Right Circulant Matrices with Fibonacci Sequence

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***Abstract*—In this paper, the right circulant matrix with Fibonacci sequence was defined, and then the formula for its Euclidean norm and eigenvalues were derived.**

***Index Terms*—right circualnt matrix with Fibonacci sequence, Euclidean norm, eigenvalues**

***MSC 2010 Codes* – 05C50, 11B50**

I. Introduction

The Fibonacci sequence is one of the most famous sequences. It has many applications and can also be seen in nature. It is defined by the linear recurrence relation

$$F_n = F_{n-1} + F_{n-2};\ n \geq 2$$

with initial values $F_0 = 0$ and $F_1 = 1$. The closed form is given by the formula

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio.

II. Preliminaries

Definition 2.1: A matrix $RCIRC_n(\vec{c}) \in M_{nxn}(\mathbb{R})$ is said to be a right circulant matrix if it is of the form

$$RCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0 \end{pmatrix}$$

The above matrix has the following structure

1) Each row is a right cyclic shift of the row above it. Thus, $RCIRC_n(\vec{c})$ is determined by the first row $(c_0,\ c_1,\ \dots, c_{n-1})$.
2) $c_{ij} = c_{kl}$ whenever $j - i = k - l$ (mod n).

Furthermore, its eigenvalues are just the Discrte Fourier Transform of its first row. That is,

$$\lambda_m = \sum_{k=0}^{n-1} c_k \omega^{-mk} \tag{1}$$

where $\omega = e^{\frac{2\pi i}{n}}$.

Definition 2.2: The first row of $RCIRC_n(\vec{c})$ is called the circulant vector.

Definition 2.3: The matrix given by

$$RCIRC_n(\vec{F}) = \begin{pmatrix} F_0 & F_1 & F_2 & \dots & F_{n-2} & F_{n-1} \\ F_{n-1} & F_0 & F_1 & \dots & F_{n-3} & F_{n-2} \\ F_{n-2} & F_{n-1} & F_0 & \dots & F_{n-4} & F_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_2 & F_3 & F_4 & \dots & F_0 & F_1 \\ F_1 & F_2 & F_3 & \dots & F_{n-1} & F_0 \end{pmatrix}$$

is called the circulant matrix with fibonacci sequence.

Definition 2.4: The Euclidean norm of an nxn matrix A, which will be denoted by $||A||_E$, is given by the formula

$$||A||_E = \sqrt{\sum_{i,j=0}^{n} a_{ij}^2}$$

Definition 2.5: The spectral norm of an nxn matrix A, which will be denoted by $||A||_2$, is given by the formula

$$||A||_2 = max\left\{|\lambda_m|\right\}$$

where m=0,1, ..., n-1 and λ_m are the eigenvalues of A.

The following Fibonacci identities and theorem will be used for solving the Euclidean norm and eigenvalues of $RCIRC_n(\vec{F})$

1) $F_k = \frac{\phi^{k+1} - (1-\phi)^{k+1}}{2\phi - 1}$
2) $\sum_{k=0}^{n-1} F_k^2 = F_{n-1}F_n$
3) $\sum_{k=0}^{n-1} F_k = F_{n-1} - 1$

Theorem 2.6: From [1], the eigenvalues of

$$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & \dots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & \dots & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & \dots & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & \dots & a & ar \\ ar & ar^2 & ar^3 & \dots & ar^{n-1} & a \end{pmatrix}$$

are $\lambda_0 = \frac{a(1-r^n)}{1-r}$ and $\lambda_m = \frac{a(1-r^n)}{1-r\omega^{-m}}$ where m=1,2, ..., n-1.

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III. Main Results

Theorem 3.1:

$$||RCIRC_n(\vec{F})||_E = \sqrt{nF_{n-1}F_n}$$

Proof:

$$||RCIRC_n(\vec{F})||_E = \sqrt{n\sum_{k=0}^{n-1} F_k^2} = \sqrt{nF_{n-1}F_n}$$

via the identity (2).

Theorem 3.2: The eigenvalues of $RCIRC_n(\vec{F})$ are the following

- $\mu_0 = F_n - 1$
- $\mu_m = \frac{1}{2\phi-1}\left[\frac{\phi-\phi^{n+1}}{1-\phi\omega^{-m}} - \frac{(1-\phi)-(1-\phi)^{n+1}}{1-(1-\phi)\omega^{-m}}\right]$

where m=1,2, ..., n-1.

Proof: For m=0,

$$\mu_0 = \sum_{k=0}^{n-1} F_k = F_n - 1$$

via identity (3).
For $m \neq 0$

$$\begin{aligned}\mu_m &= \sum_{k=0}^{n-1} F_k \omega^{-mk} \\ &= \sum_{k=0}^{n-1}\left[\frac{\phi^{k+1}-(1-\phi)^{k+1}}{2\phi-1}\right]\omega^{-mk} \\ &= \frac{1}{2\phi-1}\left[\phi\sum_{k=0}^{n-1}\phi^k - (1-\phi)\sum_{k=0}^{n-1}(1-\phi)^k\right]\omega^{-mk}\end{aligned}$$

Using Theorem 2.5, the result will be

$$\mu_m = \frac{1}{2\phi-1}\left[\frac{\phi-\phi^{n+1}}{1-\phi\omega^{-m}} - \frac{(1-\phi)-(1-\phi)^{n+1}}{1-(1-\phi)\omega^{-m}}\right]$$

IV. Conclusion

The formulae for the Euclidean norm and eigenvalues of a right circulant matrix with Fibonacci sequence ($RCIRC_n(\vec{F})$) were established.

References

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[3] www.mathworld.wolfram.com/FibonacciNumber.html