Right Circulant Matrices with Fibonacci Sequence
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Abstract—In this paper, the right circulant matrix with Fibonacci sequence was defined, and then the formula for its Euclidean norm and eigenvalues were derived.

Index Terms—right circulant matrix with Fibonacci sequence, Euclidean norm, eigenvalues

MSC 2010 Codes – 05C50, 11B50

I. INTRODUCTION

The Fibonacci sequence is one of the most famous sequences. It has many applications and can also be seen in nature. It is defined by the linear recurrence relation

\[ F_n = F_{n-1} + F_{n-2}; \quad n \geq 2 \]

with initial values \( F_0 = 0 \) and \( F_1 = 1 \). The closed form is given by the formula

\[ F_n = \frac{\phi^n - (1 - \phi)^n}{\sqrt{5}} \]

where \( \phi = \frac{1 + \sqrt{5}}{2} \), the golden ratio.

II. PRELIMINARIES

Definition 2.1: A matrix \( RCIRC_n(\vec{c}) \in M_{n \times n}(\mathbb{R}) \) is said to be a right circulant matrix if it is of the form

\[
RCIRC_n(\vec{c}) = \begin{pmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
    c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
    c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\
    c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0
\end{pmatrix}
\]

The above matrix has the following structure

1) Each row is a right cyclic shift of the row above it. Thus, \( RCIRC_n(\vec{c}) \) is determined by the first row \((c_0, c_1, \ldots, c_{n-1})\).
2) \( c_{ij} = c_{ki} \) whenever \( j - i = k - l \) (mod n).

Furthermore, its eigenvalues are just the Discrete Fourier Transform of its first row. That is,

\[ \lambda_m = \sum_{k=0}^{n-1} c_k \omega^{-mk} \quad (1) \]

where \( \omega = e^{\frac{2\pi i}{n}} \).

Definition 2.2: The first row of \( RCIRC_n(\vec{c}) \) is called the circulant vector.

Definition 2.3: The matrix given by

\[
RCIRC_n(\vec{F}) = \begin{pmatrix}
    F_0 & F_1 & F_2 & \cdots & F_{n-2} & F_{n-1} \\
    F_{n-1} & F_0 & F_1 & \cdots & F_{n-3} & F_{n-2} \\
    F_{n-2} & F_{n-1} & F_0 & \cdots & F_{n-4} & F_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    F_2 & F_3 & F_4 & \cdots & F_0 & F_1 \\
    F_1 & F_2 & F_3 & \cdots & F_{n-1} & F_0
\end{pmatrix}
\]

is called the circulant matrix with fibonacci sequence.

Definition 2.4: The Euclidean norm of an \( n \times n \) matrix \( A \), which will be denoted by \( ||A||_E \), is given by the formula

\[ ||A||_E = \sqrt{\sum_{i,j=0}^{n} a_{ij}^2} \]

Definition 2.5: The spectral norm of an \( n \times n \) matrix \( A \), which will be denoted by \( ||A||_2 \), is given by the formula

\[ ||A||_2 = \max \{ |\lambda_m| \} \]

where \( m = 0, 1, \ldots, n-1 \) and \( \lambda_m \) are the eigenvalues of \( A \).

The following Fibonacci identities and theorem will be used for solving the Euclidean norm and eigenvalues of \( RCIRC_n(\vec{F}) \).

1) \( F_k = \frac{\phi^{k+1} - (1 - \phi)^{k+1}}{2\phi - 1} \)
2) \( \sum_{k=0}^{n-1} F_k = F_{n-1}F_n \)
3) \( \sum_{k=0}^{n-1} F_k = F_{n-1} - 1 \)

Theorem 2.6: From [1], the eigenvalues of

\[
RCIRC_n(\vec{g}) = \begin{pmatrix}
    g_0 & ar & ar^2 & \cdots & ar^{n-2} & ar^{n-1} \\
    ar & g_0 & ar & \cdots & ar^{n-3} & ar^{n-2} \\
    ar^2 & ar & g_0 & \cdots & ar^{n-4} & ar^{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    ar^{n-2} & ar^{n-3} & ar^{n-4} & \cdots & g_0 & ar \\
    ar^{n-1} & ar^{n-2} & ar^{n-3} & \cdots & ar & g_0
\end{pmatrix}
\]

are \( \lambda_0 = \frac{a(1-r^n)}{1-r} \) and \( \lambda_m = \frac{a(1-r^n)}{1-r\omega^{-m}} \) where \( m = 1, 2, \ldots, n-1 \).
III. Main Results

Theorem 3.1:
\[ ||RCIRC_n(\vec{F})||_E = \sqrt{nF_{n-1}F_n} \]

Proof:
\[ ||RCIRC_n(\vec{F})||_E = \sqrt{n \sum_{k=0}^{n-1} F_k^2} = \sqrt{nF_{n-1}F_n} \]
via the identity (2).

Theorem 3.2: The eigenvalues of \( RCIRC_n(\vec{F}) \) are the following
- \( \mu_0 = F_n - 1 \)
- \( \mu_m = \frac{1}{2\phi - 1} \left[ \frac{\phi - \phi^{n+1}}{1 - \phi \omega^{-m}} - \frac{(1-\phi)(1-\phi)^{n+1}}{1 - (1-\phi)\omega^{-m}} \right] \)

where \( m=1,2, \ldots, n-1 \).

Proof: For \( m=0 \),
\[ \mu_0 = \sum_{k=0}^{n-1} F_k = F_n - 1 \]
via identity (3).
For \( m \neq 0 \)
\[ \mu_m = \frac{1}{2\phi - 1} \left[ \sum_{k=0}^{n-1} \phi^{-mk} - \sum_{k=0}^{n-1} (1-\phi)^{-mk} \right] \]
Using Theorem 2.5, the result will be
\[ \mu_m = \frac{1}{2\phi - 1} \left[ \frac{\phi - \phi^{n+1}}{1 - \phi \omega^{-m}} - \frac{(1-\phi)(1-\phi)^{n+1}}{1 - (1-\phi)\omega^{-m}} \right] \]

IV. Conclusion

The formulae for the Euclidean norm and eigenvalues of a right circulant matrix with Fibonacci sequence \( (RCIRC_n(\vec{F})) \) were established.

REFERENCES
3. www.mathworld.wolfram.com/FibonacciNumber.html