Fuzzy Normed Linear Spaces

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***Abstract*—Following the concept of fuzzy norm introduced by T. Bag and S.K. Samanta (2003), a definition of fuzzy norm is given. A critical analysis of the conditions of redefined fuzzy norm is studied. By the help of the critical analysis, a decomposition theorem of the fuzzy norm into *quasi-norm family is established.Inter relation between fuzzy normed linear space and generating space of *quasi-norm family is studied.**

***Index Terms*—Fuzzy norm, fuzzy normed linear space.**

***MSC 2010 Codes* – 46S40, 03E72**

I. Introduction

VARIOUS definitions of fuzzy norms on a linear space were introduced by different authors [1-5]. Following T. Bag and S.K. Samanta [5], we have introduced a definition of fuzzy norm in this paper. We have studied a critical analysis of the conditions of redefined fuzzy norm and proved a decomposition theorem of fuzzy norm into *quasi-norm family. A comparison study of generating space of *quasi-norm family and fuzzy normed linear space is done.

We give the ideas of different types of convergent sequences, Cauchy sequences, completeness of fuzzy normed linear spaces etc and study the relationship among these concepts.

The organization of this paper is as follows: Section II comprises some preliminary results. In section III, a definition of fuzzy norm is given and critical analysis of the conditions of redefined fuzzy norm is studied. In section IV, a decomposition theorem of fuzzy norm into a *quasi-norm family is established and their inter relations are studied.

II. Some Preliminary Results

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 2.1 [5] Let X be a linear space over the field F(real or complex). A fuzzy subset N on $X \times R$(R-set of all real numbers) is called a *fuzzy norm* on X if and only if $x\ ,y \in X$ and $c\ \in F$

(N1) $\forall t \in R$ with $t \leq 0, \quad N(x,t) = 0$

(N2) $(\forall t \in R\ ,t > 0, \quad N(x,t) = 1) \quad$ iff $x = \theta$.

(N3) $\forall t \in R\ ,t > 0,$

$$N(cx,t) = N\left(x\ ,\frac{t}{|c|}\right) \quad \text{if } c\ \neq 0.$$

(N4) $\forall s\ ,t \in R, \quad x\ ,y\ \in X,$

$$N(x+y\ ,t+s\ ) \geq \ \min\{N(x,t), N(y,s)\}$$

(N5) $N(x,\cdot)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X, N) will be referred to as a *fuzzy normed linear space*. □

Definition 2.2 [5] Let (X, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ is said to be *convergent* if $\exists\ \ x\ \in X$ such that

$$\lim_{n\to\infty} N(x_n - x, t) = 1 \quad \forall t > 0.$$

Then x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $lim\ x_n$. □

Definition 2.3 [5] A sequence $\{x_n\}$ in X is said to be a *Cauchy* sequence, if

$$\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1 \quad \forall t > 0 \text{ and } \ p = 1, 2, \ldots$$

□

Definition 2.4 [5] A fuzzy normed linear space (X, N) is said to be *complete* if every Cauchy sequence in X converge to some point in X. □

III. Redefined Fuzzy Norm and Fuzzy Normed Linear Space

In this section, we give a definition of fuzzy norm and study the critical analysis of the conditions of redefined fuzzy norm.

Definition 3.1 Let X be a linear space over the field F (real or complex) and $\star$ is a continuous t-norm. A fuzzy subset N on $X \times R$ (R-set of all real numbers) is called a *fuzzy norm* on X if and only if for $x\ ,y \in X$ and $c \in\ F$

(N1) $\forall\ t \in R$ with $t \leq\ 0,\ N(x,\ t) = 0;$

(N2) $(\forall\ t \in R,\ t > 0; \quad N(x,t) = 1) \quad$ iff $x = \theta;$

(N3) $\forall t \in R\ ,t > 0,$

$$N(cx,t) = N\left(x\ ,\frac{t}{|c|}\right) \text{ if } c \neq 0$$

(N4) $\forall s\ ,t \in R, \quad x\ ,y\ \in X;$

$$N(x+y\ ,t+s\ ) \geq \ N(x,t) * N(y,s)$$

(N5) $\lim_{t\to\infty} N(x,t) = 1.$

The triplet $(X,\ N,\ *)$ will be referred to as a *fuzzy normed linear space*. □

Note 3.1. From (N4) it is clear that N(x, t) is nondecreasing with respect to t.

Note 3.2. From (N4) it is clear that
$N(x_1 + \cdots + x_n, t_1 + \cdots + t_n) \geq N(x_1, t_1) \star \cdots \star N(x_n, t_n)$.

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* The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F.510/4/DRS/2009 (SAP-I)].

Example 3.1. Let X be the linear space over the field F and $N : X \times R \to [0,1]$ defined by

$$N(x,t) = \begin{cases} \frac{t-||x||}{t+||x||} & \text{for} \quad t > ||x|| \\ 0 & \text{for} \quad t \leq \ ||x|| \end{cases}$$

Then $(X, N, *)$ is a fuzzy normed linear space.

Proof: Conditions (N1),(N2),(N3) and (N5) are directly satisfied from the definition.

For (N4), let $x, y \in X$ and $s, t \in R$.

If $t \leq ||x||$ or $s \leq ||y||$ or both then

$$N(x+y, t+s) \geq N(x,t) * N(y,s)$$

holds obviously.

Let $t > ||x||$ and $s > ||y||$.Then

$N(x+y, t+s) = \frac{t+s-||x+y||}{t+s+||x+y||}$

$\geq \frac{t+s-||x||-||y||}{t+s+||x||+||y||}$

$\geq \ \min\{\frac{t-||x||}{t+||x||}, \frac{s-||y||}{s+||y||}$

$\geq N(x,t) * N(y,s)$.

Hence $(X, N, *)$ is a fuzzy normed linear space.

Example 3.2. Let $(X, ||.||)$ be the normed linear space over the field F(real or complex) and $N : X \times R \to [0,1]$ defined by

$$N(x,t) = \begin{cases} 0 & \text{for } t \ \leq ||x|| \\ 1 & \text{for } t > \ ||x|| \end{cases}$$

.

Then N is a fuzzy norm on X and $(X, N, *)$ is a fuzzy normed linear space.

Proof: Conditions (N1),(N2),(N3) and (N5) are directly satisfied from the definition.

For (N4),let $t, s \ \in R$ and $x, y \ \in X$.Then

$$N(x+y, t+s) = 0 \Leftrightarrow \quad t+s \leq ||x+y|| \leq ||x|| + ||y||.$$

If $||x|| < t$ then $||y|| \geq \ s$,
i.e.,if $N(x,t) = 1$ then $N(y,s) = 0$.

If $||y|| < s$ then $||x|| \geq \ t$,
i.e.,if $N(y,s) = 1$ then $N(x,t) = 0$.

Thus $\quad N(s+y, t+s) = 0 \Rightarrow N(x,t) * N(y,s) = 0$.

Similarly $\quad N(s+y, t+s) = 1 \Rightarrow N(x+y, t+s) \geq N(x,t) * N(y,s)$.

So $\quad N(s+y, t+s) \ \geq N(x,t) * N(y,s)$ in any case. Thus N is a fuzzy norm on X and (X,N,*) is a fuzzy normed linear space.

Proposition 3.1. Limit of a sequence in a fuzzy normed linear space (X,N,*) if exist is unique.

Proof: Proof is straightforward.

Proposition 3.2. Every convergent sequence is a Cauchy sequence.

Proof: Proof is straightforward.

Proposition 3.3. Let (X,N,*) be a fuzzy normed linear space and $\{x_n\}$ be a convergent sequence which converges to x in X. Then any subsequence $\{x_{n_k}\}$ also converges to x.

Proof: Proof is straightforward.

3.1 Critical analysis of the conditions of fuzzy norm.

Definition 3.1.1 Let X be any nonempty set, N be a function defined on $X \times [0, \infty) \to \ [0,1]$. Define

$$|x|_\alpha = \bigwedge\{t > 0 : N(x,t) \geq \alpha\}, \alpha \in [0,1).$$

Lemma 3.1.1 Let X be any nonempty set, $N : X \times [0, \infty) \to \ [0,1]$ and $|x|_\alpha$ is defined as in Definition 3.1.1. Then for $x \in X \quad |x|_\alpha$ is nondecreasing with respect to $\alpha \in \ [0,1)$.

Proof. Proof is straightforward.

Lemma 3.1.2 Let X be any nonempty set, N be a function defined on $X \times [0, \infty) \to \ [0,1]$ and $|x|_\alpha$ is defined as in Definition 3.1.1. Then the condition
(N5) holds iff $|x|_\alpha \ < \ \infty \ \ \forall \alpha \in \ [0,1)$.

Proof. Proof is straightforward.

Lemma 3.1.3 Let X be any nonempty set, N be a nondecreasing function defined on $X \times [0, \infty) \to \ [0,1]$ and $|x|_\alpha$ is defined as in Definition 3.1.1. Then $|x|_\alpha \ > \ 0, \forall \alpha \in (0,1) \Rightarrow$ N satisfies (N1).

Proof. Proof is straightforward.

Remark 3.1.1 But the converse of Lemma 3.1.3 is not true, which is justified by the following example.

Example 3.1.1 Let X be any nonempty set and $N : \ X \times [0 \ , \ \infty) \to [0 \ , \ 1]$ be defined by

$$N(x,t) = \begin{cases} 0 & \text{for } t = 0 \\ 1 & \text{otherwise} \end{cases}$$

Here N satisfies (N1) but $|x|_\alpha \ = \ 0, \forall \alpha \in (0,1)$.

Remark 3.1.2 If $N(x,.)$ is strictly increasing then

$$(N1) \Rightarrow |x|_\alpha \ > \ 0 \ \forall \alpha \ \in (0,1)).$$

Lemma 3.1.4 Let X be any nonempty set, N be a function defined on $X \times [0, \infty) \to \ [0,1]$ and $|x|_\alpha$ is defined as in Definition 2.1.1. Then
$(N2L) : \ N(x, \ t) = 1 \quad \forall \ \ t > 0 \ \Rightarrow |x|_\alpha = \ 0, \forall \alpha \in \ [0,1)$.

Proof. Proof is straightforward.

Remark 3.1.3 But the converse of Lemma 3.1.4 is not true, which is justified by the following example.

Example 3.1.2 Let X be any nonempty set and $N : \ X \times [0 \ , \ \infty) \to [0 \ , \ 1]$ define by

$$N(x,t) = \begin{cases} 1 & \text{for } t \leq \ 5 \\ 0 & \text{otherwise} \end{cases}$$

Here $|x|_\alpha \ = \ 0, \forall \alpha \in [0,1)$ but $(N2L)$ is not satisfied.

Remark 3.1.4 If $N(x,.)$ is assumed to be increasing then the converse of Lemma 3.1.4 holds.

Lemma 3.1.5 Let X be any nonempty set, N be a function defined on $X \times [0, \infty) \to \ [0,1]$ and $|x|_\alpha$ is defined as in Definition 3.1.1. Then
$N(cx,t) \ = \ N(x, \frac{t}{|c|})$ for $c \ \neq \ 0 \ \Rightarrow \ |cx|_\alpha = |c||x|_\alpha \ \forall \alpha \in \ [0,1), \ \forall x \ \in X$.

Proof. Proof is straightforward.

Lemma 3.1.6 Let X be any nonempty set, N be a function defined on $X \times [0, \infty) \to \ [0,1]$ and $|x|_\alpha$ is defined as in Definition 2.1.1. If N satisfies the condition (N4), then $\forall \ x \ , y \ \in \ X$ and $\forall \alpha \ , \beta \ \in \ [0,1)$,

$$|x|_\alpha + |y|_\beta \geq |x+y|_{\alpha * \beta}.$$

Proof. Let $\alpha,\ \beta\ \in (0,1)$. Then
$|x|_\alpha\ +\ |y|_\beta\ =\ \bigwedge\{t\ >\ 0\ :\ N(x,t)\ \geq\ \alpha\} + \bigwedge\{s\ >\ 0\ :\ N(y,s) \geq \beta\}$

$= \bigwedge\{t{+}s > 0 : N(x,t) \geq \alpha, N(y,s) \geq \beta\}$

Now $N(x+y,t+s) \geq\ N(x,t) * N(y,s) \geq\ \alpha * \beta$.

Thus $|x|_\alpha\ +\ |y|_\beta\ \ \geq \bigwedge\{t+s > 0 : N(x+y,t+s) \geq \alpha * \beta\} = |x+y|_{\alpha*\beta}$.

Remark 3.1.5 But The converse of Lemma 2.1.6 is not true, which is justified by the following example.

Example 3.1.3 Let X be any nonempty set and $N :\ X \times [0\ ,\ \infty) \rightarrow [0\ ,\ 1]$ define by

$$N(x,t) = \begin{cases} 1 & \text{for } t \leq\ 5 \\ 0 & \text{otherwise} \end{cases}$$

Here $|x|_\alpha\ +\ |y|_\beta\ \ = |x+y|_{\alpha*\beta}\ \ \forall\alpha \in\ [0,1), \forall\beta \in\ [0,1)$ holds but $N(x,3)*N(y,4) = 1\ >\ N(x+y,3+4) = 0$.

Remark 3.1.6 In Lemma 3.1.6, if the t-norm $*$ approaches to $'min'$,then $\alpha * \alpha \rightarrow \alpha$.Hence in particular, if $* =' min'$ then

$|x|_\alpha + |y|_\alpha \geq |x+y|_\alpha\ \forall\alpha \in\ [0,1)\ \forall\ x,y,\in\ X$.

IV. A Decomposition Theorem of Fuzzy Norm into a Family of Quasi Norms

A decomposition theorem of fuzzy norm into a *quasi-norm family is established and their inter relations are studied in this section.

Theorem 4.1. Let $(X,N,*)$ be a fuzzy normed linear space. For $\alpha \in\ [0,1)$ we define

$$|x|_\alpha = \bigwedge\{t > 0 : N(x,t) \geq \alpha\}$$

Then

(q1) $|x|_\alpha \geq\ 0\ \ \forall x \in X,\ \forall\alpha \in [0,1)$ and $|x|_0 =\ 0\ \ \forall x \in X$.
(q2) $(|x|_\alpha =\ 0\ \ \forall\alpha \in [0,1))$ iff $x = \theta$.
(q3) $|cx|_\alpha = |c||x|_\alpha\ \ \forall\alpha \in (0,1)$.
(q4) $\forall\alpha\ \beta\ \in [0,1)\ \ |x+y|_{\alpha*\beta}\ \ \leq\ |x|_\alpha + |y|_\beta$.
(q5) If $\alpha\ \geq\ \beta$ then $|x|_\alpha \geq\ |x|_\beta$.

Proof: Proofs follow from Lemma 3.1.1 to 3.1.6.

Note 4.1. If we assume **(N6)** $N(x,t) > 0\ \forall\ t > 0\ \Rightarrow\ x = \theta$, then **(q6)** $|x|_\alpha =\ 0$ iff $x = \theta\ \ \ \forall\alpha \in\ (0,1)$.

Note 4.1. If $(X,N,*)$ is a fuzzy normed linear space, then we call $q^* = \{|.|_\alpha : \alpha \in (0,1)\}$ a **-quasi norm family* and $(X,\ q^*)$ a *generating space* of *-quasi norm family(gs*qnf).

Note 4.3. In Example 3.1 and Example 3.2, (X,N,*) is a fuzzy normed linear space satisfying (N6).

Note 4.4. Let $X = R$ and $|x|_\alpha = \alpha.||x||,\ \ \ \alpha \in\ (0,1)$ be an ascending family of norms on X. Define

$$N'(x,t) = \begin{cases} \bigvee\{\alpha > 0 : |x|_\alpha \leq t\} & \text{for } t > 0 \\ 0 & \text{for } t \leq\ 0 \end{cases}$$

.

Let $t > x > 0, s > y > 0$ and *(a,b)=a.b, then * is a t-norm.
Now $N'(x,t)*N'(y,s) = \bigvee\{\alpha > 0 : |x|_\alpha \leq t\} * \bigvee\{\alpha > 0 : |y|_\alpha \leq s\}$.

$\Rightarrow N'(x,t)*N'(y,s) = \frac{t}{|x|} * \frac{s}{|y|}$
$= \frac{ts}{xy}$

$N'(x+y,t+s) = \bigvee\{\alpha > 0 : |x+y|_\alpha \leq t+s\}$
$= \frac{t+s}{x+y}$.

Now $N'(x,t)*N'(y,s) - N'(x+y,t+s) = \frac{ts}{xy} - \frac{t+s}{x+y}$
$= \frac{tx(s-y)+sy(t-x)}{xy(x+y)}\ > 0$

$\Rightarrow N'(x,t)*N'(y,s) > N'(x+y,t+s)$.

Thus $(X,N',*)$ is not a fuzzy normed linear space. So the converse of the Theorem 3.1 is not true.

Definition 4.1 $|.|_\alpha$ is said to be continuous with respect to $\alpha \in\ \ [0,1)$ if for any sequence $\{\alpha_n\}$ in $[0,1)$ such that $\alpha_n \rightarrow \alpha$ implies $|x|_{\alpha_n} \rightarrow |x|_\alpha\ \ \forall x \in X$.

Theorem 4.2. Let X be a linear space and $q^* = \{|.|_\alpha : \alpha \in [0,1)\}$ be a $*$-quasi norm family on X satisfying (q6). We further assume that $|.|_\alpha$ is continuous with respect to α. We defined

$$N'(x,t) = \begin{cases} \bigvee\{\alpha \in [0,1) : |x|_\alpha \leq t\} & \text{for } (x,t) \neq (\theta,0) \\ 0 & \text{for } \ (x,t) = (\theta,0) \end{cases}$$

.

Then $(X,N',*)$ is a fuzzy norm linear space.

Proof: Condition $(N'1)$, $(N'2)$, $(N'3)$ and $(N'5)$ directly follow from the definition.

For $(N'4)$,let $x,y\ \in X$ and $t,s\ \in\ R$.Then

$$N'(x+y,t+s) = \bigvee\{\alpha \in [0,1) : |x+y|_\alpha \leq t+s\}$$

Now $N'(x,t) = \bigvee\{\alpha \in (0,1) : |x|_\alpha \leq t\} = \alpha'$(say) and $N'(y,s) = \bigvee\{\beta \in (0,1) : |y|_\beta \leq s\} = \beta'$(say) where $|x|_{\alpha'} = t$ and $|y|_{\beta'} = s$.

[Since $|.|_\alpha$ is continuous with respect to $\alpha \in\ [0,1)$.]

Now

$N'(x,t)*N'(y,s) = \alpha' * \beta'$ and

$$|x+y|_{\alpha'*\beta'} \leq |x|_{\alpha'} + ||y||_{\beta'} =\ t+s\ ;$$

$\Rightarrow N'(x+y,t+s) \geq \alpha' * \beta' =\ N'(x,t)*N'(y,s)$

$\Rightarrow\ \ (X,\ N',*)$ is a fuzzy normed linear space.

Definition 4.2 Let $(X,\ q^*)$ a generating space of *-quasi norm family and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* if $\exists\ \ x\ \in X$ such that

$$\lim_{n\rightarrow\infty} |x_n - x|_\alpha = 0\ \forall\alpha \in [0,1).$$

Then x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $lim\ x_n$.

Definition 4.3 Let $(X,\ q^*)$ a generating space of *-quasi norm family and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be a *Cauchy* sequence ,if

$$\lim_{n\rightarrow\infty} |x_{n+p} - x_n|_\alpha = 0\ \ \forall\alpha \in [0,1),\ \ \ \forall\ \ \ p = 1,2,\ldots$$

Definition 4.4 A generating space of *-quasi norm family $(X,\ q^*)$ is said to be *complete* if every Cauchy sequence in X converge to some point in X.

Proposition 4.1 Let (X,N,*) be a fuzzy normed linear space satisfying (N6).If $\{x_n\}$ be a sequence in X,then $\{x_n\} \rightarrow x$ with respect to N $\Leftrightarrow \{x_n\} \rightarrow x$ with respect to q^*.

Proof: Let $\{x_n\}$ be a sequence in X,such that $\{x_n\} \rightarrow x$ with respect to N.

$\Rightarrow \lim_{n\rightarrow\infty} N(x_n - x,t) = 1\ \ \ \ \forall t > 0$.

We choose $\alpha \in [0,1)$.

$\Rightarrow \lim_{n\rightarrow\infty} N(x_n - x,t) > \alpha\ \ \ \ \forall t > 0$.

$\Rightarrow \exists$ a positive integer $n_0(\alpha,t)$ such that

$N(x_n - x, t) > \alpha \quad \forall n \geq \ n_0(\alpha, t)$.
$\Rightarrow |x_n - x|_\alpha \leq t \ \forall n \geq \ n_0(\alpha, t)$.
$\Rightarrow \lim\limits_{n\to\infty} |x_n - x|_\alpha \leq t \ \forall \ t > 0$.
$\Rightarrow \lim\limits_{n\to\infty} |x_n - x|_\alpha = 0$.

Thus $\{x_n\} \to x$ with respect to q^*.

Next we suppose $\lim\limits_{n\to\infty} |x_n - x|_\alpha = 0 \ \ \forall \ \alpha \ \in (0,1)$.

Then corresponding to any $t > 0$ $\exists$ a positive integer $n_0(\alpha, t)$ such that
$|x_n - x|_\alpha < t \ \forall n \geq \ n_0(\alpha, t)$
$\Rightarrow \ N(x_n - x, t) \geq \ \alpha \quad \forall n \geq \ n_0(\alpha, t)$.
$\Rightarrow \lim\limits_{n\to\infty} N(x_n - x, t) \geq \ \alpha \quad \forall \alpha \in (0, 1)$.
$\Rightarrow \lim\limits_{n\to\infty} N(x_n - x, t) = 1 \quad \forall t > 0$.

Hence proved. □

Proposition 4.2 Let (X,N,*) be a fuzzy normed linear space satisfying (N6) and $\{x_n\}$ be a Cauchy sequence in (X,N,*) if and only if it is a Cauchy sequence in (X, q^*).

Proof: Let $\{x_n\}$ be a Cauchy sequence in X, then
$\lim\limits_{n\to\infty} N(x_{n+p} - x_n, t) = 1 \ \forall \ t > 0 \ , \ \forall \quad p = 1, 2, \ldots$
$\Leftrightarrow$ for each $\alpha \in \ [0, 1) \ \ \exists$ a positive integer $n_0(\alpha, t)$ such that
$N(x_{n+p} - x_n, t) > \alpha \ \forall \ n \geq \ n_0(\alpha, t) \ \ \forall \ \ p = 1, 2, \ldots;$
$\Leftrightarrow |(x_{n+p} - x_n|_\alpha = 0 \ \ \forall \ \ p = 1, 2, \ldots;$
$\Leftrightarrow \{x_n\}$ is a Cauchy sequence in (X, q^*).

Proposition 4.3 If $(X, N, *)$ be a complete fuzzy normed linear space then (X, q^*) is a complete generating space of *-quasi norm family.

Proof: Proof is straightforward.

V. Conclusion

Following Bag and Samanta (2003), we have introduced, in this paper, a more general definition of fuzzy norm. Actually proceeding through critical analysis on the conditions of fuzzy norm we arrive at a very useful decomposition theorem of a fuzzy norm in its full generality, into a $*$-quasi norm family and from which it is deduced that a fuzzy norm with particular t-norm viz. 'min' is decomposable to a family of crisp norm. We think there is a wide scope to study fuzzy normed linear spaces in general setting.

Acknowledgment

The authors are grateful to the referees for their valuable suggestions in rewriting the paper in the present form. The authors are also grateful to the Editor-in-Chief for his valuable comments to standardize it.

References

[1] G. S. Rhie, B. M. Choi and S. K. Dong, ”On the completeness of fuzzy normed linear space”, *Math. Japonica*, vol. 45, no. 1, pp. 33-37, 1997.
[2] S. V. Krishna and K. K. M. Sarma, “Separation of fuzzy normed linear space”, *Fuzzy Sets and Systems*, vol. 63, pp. 207-217, 1994.
[3] C. Felbin, “Finite dimensional fuzzy normed linear space”, *Fuzzy Sets and Systems*, vol. 48, pp. 239-248, 1992.
[4] S. C. Chang and J. N. Mordeson, “Fuzzy linear operators and Fuzzy normed linear space”, *Bull.Cal.Math.Soc.*, vol. 86, pp. 429-436, 1994.
[5] T. Bag, S. K. Samanta, “Finite dimensional fuzzy normed linear spaces”, *The Journal of Fuzzy Mathematics*, vol. 3, pp. 687-705, 2003.