}{2} \leq \beta(\frac{1}{4})(\frac{1}{4})$
$= \beta(\frac{d(x,u)+d(y,v)}{2})(\frac{d(x,u)+d(y,v)}{2})$,
which is also absurd for any $\beta \in S$.

Hence the inequality (1.14.1) fails to hold.

Thus Theorem 1.10 and Theorem 1.14 are not applicable.

Remark 2.6: Theorem 1.9 follows as a corollary to Theorem 2.1 and Theorem 1.10 follows as a corollary to Theorem 2.2 (Theorem 2.3), by choosing $\varphi(t) = t,\ \beta(t) = k$ where $k \in [0,1)$ and $g = I_X$, the identity map of X in Theorem 2.1 (respectively Theorem 2.2 and Theorem 2.3).

Remark 2.7: Theorem 1.13 follows as a corollary to Theorem 2.1 and Theorem 1.14 follows as a corollary to Theorem 2.2 (Theorem 2.3), by choosing $\varphi(t) = t$ and $g = I_X$, the identity map of X in Theorem 2.1 (respectively Theorem 2.2 and Theorem 2.3).

Therefore, by Example 2.5, Remark 2.6 and Remark 2.7, it follows that Theorem 2.3 is a generalization of both the theorems Theorem 1.10 and Theorem 1.14.

The following theorem gives the criteria for the existence of unique coupled common fixed point.

Theorem 2.8: In addition to the hypotheses of Theorem 2.1 (respectively Theorem 2.2 and Theorem 2.3) suppose that for every (x,y) and (z,t) in $X \times X$, there exists (u,v) in $X \times X$ such that $(F(u,v), F(v,u))$ is comparable to $(F(x,y), F(y,x))$ and $(F(z,t), F(t,z))$ and if g is one-one, then F and g have a unique coupled common fixed point. $i.e.$, there exists a unique $(x,y) \in X \times X$ such that $x = gx = F(x,y)$ and $y = gy = F(y,x)$.

Proof:

From Theorem 2.1 (respectively Theorem 2.2 and Theorem 2.3), the set of coupled coincidence points of F and g is nonempty.

Suppose that (x,y) and (z,t) are two coupled coincidence points of F and g. *i.e.,* $gx = F(x,y),\ gy = F(y,x)$ and $gz = F(z,t),\ gt = F(t,z)$.

Now, we prove that $gx = gz$ and $gy = gt$.

Let $(u,v) \in X \times X$ such that $(F(u,v), F(v,u))$ is comparable to $(F(x,y), F(y,x))$ and $(F(z,t), F(t,z))$.

Now, we construct sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$u_0 = u,\ v_0 = v,\ gu_{n+1} = F(u_n, v_n),\ gv_{n+1} = F(v_n,\ u_n). \tag{2.8.1}$$

Further, we set $x_0 = x,\ y_0 = y,\ z_0 = z,\ t_0 = t$ and define sequences $\{gx_n\},\ \{gy_n\},\ \{gz_n\},\ \{gt_n\}$ by

$$gx_{n+1} = F(x_n, y_n),\ gy_{n+1} = F(y_n,\ x_n)$$
$$gz_{n+1} = F(z_n, t_n),\ gt_{n+1} = F(t_n,\ z_n).$$

Since $(F(x,y), F(y,x)) = (gx, gy)$ and $(F(u,v), F(v,u)) = (gu_1, gv_1)$ are comparable, without loss of generality, we assume that

$$gx \preceq gu_1 \text{ and } gy \succeq gv_1. \tag{2.8.2}$$

Now, we prove that

$$gx \preceq gu_n \text{ and } gy \succeq gv_n,\ \forall n \geq 0. \tag{2.8.3}$$

From (2.8.2), clearly (2.8.3) is true for $n = 1$.

Assume that (2.8.3) is true for some n.

Now,
$gu_{n+1} = F(u_n, v_n) \succeq F(x,y) = gx$
and
$gv_{n+1} = F(v_n, u_n) \preceq F(y,x) = gy$.
Thus by mathematical induction (2.7.3) is true for all n =1, 2, 3, ...
Since $gx \preceq gu_{n-1}$ and $gy \succeq gv_{n-1}$, from (2.1.1), we have

$$d(gx, gu_n)) = d(F(x,y), F(u_{n-1}, v_{n-1}))$$
$$\leq \beta(\tfrac{d(gx,gu_{n-1})+d(gy,gv_{n-1})}{2})\varphi(\tfrac{d(gx,gu_{n-1})+d(gy,gv_{n-1})}{2}). \tag{2.8.4}$$

Similarly

$$d(gy, gv_n)) = d(F(y,x), F(v_{n-1}, u_{n-1}))$$
$$\leq \beta(\tfrac{d(gy,gv_{n-1})+d(gx,gu_{n-1})}{2})\varphi(\tfrac{d(gy,gv_{n-1})+d(gx,gu_{n-1})}{2}). \tag{2.8.5}$$

Now adding (2.8.4) and (2.8.5) we get

$$d(gx, gu_n) + d(gy, gv_n))$$
$$\leq 2\beta(\tfrac{d(gx,gu_{n-1})+d(gy,gv_{n-1})}{2})\varphi(\tfrac{d(gx,gu_{n-1})+d(gy,gv_{n-1})}{2})$$
$$< d(gx, gu_{n-1}) + d(gy, gv_{n-1}).$$

$i.e.$, $\{d(gx, gu_n) + d(gy, gv_n)\}$ is a decreasing sequence, and hence converges to r(say), $r \geq 0$.

Now, we prove that $r = 0$. Assume that $r > 0$.

From (2.8.4) and (2.8.5), we have

$$d(gx, gu_n) + d(gy, gv_n))$$
$$\leq \beta\big(\tfrac{d(gx,gu_{n-1})+d(gy,gv_{n-1})}{2}\big)\big(d(gx, gu_{n-1}) + d(gy, gv_{n-1})\big)$$
$$< d(gx, gu_{n-1}) + d(gy, gv_{n-1}).$$

On taking limits as $n \to \infty$, we get

$$r \leq \lim_{n\to\infty} \beta\Big(\frac{d(gx, gu_{n-1}) + d(gy, gv_{n-1})}{2}\Big) r \leq r.$$

Hence,

$$\lim_{n\to\infty} \beta\Big(\frac{d(gx, gu_{n-1}) + d(gy, gv_{n-1})}{2}\Big) = 1.$$

Thus, $d(gx, gu_{n-1}) + d(gy, gv_{n-1}) \to 0$ as $n \to \infty$, since $\beta \in S$.

Hence $\lim_{n\to\infty} d(gx, u_n) = 0$ and $\lim_{n\to\infty} d(gy, v_n) = 0$.

Similarly, we get $d(gz, u_n) \to 0$ and $d(gt, v_n) \to 0$ as $n \to \infty$.

By uniqueness of the limit, we get $gx = gz$ and $gy = gt$.

Since g is one-one, we have $x = z$ and $y = t$.

Thus the coupled coincidence point of F and g is unique.

Hence, the set of all coupled coincidence points of F and g is nonempty and it is singleton. Suppose that it is (x, y).

For argumental simplicity, take $u = gx$ and $v = gy$.

Since F and g are compatibility, we have $g(F(x, y)) = F(gx, gy)$ and $g(F(y, x)) = F(gy, gx)$.

Hence,

$gu = ggx = g(F(x, y)) = F(gx, gy) = F(u, v)$

and

$gv = ggy = g(F(y, x)) = F(gy, gx) = F(v, u)$.

Hence, (u, v) is a coupled coincidence point.

But the coupled coincidence point is unique, hence $x = u$ and $y = v$.

Hence $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

i.e., (x, y) is a coupled fixed point of F and g.

Thus F and g have a unique coupled common fixed point, since F and g have unique coupled fixed point.

This completes the proof. □

Example 2.9: Let $X = [0, 1]$ with the usual metric and the usual order.

Clearly $(X, \leq)$ is a partially ordered set and $(X,\ d)$ is a complete metric space.

We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x^2-y^2}{8} & \text{if } x \geq y \\ 0 & \text{if } \ x < y; \quad x, y \in X \end{cases}$$

Also, we define $g : X \to X$ by $g(x) = x^2,\ x \in X$.

Here F and g are continuous, g is non-decreasing and F and g are compatible maps.

$\varphi : [0,\infty) \to [0,\infty)$ is defined by

$$\varphi(t) = \frac{3}{2} t, \forall t \geq 0.$$

$\beta : [0,\infty) \to [0, 1)$ is defined by

$$\beta(t) = \begin{cases} \frac{1}{1+t} & if\ t > 0 \\ 0 & if\ t = 0. \end{cases}$$

Then $\beta \in S$.

Also, we choose $x_0 = 0$ and $y_0 = c (> 0)$ in X.

Then, $gx_0 = g0 = 0 = F(0, c) = F(x_0, y_0)$ and $gy_0 = gc = c^2 \geq \frac{c^2}{8} = F(c, 0) = F(y_0, x_0)$.

Let $gx \geq gu,\ gy \leq gv$. Now, we verify the inequality (2.1.1) in the following cases:

Case (i) : $x \geq y$ and $u \geq v$.

In this case,

$$\begin{aligned} d(F(x, y), F(u, v)) &= \tfrac{1}{8}\big|(x^2 - y^2) - (u^2 - v^2)\big| \\ &\leq \tfrac{1}{8}[(x^2 - u^2) + (v^2 - y^2)]. \end{aligned}$$

and

$$\beta\big(\tfrac{d(gx,gu)+d(gy,gv)}{2}\big)\varphi\big(\tfrac{d(gx,gu)+d(gy,gv)}{2}\big)$$
$$= \tfrac{1}{1+[(x^2-u^2)+(v^2-y^2)]} \tfrac{3}{2} \tfrac{[(x^2-u^2)+(v^2-y^2)]}{2}.$$

Let us take $x^2 - u^2 = t_1$ and $v^2 - y^2 = t_2$.

Since $t_1 + t_2 \leq 5, \quad t_1,\ t_2 \in X$, hence

$\frac{1}{8}[t_1 + t_2] \leq \frac{1}{1+t_1+t_2} \frac{3}{2} \frac{t_1+t_2}{2}$ so that inequality (2.1.1) holds.

Case (ii) : $x \geq y$ and $u \leq v$.

Here

$$\begin{aligned} d(F(x, y), F(u, v)) &= \tfrac{1}{8}(x^2 - y^2) \\ &= \tfrac{1}{8}(x^2 - u^2 + u^2 - y^2) \\ &\leq \tfrac{1}{8}(x^2 - u^2 + v^2 - y^2), \text{ since } u \leq v. \end{aligned}$$

Now, proceeding as in Case (i), the inequality (2.1.1) satisfied.

Inequality (2.1.1) clearly holds in other cases also, $(i.e.,\ x < y,\ u \geq y$, and $x < y,\ u, v)$.

Hence all the hypotheses of Theorem (2.8) are satisfied and $(0, 0)$ is a coupled coincidence point.

In fact, $(0, 0)$ is the unique coupled common fixed point. □

III. Conclusion

In this paper, we proved the existence of coupled coincidence points for a pair of compatible maps satisfying Geraghty contraction in ordered metric spaces with different hypotheses (Theorem 2.1, Theorem 2.2 and Theorem 2.3).

(i) Example 2.5 and Remark 2.6 suggest that Theorem 2.3 is a generalization of Theorem1.10;
(ii) Example 2.5 and Remark 2.7 suggest that Theorem 2.3 is a generalization of Theorem1.14;
(iii) Theorem 1.9, Theorem 1.13 follow as corollaries to Theorem 2.1.

Existence of unique coupled common fixed point is proved in Theorem 2.8. Examples are provided to illustrate our results.

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