Coupled Common Fixed Points for a Pair of Compatible Maps satisfying Geraghty Contraction in Partially Ordered Metric Spaces

G. V. R. Babu* and P. Subhashini**

Abstract—In this paper, we prove the existence of coupled coincidence and coupled common fixed points for a pair of compatible maps satisfying Geraghty contraction in partially ordered complete metric spaces. Our results generalize the results of Bhaskar and Lakshmikantham (2006) and Choudhury and Kundu (2012). Examples are provided to illustrate our results.

Index Terms—Partially ordered set, mixed \(g\)-monotone property, coupled coincidence point, coupled common fixed point.

MSC 2010 Codes – 47H10, 54H25.

I. INTRODUCTION


In 2006, Bhaskar and Lakshmikantham [3] established the existence of coupled fixed points for mixed monotone operators in partially ordered metric spaces.


In 2009, Lakshmikantham and Ćirić [6] introduced the concept of commuting maps in the context of coupled fixed points and proved coupled coincidence and coupled common fixed points in partially ordered metric spaces.

In 2010, Choudhury and Kundu [7] generalized the concept of commuting maps in the context of coupled fixed points by introducing compatible maps and established the existence of coupled coincidence points in partially ordered metric spaces.

Many researchers, namely Choudhury, Metiya and Kundu [8]; Alsulami, Alotaibi[9]; Berinde[10]; Harjani, Lopez, Sadarangani[11]; Hemant Kumar, Shatanawi[12]; Karapınar, Luong, Thuan[13]; Luong, Thuan[14]; Luong, Thuan, Hai[15]; Sabeghadam, Maslha[16] worked in this direction.

Throughout this paper, \(R\) denotes the set of all real numbers and \(R^+ = [0, \infty)\).

\*G. V. R. Babu is with the Department of Mathematics, Andhra University, Visakhapatnam-530 003, INDIA. e-mail: gvr_babu@hotmail.com

** P. Subhashini is with the Department of Mathematics, Andhra University, Visakhapatnam-530 003, INDIA. Permanent address: Department of Mathematics, P. R. Govt. College (A), Kakinada - 533 001, India. e-mail: subhashini.pandu@rediffmail.com

Definition 1.1: [3] Let \((X, \preceq)\) be a partially ordered set and \(F : X \times X \to X\) be a map. We say that \(F\) has the mixed monotone property if \(F(x, y)\) is non-decreasing in \(x\) and is non-increasing in \(y\). i.e, for any \(x, y \in X\),

\[
x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),
\]

\[
y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).
\]

\[\square\]

Definition 1.2: [3] Let \(X\) be a non-empty set and \(F : X \times X \to X\) be a map. An element \((x, y) \in X \times X\) is said to be a coupled fixed point of \(F\) if \(F(x, y) = x\) and \(F(y, x) = y\).

Definition 1.3: [6] Let \((X, \preceq)\) be a partially ordered set and \(F : X \times X \to X\) and \(g : X \to X\) be two maps. We say that \(F\) has the mixed \(g\)-monotone property if for any \(x, y \in X\),

\[
x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y),
\]

\[
y_1, y_2 \in X, \quad g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2).
\]

\[\square\]

Definition 1.4: [6] Let \(X\) be a non-empty set and \(F : X \times X \to X\) and \(g : X \to X\) be two maps. An element \((x, y) \in X \times X\) is said to be a coupled coincidence point of \(F\) and \(g\) if

\[
F(x, y) = g(x) \quad \text{and} \quad F(y, x) = g(y)
\]

and \((x, y)\) is said to be a coupled common fixed point of \(F\) and \(g\) if

\[
F(x, y) = g(x) = x \quad \text{and} \quad F(y, x) = g(y) = y.
\]

\[\square\]

Definition 1.5: [6] Let \(X\) be a non-empty set and \(F : X \times X \to X\) and \(g : X \to X\) be two maps. We say that \(F\) and \(g\) are commutative if

\[
g(F(x, y)) = F(g(x), g(y))\]

for all \(x, y \in X\).

\[\square\]

Definition 1.6: [7] Let \((X, d)\) be a metric space. The mappings

\[
F : X \times X \to X \quad \text{and} \quad g : X \to X
\]

are said to be compatible if

\[
\lim_{n \to \infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0
\]
and
\[
\lim_{n \to \infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0,
\]
whenever \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(X\) such that
\[
\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} g(x_n) = x
\]
and
\[
\lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} g(y_n) = y
\]
for all \(x, y \in X\). □

Remark 1.7: Let \((X, d)\) be a metric space. If the mappings \(F : X \times X \to X\) and \(g : X \to X\) are commutative then \(F\) and \(g\) are compatible but its converse need not be true.

Example 1.8: Let \(X = \mathbb{R}\) with the usual metric.

We define \(F : X \times X \to X\) by
\[
F(x, y) = x^2 - y^2, \quad x, y \in X
\]
and \(g : X \to X\) by
\[
g(x) = x^2, \quad x \in X.
\]
Let \(\{x_n\}\) and \(\{y_n\}\) be two sequences in \(X\) such that
\[
\lim_{n \to \infty} x_n = l_1 \quad \text{and} \quad \lim_{n \to \infty} x_n = l_2.
\]
Let
\[
\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} g(x_n)
\]
and
\[
\lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} g(y_n).
\]
Then it follows that \(l_1 = 0\) and \(l_2 = 0\) so that
\[
d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) \to 0 \quad \text{as} \quad n \to \infty
\]
and
\[
d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence, the mappings \(F\) and \(g\) are compatible in \(X\).

Now, \(F(gx, gy) = x^4 - y^4\) and \(g(F(x, y)) = (x^2 - y^2)^2\). Hence \(F(gx, gy) \neq g(F(x, y))\) for all \(x, y \in X\). Thus \(F\) and \(g\) are not commutative. □

The following are the main theorems of Bhaskar and Lakshmikantham [3].

Theorem 1.9: [3] Let \((X, \preceq)\) be a partially ordered set and suppose there exists a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be a continuous map having the mixed monotone property on \(X\). Assume that there exists \(k \in [0, 1)\) with
\[
d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v)) \tag{1.9.1}
\]
for \(x, y, u, v \in X\) with \(x \preceq u, y \preceq v\).

If there exist \(x_0, y_0 \in X\) such that \(x_0 \preceq F(x_0, y_0)\) and \(y_0 \preceq F(y_0, x_0)\), then there exist \(x, y \in X\) such that \(x = F(x, y)\) and \(y = F(y, x)\). □

Theorem 1.10: [3] Let \((X, \preceq)\) be a partially ordered set and suppose there exists a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be a mapping having the mixed monotone property on \(X\). Assume that there exists \(k \in [0, 1)\) with
\[
d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v)) \tag{1.10.1}
\]
for \(x, y, u, v \in X\) with \(x \preceq u, y \preceq v\).

Further assume that \(X\) has the following properties:

(i) if \(\{x_n\}\) is a non-decreasing sequence in \(X\) with \(x_n \to x\), then \(x_n \preceq x\) for all \(n = 1, 2, 3, \ldots\);
(ii) if \(\{y_n\}\) is a non-increasing sequence in \(X\) with \(y_n \to y\), then \(y_n \preceq y\) for all \(n = 1, 2, 3, \ldots\)

If there exist \(x_0, y_0 \in X\) such that \(x_0 \preceq F(x_0, y_0)\) and \(y_0 \preceq F(y_0, x_0)\), then there exist \(x, y \in X\) such that
\[
x = F(x, y) \quad \text{and} \quad y = F(y, x). \quad \square
\]
\[ \text{Theorem 1.14:} \quad [19] \text{Let} \ (X, \preceq) \text{be a partially ordered set and suppose that there exists a metric} \ d \text{ on} \ X \text{ such that} \ (X, d) \text{ is a complete metric space. Let} \ F : X \times X \to X \text{ be a mapping having the mixed monotone property and there exists} \ \beta \in S \text{ such that} \]
\[ d(F(x,y), F(u,v)) \leq \beta \left( \frac{d(x,u) + d(y,v)}{2} \right) \left( \frac{d(x,u) + d(v,y)}{2} \right) \]
\[ (1.14.1) \]
\text{for} \ x, y, u, v \in X \text{ with} \ x \succeq u, \ y \preceq v. \text{ Also suppose that} X \text{ has the following properties:}
\begin{enumerate}
\item if \ \{x_n\} \text{ is a non-decreasing sequence in} X \text{ with} \ x_n \to x, \text{ then} \ x_n \preceq x \text{ for all} \ n = 1, 2, 3, \ldots ;
\item if \ \{y_n\} \text{ is a non-increasing sequence in} X \text{ with} \ y_n \to y, \text{ then} \ y_n \succeq y \text{ for all} \ n = 1, 2, 3, \ldots .
\end{enumerate}
\text{If there exist} \ x_0, y_0 \in X \text{ such that}
\[ x_0 \preceq F(x_0, y_0) \text{ and} \ y_0 \succeq F(y_0, x_0), \]
\text{then} \ F \text{ has a coupled fixed point in} X, \text{ extit i.e., there exist}
\[ x, y \in X \text{ such that} \ x = F(x,y) \text{ and} \ y = F(y,x). \square \]
\text{In this paper we prove the existence of coupled coincidence and coupled common fixed points for a pair of compatible maps satisfying Geraghty contraction in partially ordered complete metric spaces. Our results generalize the results of Bhaskar, Lakshmikantham [3] and Choudhury, Kundu [19].}

\text{In the following, we write} \ \Phi = \{ \psi / \psi : R^+ \to R^+ \text{ satisfying} \ \psi(t) \leq t \text{ for} \ t > 0 \text{ and} \ \psi(t) = 0 \text{ if and only if} \ t = 0 \}. \]

\text{II. MAIN RESULTS}

\[ \text{Theorem 2.1:} \text{Let} \ (X, \preceq) \text{be a partially ordered set and suppose that there exists a metric} \ d \text{ on} \ X \text{ such that} \ (X, d) \text{ is a complete metric space. Let} \ F : X \times X \to X, \ g : X \to X \text{ be two mappings such that} F \text{ is a continuous map having the mixed} \ g-\text{monotone property on} \ X. \text{ Assume that there exists} \ \beta \in S \text{ and} \ \psi \in \Phi \text{ such that} \]
\[ d(F(x,y), F(u,v)) \leq \beta \left( \frac{d(gx,gu) + d(gy,gv)}{2} \right) \cdot \psi \left( \frac{d(gx,gu) + d(gy,gv)}{2} \right) \]
\[ (2.1.1) \]
\text{for} \ x, y, u, v \in X \text{ with} \ gx \succeq gu, \ gy \preceq gv.
\text{Suppose that} F(X \times X) \subseteq g(X), \text{ g is continuous,} \ F \text{ and} \ g \text{ are compatible maps.}
\text{If there exist} \ x_0, y_0 \in X \text{ such that} \ gx_0 \preceq F(x_0, y_0) \text{ and} \ y_0 \succeq F(y_0, x_0) \text{ then} \ F \text{ and} \ g \text{ have a coupled coincidence point in} X, \text{ i.e., there exist} \ x, y \in X \text{ such that} gx = F(x, y) \text{ and} \ gy = F(y, x). \]
\[ \text{Proof.} \text{ Let} \ x_0, y_0 \in X \text{ with} \]
\[ gx_0 \preceq F(x_0, y_0) \text{ and} \ y_0 \succeq F(y_0, x_0). \]
\text{If} \ gx_0 = F(x_0, y_0) \text{ and} \ y_0 = F(y_0, x_0) \text{ then} \ (x_0, y_0) \text{ is a coupled coincidence point of} \ F \text{ and we are through.}
\text{Otherwise, since} F(X \times X) \subseteq g(X) \text{ we define sequences} \ \{gx_n\} \text{ and} \ \{gy_n\} \text{ in} X \text{ such that} \ gx_{n+1} = F(x_n, y_n) \text{ and} \ gy_{n+1} = F(y_n, x_n) \text{ for all} \ n = 0, 1, 2, \ldots .
\text{Now, we show that} \ \{gx_n\} \text{ is an increasing sequence and} \ \{gy_n\} \text{ is a decreasing sequence, i.e.,}
\[ gx_n \preceq x_{n+1} \text{ and} \ gy_n \succeq y_{n+1} \forall n = 0, 1, \ldots \]

\[ (2.1.2) \]
The proof is by mathematical induction.
\text{We have} \ gx_0 \preceq F(x_0, y_0) \text{ and} \ y_0 \succeq F(y_0, x_0).
\text{Hence} \ gx_0 \preceq gx_1 \text{ and} \ y_0 \succeq gy_1 \text{ so that} (2.1.2) \text{ is true for} \ n = 0.
\text{Assume that} (2.1.2) \text{ is true for some} n. \text{ By using the mixed} \ g-\text{monotone property of} \ F \text{, we have}
\[ gx_{n+2} = F(x_{n+1}, y_{n+1}) \preceq F(x_n, y_n) = gx_{n+1} \]
\text{and}
\[ gy_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_n) = gy_{n+1}. \]
\text{Hence} (2.1.2) \text{ is true for} \ n + 1. \text{ Thus by the mathematical induction, (2.1.2) follows.}
\text{Let us denote} \ \delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}).
\text{Case (i): Suppose that} gx_n = gx_{n+1} \text{ and} gy_n = gy_{n+1} \text{ for some} n. \text{ i.e.,}
\[ gx_{n+1} = F(x_n, y_n) = gx_n \]
\text{and}
\[ gy_{n+1} = F(y_n, x_n) = gy_n. \]
\text{Hence} \ (x_n, y_n) \text{ is a coupled coincidence point.}
\text{Case (ii): Suppose that} gx_n \neq gx_{n+1} \text{ or} gy_n \neq gy_{n+1} \text{ for all} \ n.
\text{We have}
\[ gx_n \preceq gx_{n+1} \text{ and} \ gy_n \succeq gy_{n+1}, \forall n = 0, 1, 2, \ldots . \]
\text{Now,}
\[ d(gx_{n+1}, gx_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq \beta \left( \frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2} \right) \]
\[ < d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}). \]
\text{Again,}
\[ d(gy_{n+1}, gy_n) = d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \leq \beta \left( \frac{d(gy_n, gy_{n+1}) + d(gx_n, gx_{n+1})}{2} \right) \]
\[ < d(gy_n, gy_{n-1}) + d(gx_n, gx_{n-1}). \]
\text{Adding} (2.1.3) \text{ and} (2.1.4), \text{ we have}
\[ d(gx_{n+1}, gx_n) + d(gy_n, gy_{n+1}) < d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n). \]
\text{Hence} \ \{\delta_n\} \text{ is non-negative decreasing sequence of reals and hence it converges to a real number} r(\text{say}), r \geq 0.
\text{If possible, we assume that} \ r > 0.
\text{From} (2.1.3) \text{ and} (2.1.4), \text{ we have}
\[ d(gx_{n+1}, gx_n) + d(gy_n, gy_{n+1}) < 2\beta \left( \frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2} \right) \]
\[ \leq \beta \left( \frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2} \right) \]
\[ < d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n). \]
\text{On taking limits as} \ n \to \infty, \text{ we get}
\[ r \leq \lim_{n \to \infty} \beta \left( \frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2} \right) r \leq r. \]
\text{Hence}
\[ \lim_{n \to \infty} \beta \left( \frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2} \right) = 1. \]
Since $\beta \in S$, we have
\[ d(gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n) \to 0, \text{ as } n \to \infty, \]
which is a contradiction for $r > 0$.

Hence, $r = 0$.

Next, we prove that \{gx_n\} and \{gy_n\} are Cauchy sequences.

Otherwise, assume that at least one of the sequences \{gx_n\} and \{gy_n\} is not a Cauchy sequence.

Then there exists an $\epsilon > 0$ for which we can find subsequences \{m(k)\} and \{n(k)\} of positive integers with $n(k) > m(k)$ such that
\[ d_k = d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) \geq \epsilon. \quad (2.1.5) \]

Further, we choose $n(k)$ to be the smallest positive integer such that $n(k) > m(k)$ satisfying (2.1.5). Hence $d_k \geq \epsilon$ and
\[ d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) < \epsilon. \quad (2.1.6) \]

From the triangle inequality and (2.1.6), we have
\[ \epsilon \leq d_k \leq d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) \]
\[ < \epsilon + d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}). \quad (2.1.7) \]

Now, we prove that $\lim_{k \to \infty} d_k = \epsilon$.

On taking limit superior as $k \to \infty$ in (2.1.7), we get
\[ \epsilon \leq \lim sup_{k \to \infty} d_k \leq \epsilon. \]

Hence,
\[ \lim sup_{k \to \infty} d_k = \epsilon. \quad (2.1.8) \]

Again, on taking limit inferior as $k \to \infty$ in (2.1.5) and from (2.1.8), we get $\epsilon \leq \lim inf_{k \to \infty} d_k \leq \lim sup_{k \to \infty} d_k = \epsilon$. Hence
\[ \lim inf_{k \to \infty} d_k = \epsilon. \quad (2.1.9) \]

Thus, from (2.1.8) and (2.1.9), $\lim_{k \to \infty} d_k$ exists and
\[ \lim_{k \to \infty} d_k = \epsilon. \quad (2.1.10) \]

Again, for all $k \geq 0$, we have
\[ d_k = d(gx_{m(k)}, gx_n(k)) + d(gy_{m(k)}, gy_n(k)) \]
\[ \leq d(gx_{m(k)}, gx_{n(k)+1}) + d(gx_{m(k)+1}, gx_{n(k)+1}) \]
\[ + d(gx_{n(k)+1}, gx_n(k)) + d(gy_{m(k)}, gy_{n(k)+1}) \]
\[ + d(gy_{m(k)+1}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_n(k)). \quad (2.1.11) \]

Now, since $gx_{m(k)} \preceq gx_n(k)$ and $gy_{m(k)} \preceq gy_n(k)$, we have
\[ d(gx_{m(k)+1}, gx_{n(k)+1}) = d(F(x_{m(k)}, y_m(n)), F(x_{n(k)}, y_n(n))) \]
\[ \leq \beta(d(gx_{m(k)}, gx_n(k)) + d(gy_{m(k)}, gy_n(k))) \]
\[ = \beta\left(\frac{d(gx_{m(k)}, gx_n(k)) + d(gy_{m(k)}, gy_n(k))}{2}\right) \]
\[ \leq \beta\left(\frac{d_k}{2}\right) d_k. \quad (2.1.12) \]

Similarly, we can prove that
\[ d(gy_{m(k)+1}, gy_{n(k)+1}) \leq \beta\left(\frac{d_k}{2}\right) d_k. \quad (2.1.13) \]

Adding (2.1.12) and (2.1.13), we get
\[ d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1}) \leq \beta\left(\frac{d_k}{2}\right) d_k. \]

Now, from (2.1.11)
\[ d_k \leq d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1}) \leq \beta\left(\frac{d_k}{2}\right) d_k. \]

On taking limits as $k \to \infty$ and by using (2.1.10), we get
\[ \epsilon \leq \lim_{k \to \infty} \beta\left(\frac{d_k}{2}\right) \leq \epsilon. \]

Hence,
\[ \lim_{k \to \infty} \beta\left(\frac{d_k}{2}\right) = 1. \]

Therefore, $d_k \to 0$ as $k \to \infty$, a contradiction to (2.1.10).

Hence, \{gx_n\} and \{gy_n\} are Cauchy sequences in $X$.

Since $X$ is complete, there exist $x, y \in X$ such that
\[ \lim_{n \to \infty} gx_n = x \quad \text{and} \quad \lim_{n \to \infty} gy_n = y. \]

We have
\[ x = \lim_{n \to \infty} gx_{n+1} = \lim_{n \to \infty} F(x_n, y_n) \]
\[ \text{and} \]
\[ y = \lim_{n \to \infty} gy_{n+1} = \lim_{n \to \infty} F(y_n, x_n). \]

Now, since $F$ and $g$ are compatible, we have
\[ \lim_{n \to \infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0 \quad (2.1.14) \]
\[ \text{and} \]
\[ \lim_{n \to \infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0. \quad (2.1.15) \]

Now, we prove that $gx = F(x, y)$ and $gy = F(y, x)$.

For all $n \geq 0$, we have
\[ d(gx, F(gx_n, gy_n)) \leq d(gx, F(x_n, y_n)) \]
\[ + d(F(x_n, y_n), F(gx_n, gy_n)). \]

On taking limits as $n \to \infty$ and from (2.1.14) we get
\[ d(gx, F(x, y)) = 0, \]

since $F$ and $g$ are continuous.

Similarly it is easy to see that
\[ d(gy, F(y, x)) = 0. \]

Hence $gx = F(x, y)$ and $gy = F(y, x)$.

Thus $(x, y)$ is a coupled coincidence point of $F$ and $g$.

This completes the proof of the theorem. $\square$

**Theorem 2.2:** Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$, $g : X \to X$ be two mappings such that $F$ has the mixed $g-$monotone property on $X$. Assume that there exists $\beta \in S$ and $\varphi \in \Phi$ such that
\[ d(F(x, y), F(u, v)) \leq \beta\left(\frac{d(xu, gu) + d(yv, gv)}{2}\right) \]
\[ \varphi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right) \]
for $x, y, u, v \in X$ with $gx \succeq gu$, $gy \succeq gv$. \hspace{1cm} (2.2.1)
Suppose that $F(X \times X) \subseteq g(X)$, $g$ is continuous, $g$ is non-decreasing and $F$ and $g$ are compatible maps. Further, assume that:

(a) if $\{x_n\}$ is a non-decreasing sequence in $X$ with $x_n \to x$, then $x_n \leq x$ for all $n = 1, 2, 3, \ldots$;

(b) if $\{y_n\}$ is a non-increasing sequence in $X$ with $y_n \to y$, then $y_n \geq y$ for all $n = 1, 2, 3, \ldots$.

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \preceq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point in $X$, i.e., there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

**Proof:** From Theorem 2.1, we have

$$\lim_{n \to \infty} gx_n = x \quad \text{and} \quad \lim_{n \to \infty} gy_n = y.$$ Hence, from the hypotheses (a) and (b), we have $gx_n \leq x$ and $gy_n \geq y$, $\forall n = 1, 2, 3, \ldots$.

Since $g$ is non-decreasing, we have $ggx_n \leq gx$ and $ggx_n \geq gy$, $\forall n = 1, 2, 3, \ldots$.

Since $g$ is continuous and $F$ and $g$ are compatible mappings, we have

$$\lim_{n \to \infty} gx_n = gx = \lim_{n \to \infty} gF(x_n, y_n) = \lim_{n \to \infty} F(gx_n, gy_n)$$

and

$$\lim_{n \to \infty} gy_n = gy = \lim_{n \to \infty} gF(y_n, x_n) = \lim_{n \to \infty} F(gy_n, gx_n).$$

Now, we have

$$d(gx, F(x, y)) \leq d(gx, gx_{n+1}) + d(gy_{n+1}, F(x, y)).$$

On taking limits as $n \to \infty$, we get

$$d(gx, F(x, y)) \leq \lim_{n \to \infty} d(gx, gx_{n+1}) + \lim_{n \to \infty} d(F(x_n, y_n), F(x, y))$$

$$= \lim_{n \to \infty} d(F(x_n, y_n), F(x, y)).$$

(2.2.2)

Now from (2.2.1)

$$d(F(x, y), F(gx_n, gy_n))$$

$$\leq \beta \left( \frac{d(gx, gx_n) + d(\varphi(\gamma, gy_n))}{2} \right) \varphi \left( \frac{d(gx, gx_n) + d(\varphi(\gamma, gy_n))}{2} \right)$$

$$< \frac{d(gx, gx_n) + d(\varphi(\gamma, gy_n))}{2}.$$ On taking limits $n \to \infty$, we get

$$\lim_{n \to \infty} d(F(x, y), F(gx_n, gy_n)) = 0.$$

Hence from (2.2.2) we have $gx = F(x, y)$. Similarly, we can prove that $gy = F(y, x)$.

Thus $(x, y)$ is a coupled coincidence point of $F$ and $g$.

This completes the proof. \(\Box\)

We relax the conditions continuity and non-decreasing property of $g$, compatibility of $F$ and $g$ and completeness of $X$ in Theorem 2.2 by assuming $g(X)$ is a complete subspace of $X$, and prove the following theorem.

**Theorem 2.3:** Let $(X, \preceq, d)$ be a partially ordered metric space. Let $F : X \times X \to X$, $g : X \to X$ and $F$ has the mixed $g$-monotone property and $F(X \times X) \subseteq g(X)$, $g(X)$ is a complete subspace of $X$. Assume that there exists $\beta \in S$ and $\varphi \in \Phi$ such that $d(F(x, y), F(u, v)) \leq \beta \left( \frac{d(gx, gy) + d(gu, gv)}{2} \right)$

$$\varphi \left( \frac{d(gx, gy) + d(gu, gv)}{2} \right)$$

for $x, u, v, y \in X$ with $gx \geq gu$, $gy \leq gv$. (2.3.1)

Further assume that $X$ has the following property:

(a) if $\{x_n\}$ is a non-decreasing sequence in $X$ with $x_n \to x$, then $x_n \leq x$ for all $n = 1, 2, 3, \ldots$;

(b) if $\{y_n\}$ is a non-increasing sequence in $X$ with $y_n \to y$, then $y_n \geq y$ for all $n = 1, 2, 3, \ldots$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \preceq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, i.e., $F$ and $g$ have a coupled coincidence point.

**Proof:** On proceeding as in proof of Theorem 2.1, the sequences $\{gx_n\}$ and $\{gy_n\}$ defined by

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n)$$

are Cauchy in $g(X)$.

Since $g(X)$ is complete, there exist $x, y \in X$ such that

$$\lim_{n \to \infty} gx_n = gx$$

and

$$\lim_{n \to \infty} gy_n = gy.$$ Hence, by our assumptions (a) and (b), we have $gx_n \preceq gx$ and $gy_n \preceq gy$ for all $n \geq 0$.

Now, we prove that $gx = F(x, y)$ and $gy = F(y, x)$.

We consider $d(gx_{n+1}, F(x, y)) = d(F(x_n, y_n), F(x, y))$

$$\leq \beta \left( \frac{d(gx_n, gx_n) + d(\varphi(\gamma, gy_n))}{2} \right) \varphi \left( \frac{d(gx_n, gx_n) + d(\varphi(\gamma, gy_n))}{2} \right)$$

$$< \frac{d(gx_n, gx_n) + d(\varphi(\gamma, gy_n))}{2}.$$ (2.3.2)

On taking limits as $n \to \infty$, in (2.3.2), we get $d(gx, F(x, y)) = 0$.

Similarly, we get $d(gy, F(y, x)) = 0$.

Hence, $F(x, y) = gx$ and $F(y, x) = gy$.

Thus, $(x, y)$ is a coupled coincidence point.

This completes the proof. \(\Box\)

**Remark 2.4:** Theorem 2.3 (Theorem 2.1 and Theorem 2.2) does not guarantee the uniqueness of the coupled fixed point.

Now we present an example in support of this statement. \(\Box\)

**Example 2.5:** Let $X = \{0, \frac{1}{2}, 2\}$ with the usual metric.

We consider the following relation on $X$.

For any $x, y \in X$, $x \preceq y \iff x = y$ or $\left(0, \frac{1}{2}\right)$.

Clearly $(X, d)$ is a complete metric space and $(X, \preceq)$ is a partially ordered set.

We have

$$\preceq := \{(0, 0), \left(\frac{1}{2}, \frac{1}{2}\right), (2, 2), (0, \frac{1}{2})\}.$$ Set

$$A = \{(0, 0), (0, \frac{1}{2}), (0, 2), (2, 0), (2, 2), (2, \frac{1}{2})\}$$

$$B = \{\left(\frac{1}{2}, 0\right), (\frac{1}{2}, 2)\}$$

$$C = \{\left(\frac{1}{2}, \frac{1}{2}\right)\}.$$
We define $F : X \times X \to X$ by
\[
F(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in A \\
\frac{1}{2} & \text{if } (x, y) \in B \\
2 & \text{if } (x, y) \in C
\end{cases}
\]
and $g : X \to X$ by
\[
g0 = 0, \quad g\frac{1}{2} = 2, \quad g2 = \frac{1}{2}.
\]

clearly, $F$ has the mixed $g$-monotone property and $g(X)$ is a complete subspace of $X$ and properties (a), (b) of Theorem 2.3 hold. By choosing $x_0 = 0$, $y_0 = 2$ we have
\[
x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \preceq F(y_0, x_0).
\]
The inequality (2.3.1) clearly holds, since for every $(x, y), (u, v) \in X \times X$ whenever $gx \succeq gu$ and $gy \preceq gv$, we have $d(F(x, y), F(u, v)) = 0$.

So $F$ satisfies all the hypotheses of Theorem 2.3 and $F, g$ have two coincidence points $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ even though $X$ is complete.

Further, let us choose $((x, y), (u, v)) = ((\frac{1}{2}, 0), (0, 0))$.

In this case, $x \geq u$ and $y \leq v$ and
\[
d(F(x, y), F(u, v)) = \frac{1}{2} \leq \frac{k}{2} \leq \frac{k}{2} [d(x, u) + d(y, v)],
\]
which is absurd for any $0 \leq k < 1$.

Hence the inequality (1.10) fails to hold. Further
\[
d(F(x, y), F(u, v)) = \frac{1}{2} \leq \frac{\beta}{4} (\frac{d(x, u) + d(y, v)}{2}),
\]
which is also absurd for any $\beta \in S$.

Hence the inequality (1.14.1) fails to hold.

Thus Theorem 1.10 and Theorem 1.14 are not applicable.

Remark 2.6: Theorem 1.9 follows as a corollary to Theorem 2.1 and Theorem 1.10 follows as a corollary to Theorem 2.2 (Theorem 2.3), by choosing $\varphi(t) = t$, $\beta(t) = k$ where $k \in [0, 1]$ and $g = I_X$, the identity map of $X$ in Theorem 2.1 (respectively Theorem 2.2 and Theorem 2.3).

Remark 2.7: Theorem 1.13 follows as a corollary to Theorem 2.1 and Theorem 1.14 follows as a corollary to Theorem 2.2 (Theorem 2.3), by choosing $\varphi(t) = t$ and $g = I_X$, the identity map of $X$ in Theorem 2.1 (respectively Theorem 2.2 and Theorem 2.3).

Therefore, by Example 2.5, Remark 2.6 and Remark 2.7, it follows that Theorem 2.3 is a generalization of both the theorems Theorems 1.10 and 1.14.

The following theorem gives the criteria for the existence of unique coupled common fixed point.

Theorem 2.8: In addition to the hypotheses of Theorem 2.1 (respectively Theorem 2.2 and Theorem 2.3) suppose that for every $(x, y)$ and $(z, t)$ in $X \times X$, there exists $(u, v)$ in $X \times X$ such that $(F(u, v), (F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$ and if $g$ is one-one, then $F$ and $g$ have a unique coupled common fixed point. i.e., there exists a unique $(x, y) \in X \times X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Proof:
On taking limits as \( n \to \infty \), we get
\[
r \leq \lim_{n \to \infty} \beta \left( \frac{d(gx, gu_{n-1}) + d(gy, gv_{n-1})}{2} \right) r \leq r.
\]
Hence,
\[
\lim_{n \to \infty} \beta \left( \frac{d(gx, gu_{n-1}) + d(gy, gv_{n-1})}{2} \right) = 1.
\]
Thus, \( d(gx, gu_{n-1}) + d(gy, gv_{n-1}) \to 0 \) as \( n \to \infty \), since \( \beta \in S \).

Hence \( \lim d(gx, u_n) = 0 \) and \( \lim d(gy, v_n) = 0 \).
Similarly, we get \( d(gz, u_n) \to 0 \) and \( d(gt, v_n) \to 0 \) as \( n \to \infty \).

By uniqueness of the limit, we get \( gx = gz \) and \( gy = gt \).
Since \( g \) is one-one, we have \( x = z \) and \( y = t \).
Thus the coupled coincidence point of \( F \) and \( g \) is unique.
Hence, the set of all coupled coincidence points of \( F \) and \( g \) is nonempty and it is singleton. Suppose that it is \( (x, y) \).
For argumental simplicity, take \( u = gx \) and \( v = gy \).
Since \( F \) and \( g \) are compatibility, we have
\[
g(F(x, y)) = F(gx, gy) \quad \text{and} \quad (F(y, x)) = F(gy, gx).
\]
Hence,
\[
gu = ggx = F(gx, gy) \quad \text{and} \quad (guy) = F(gy, gx).
\]
Thus \( F \) and \( g \) have a unique coupled common fixed point, since \( F \) and \( g \) have unique coupled fixed point.

This completes the proof. \( \square \)

**Example 2.9:** Let \( X = [0, 1] \) with the usual metric and the usual order.

Clearly \( (X, \leq) \) is a partially ordered set and \( (X, d) \) is a complete metric space.

We define \( F : X \times X \to X \) by
\[
F(x, y) = \begin{cases} 
\frac{x^2 - y^2}{8} & \text{if } x \geq y \\
0 & \text{if } x < y; \ x, y \in X
\end{cases}
\]

Also, we define \( g : X \to X \) by \( g(x) = x^2, \ x \in X \).
Here \( F \) and \( g \) are continuous, \( g \) is non-decreasing and \( F \) and \( g \) are compatible maps.

\( \varphi : [0, \infty) \to [0, \infty) \) is defined by
\[
\varphi(t) = \frac{3}{2} t, \forall t \geq 0.
\]
\( \beta : [0, \infty) \to [0, 1) \) is defined by
\[
\beta(t) = \begin{cases} 
\frac{1}{t+1} & \text{if } t > 0 \\
0 & \text{if } t = 0.
\end{cases}
\]

Then \( \beta \in S \).

Also, we choose \( x_0 = 0 \) and \( y_0 = c(> 0) \) in \( X \).

Then, \( gx_0 = g0 = 0 = F(0, c) = F(x_0, y_0) \) and \( gy_0 = gc = c^2 \geq \frac{c^2}{8} = F(c, 0) = F(y_0, x_0) \).

Let \( gx \geq gu, \ gy \leq gv \). Now, we verify the inequality (2.1.1) in the following cases:

**Case (i):** \( x \geq y \) and \( u \geq v \).

In this case,
\[
d(F(x, y), F(u, v)) = \frac{1}{8} \left( |x^2 - y^2| - |u^2 - v^2| \right)
\leq \frac{1}{8} \left( |x^2 - u^2| + |v^2 - y^2| \right).
\]
and
\[
\beta \left( \frac{d(gx, gu) + d(gy, gv)}{2} \right) = \frac{1}{1 + (x^2 - u^2 + v^2 - y^2)} \geq \frac{1}{2} \left( |x^2 - u^2| + |v^2 - y^2| \right).
\]

Let us take \( x^2 - u^2 = t_1 \) and \( v^2 - y^2 = t_2 \).
Since \( t_1 + t_2 \leq \frac{1}{3} \left( t_1 + t_2 \right) \) so that inequality (2.1.1) holds.

**Case (ii):** \( x \geq y \) and \( u \leq v \).

Here
\[
d(F(x, y), F(u, v)) = \frac{1}{8} \left( |x^2 - y^2| \right)
\leq \frac{1}{8} \left( x^2 - u^2 + v^2 - y^2 \right), \quad \text{since } u \leq v.
\]

Now, proceeding as in Case (i), the inequality (2.1.1) is satisfied.

Inequality (2.1.1) clearly holds in other cases also, (i.e., \( x < y, u \geq y, \) and \( x < y, u, v \)).

Hence all the hypotheses of Theorem (2.8) are satisfied and \((0, 0)\) is a coupled coincidence point.

In fact, \((0, 0)\) is the unique coupled common fixed point. \( \square \)

**III. CONCLUSION**

In this paper, we proved the existence of coupled coincidence points for a pair of compatible maps satisfying Geraghty contraction in ordered metric spaces with different hypotheses (Theorem 2.1, Theorem 2.2 and Theorem 2.3).

(i) Example 2.5 and Remark 2.6 suggest that Theorem 2.3 is a generalization of Theorem 1.10.
(ii) Example 2.5 and Remark 2.7 suggest that Theorem 2.3 is a generalization of Theorem 1.14.
(iii) Theorem 1.9, Theorem 1.13 follow as corollaries to Theorem 2.1.

Existence of unique coupled common fixed point is proved in Theorem 2.8. Examples are provided to illustrate our results.

**REFERENCES**


