A Three Species Ecological Model with a Prey, Predator and a Competitor to both the Prey and Predator

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Abstract—The present paper is devoted to an analytical investigation of three species ecological model with a Prey ($N_1$), a predator ($N_2$) and a competitor ($N_3$) to both the prey ($N_1$) and Predator ($N_2$). In addition to that, the species are provided with alternative food. The model is characterized by a set of first order non-linear ordinary differential equations. All the eight equilibrium points of the model are identified and the local and global stability of the interior equilibrium state is discussed. Further exact solutions of linearized system of equations have been derived. The stability analysis is supported by Numerical simulation using Matlab.

Index Terms—Prey, Predator, Competitor, Equilibrium points, Normal Steady State, Local stability, Global stability.

MSC 2010 Codes — 93A30, 00A71.

I. INTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [1] and by Volterra [2]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Paul Colinvaux [3], Freedman [4], Kapur [5, 6] etc. Recently Archana Reddy [7] discussed the stability analysis of two interacting species with harvesting of both species.

Lakshmi Narayan and Pattabhiramacharyulu [8, 9] and Shiva Reddy [10], [11] et al., Ravindra Reddy [12],[13],[14] et al. Shanker [15] et al and Papa Rao [16] have discussed different prey-predator models in detail. Inspired from that, we discussed a more general three species model. The model is characterized by a set of first order ordinary differential equations. All the eight equilibrium points of the model are identified and stability criteria for interior equilibrium states are discussed.

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II. BASIC EQUATIONS

The model equations for a three species prey-predator and competitor to the prey and predator system is given by the following system of first order ordinary differential equations employing the following notation:

\[\begin{align*}
\dot{N}_1 &= a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 - \alpha_{13} N_1 N_3 \\
\dot{N}_2 &= a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 - \alpha_{23} N_3 N_2 \\
\dot{N}_3 &= a_3 N_3 - \alpha_{33} N_3^2 - \alpha_{31} N_1 N_3 - \alpha_{32} N_2 N_3
\end{align*}\]  

(2.1)

where $N_1$, $N_2$ and $N_3$ are the population of the prey, predator and competitor respectively. Here, $\alpha_{11}$ is rate of decrease of the prey due to insufficient food and inter species competition, $\alpha_{12}$ is rate of decrease of the prey due to the inhibition by the predator, $\alpha_{21}$ is rate of increase of the predator due to successful attacks on the prey, $\alpha_{22}$ is rate of decrease of the predator due to insufficient food other than the prey and inter species competition, $\alpha_{23}$ is rate of decrease of the predator due to the competition with the third species, $\alpha_{33}$ is rate of decrease of the Competitor to both prey and predator due to insufficient food and inter species competition, $\alpha_{32}$ is rate of decrease of the competitor due to the competition with the predator, $\alpha_{13}$ is rate of decrease of the prey due to the competition with the both prey and predator and $\alpha_{31}$ is rate of decrease of the competitor due to the competition with the prey.

III. EQUILIBRIUM STATES

The system under investigation has 8 equilibrium states. They are:

\[\begin{align*}
E_1 : \text{The extinct state} \\
N_1 &= 0, \\
N_2 &= 0, \\
N_3 &= 0
\end{align*}\]  

(3.1)
\( \mathbf{E}_2 \): The state in which only the predator survives and the prey and competitor to the prey and predator are extinct

\[
\tilde{N}_1 = 0, \\
\tilde{N}_2 = \frac{a_2}{\alpha_{22}}, \\
\tilde{N}_3 = 0
\]  

(3.2)

\( \mathbf{E}_3 \): The state in which both the prey and the predators are extinct and competitor to the prey and predator survive

\[
\tilde{N}_1 = 0, \\
\tilde{N}_2 = 0, \\
\tilde{N}_3 = \frac{a_3}{\alpha_{33}}
\]  

(3.3)

\( \mathbf{E}_4 \): The state in which both the predator and competitor to the prey and predator are extinct and prey alone survive

\[
\tilde{N}_1 = \frac{a_1}{\alpha_{11}}, \\
\tilde{N}_2 = 0, \\
\tilde{N}_3 = 0
\]  

(3.4)

\( \mathbf{E}_5 \): The state in which both the prey and the predators exist and competitors to the prey predator are extinct.

\[
\tilde{N}_1 = \frac{(a_1\alpha_{22} - a_2\alpha_{12})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\
\tilde{N}_2 = \frac{(a_2\alpha_{11} + a_1\alpha_{21})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \\
\tilde{N}_3 = 0
\]  

(3.5)

This case arises only when

\[
a_2\alpha_{22} > a_1\alpha_{12}
\]  

(3.6)

\( \mathbf{E}_6 \): The state in which both prey and competitor to the prey and predator exist and predator extinct

\[
\tilde{N}_1 = \frac{(a_1\alpha_{33} - a_3\alpha_{13})}{(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31})}, \\
\tilde{N}_2 = 0, \\
\tilde{N}_3 = \frac{(a_3\alpha_{11} - a_1\alpha_{31})}{(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31})}.
\]  

(3.7)

The equilibrium state exist only when

\[
a_1\alpha_{33} > a_3\alpha_{13} \\
a_3\alpha_{11} > a_1\alpha_{31}
\]  

(3.8)

\( \mathbf{E}_7 \): The state in which both predator and competitor to the prey and predator exist and prey extinct

\[
\tilde{N}_1 = 0, \\
\tilde{N}_2 = \left( \frac{a_2\alpha_{33} - a_3\alpha_{23}}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}} \right), \\
\tilde{N}_3 = \left( \frac{a_3\alpha_{22} - a_2\alpha_{32}}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}} \right)
\]  

(3.9)

The equilibrium state exist only when

\[
a_2\alpha_{33} > a_3\alpha_{23}, a_3\alpha_{22} > a_2\alpha_{32}, \alpha_{22}\alpha_{33} > \alpha_{23}\alpha_{32}
\]  

(3.10)

\( \mathbf{E}_8 \): The state in which the prey, predator and competitor to the prey predator exist

\[
\tilde{N}_1 = \frac{A}{D}, \\
\tilde{N}_2 = \frac{B}{D}, \\
\tilde{N}_3 = \frac{C}{D}
\]  

(3.11)

where

\[
D = \alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22})
\]

\[
A = \alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) - \alpha_{12}(\alpha_{21}\alpha_{33} - \alpha_{31}\alpha_{23}) + \alpha_{13}(\alpha_{21}\alpha_{32} - \alpha_{31}\alpha_{22})
\]

\[
B = (a_2\alpha_{11}\alpha_{33} + a_3\alpha_{21}\alpha_{33} + a_1\alpha_{31}\alpha_{23}) - (a_3\alpha_{11}\alpha_{23} + a_2\alpha_{21}\alpha_{13} + a_1\alpha_{13}\alpha_{21})
\]

\[
C = (a_3\alpha_{11}\alpha_{22} + a_2\alpha_{21}\alpha_{12} + a_1\alpha_{12}\alpha_{31}) - (a_1\alpha_{11}\alpha_{32} + a_2\alpha_{21}\alpha_{32} + a_3\alpha_{32}\alpha_{22})
\]

(3.12)

IV. THE STABILITY OF THE EQUILIBRIUM STATES

Let

\[
N = (N_1, N_2, N_3)^T = \tilde{N} + U
\]  

(4.1)

where \( U = (u_1, u_2, u_3)^T \) is the perturbation over the equilibrium state \( \tilde{N} = (\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)^T \). The basic equations (2.1) are linearized to obtain the equations for the perturbed state.

\[
\dot{U} = AU
\]  

(4.2)

where

\[
A = \begin{bmatrix}
A_{11} & -\alpha_{12}N_1 & -\alpha_{13}N_1 \\
\alpha_{21}N_2 & A_{22} & -\alpha_{23}N_2 \\
-\alpha_{31}N_3 & -\alpha_{32}N_3 & A_{33}
\end{bmatrix}
\]

with

\[
A_{11} = a_1 - 2\alpha_{11}N_1 - \alpha_{12}N_2 - \alpha_{13}N_3 \\
A_{22} = a_2 - 2\alpha_{22}N_2 + \alpha_{21}N_1 - \alpha_{23}N_3 \\
A_{33} = a_3 - 2\alpha_{33}N_3 - \alpha_{31}N_1 - \alpha_{32}N_2
\]

(4.3)
The equilibrium state is stable, if three roots of the equation (4.3) are negative in case they are real or the roots have negative real parts if they are complex.

A. Stability of the Equilibrium State \( E_8 \)

First, we consider the local stability of the equilibrium \( E_8 \).

The variational matrix of the system (2.1) at interior equilibrium state \( E_8 \) is

\[
A = \begin{bmatrix}
-\alpha_{11}N_1 & -\alpha_{12}N_1 & -\alpha_{13}N_1 \\
\alpha_{21}N_2 & -\alpha_{22}N_2 & -\alpha_{23}N_2 \\
-\alpha_{31}N_1 & -\alpha_{32}N_3 & -\alpha_{33}N_3
\end{bmatrix}
\]  

(4.1.1)

The characteristic equation of interior equilibrium state \( E_8(N_1, N_2, N_3) \) is

\[
\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0
\]  

(4.1.2)

where

\[
b_1 = \alpha_{11}N_1 + \alpha_{22}N_2 + \alpha_{33}N_3
\]

\[
b_2 = \left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right)N_1N_3 + \left(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}\right)N_1N_2
\]

\[
+ \left(\alpha_{33}\alpha_{22} - \alpha_{32}\alpha_{23}\right)N_2N_3
\]

\[
b_3 = \left(\alpha_{11}\alpha_{33}\alpha_{22} + \alpha_{12}\alpha_{32}\alpha_{23} + \alpha_{13}\alpha_{23}\alpha_{31}\right)
\]

\[
\times N_1N_2N_3
\]

By Routh-Hurwitz criterion, all eigen values of the above characteristic equation have negative real parts if and only if

\[
b_1 > 0, \quad (b_1b_2 - b_3) > 0 \text{ and } b_1(b_2b_3 - b_3) > 0
\]

Clearly, \( b_1 > 0 \) and \( b_3 > 0 \) if

\[
(\alpha_{11}\alpha_{33}\alpha_{22} + \alpha_{12}\alpha_{32}\alpha_{23} + \alpha_{13}\alpha_{23}\alpha_{31})
\]

\[
> (\alpha_{13}\alpha_{31}\alpha_{22} + \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{21}\alpha_{32}\alpha_{13})
\]

And based on certain algebraic deductions applicable in this case, it can be verified that

\[
b_1(b_2b_3 - b_3) > 0
\]

Therefore, the roots of (4.1.1) are real and negative or complex conjugates having negative real parts.

Thus, the system is locally stable for interior equilibrium point \( E_8(N_1, N_2, N_3) \).

B. Solution of the Linearized System of Equations

The solution of linearized system of equations (4.2) is given by

\[
u_1 = A_1e^{s_1t} + B_1e^{s_2t} + C_1e^{s_3t}
\]  

(4.2.1)

\[
u_2 = A_2e^{s_1t} + B_2e^{s_2t} + C_2e^{s_3t}
\]  

(4.2.2)

\[
u_3 = A_3e^{s_1t} + B_3e^{s_2t} + C_3e^{s_3t}
\]  

(4.2.3)

where \( s_1, s_2 \) and \( s_3 \) are roots of the equation (4.1.1).
$K_5 = u_{10}(s_1 + \alpha_{22}N_2)(s_1 + \alpha_{33}N_3) - u_{20}\alpha_{12}N_1(s_1 + \alpha_{33}N_3)$
\[-u_{20}\alpha_{12}N_1(s_1 + \alpha_{33}N_3)
\]- $u_{30}\alpha_{13}N_1(s_1 + \alpha_{22}N_2)$

$K_6 = u_{10}\alpha_{20}\alpha_{32}N_2 N_3$
\[+ u_{30}\alpha_{12}\alpha_{32}N_1 N_2 + u_{20}\alpha_{12}\alpha_{32}N_1 N_3\]

$K_7 = u_{20}(s_2 + \alpha_{11}N_1)(s_2 + \alpha_{33}N_3)$
\[-u_{30}\alpha_{12}N_2(s_2 + \alpha_{11}N_1)
\]- $u_{10}\alpha_{21}N_2(s_2 + \alpha_{33}N_3)$

$K_8 = u_{10}\alpha_{20}\alpha_{32}N_2 N_3$
\[+ u_{30}\alpha_{13}\alpha_{32}N_1 N_3 - u_{20}\alpha_{13}\alpha_{31}N_1 N_3\]

$K_9 = u_{20}(s_2 + \alpha_{11}N_1)(s_2 + \alpha_{33}N_3)$
\[-u_{30}\alpha_{23}N_2(s_2 + \alpha_{11}N_1)
\]- $u_{10}\alpha_{21}N_3(s_2 + \alpha_{33}N_3)$

$K_{10} = u_{10}\alpha_{20}\alpha_{32}N_2 N_3 + u_{30}\alpha_{13}\alpha_{21}N_1 N_2$
\[-u_{20}\alpha_{13}\alpha_{31}N_1 N_3\]

$K_{11} = u_{20}(s_2 + \alpha_{11}N_1)(s_3 + \alpha_{33}N_3)$
\[-u_{30}\alpha_{23}N_2(s_3 + \alpha_{11}N_1)
\]- $u_{10}\alpha_{21}N_3(s_3 + \alpha_{33}N_3)$

$K_{12} = u_{10}\alpha_{20}\alpha_{32}N_2 N_3 + u_{30}\alpha_{13}\alpha_{21}N_1 N_2$
\[-u_{20}\alpha_{13}\alpha_{31}N_1 N_3\]

$K_{13} = u_{30}(s_1 + \alpha_{22}N_2)(s_1 + \alpha_{33}N_3)$
\[-u_{20}\alpha_{23}N_3(s_1 + \alpha_{22}N_2)
\]- $u_{10}\alpha_{31}N_3(s_1 + \alpha_{33}N_3)$

$K_{14} = u_{30}\alpha_{22}\alpha_{32}N_2 N_3 + u_{20}\alpha_{12}\alpha_{32}N_1 N_3$
\[-u_{10}\alpha_{21}\alpha_{32}N_2 N_3\]

$K_{15} = u_{30}(s_2 + \alpha_{11}N_1)(s_2 + \alpha_{33}N_3)$
\[-u_{20}\alpha_{23}N_3(s_2 + \alpha_{11}N_1)
\]- $u_{10}\alpha_{31}N_3(s_2 + \alpha_{33}N_3)$

$K_{16} = u_{30}\alpha_{22}\alpha_{32}N_2 N_3 + u_{10}\alpha_{21}\alpha_{32}N_2 N_3$
\[-u_{20}\alpha_{12}\alpha_{32}N_1 N_3\]

$K_{17} = u_{30}(s_3 + \alpha_{22}N_2)(s_3 + \alpha_{33}N_3)$
\[-u_{20}\alpha_{32}N_3(s_3 + \alpha_{11}N_1)
\]- $u_{10}\alpha_{31}N_3(s_3 + \alpha_{22}N_2)$

$K_{18} = u_{30}\alpha_{12}\alpha_{21}N_1 N_2 + u_{20}\alpha_{12}\alpha_{31}N_1 N_3$
\[-u_{10}\alpha_{21}\alpha_{32}N_2 N_3\]

where $u_{10}, u_{20}$ and $u_{30}$ are the initial strengths of $u_1, u_2$ and $u_3$ respectively.

V. GLOBAL STABILITY

**Theorem:** The equilibrium point $E_5(N_1, N_2, N_3)$ is globally asymptotically stable.

**Proof:** Let us consider the following Lyapunov function

\[V(N_1, N_2, N_3) = \left\{ N_1 - \bar{N}_1 - N_1 \ln \left( \frac{N_1}{\bar{N}_1} \right) \right\} + \left\{ N_2 - \bar{N}_2 - N_2 \ln \left( \frac{N_2}{\bar{N}_2} \right) \right\} + \left\{ N_3 - \bar{N}_3 - N_3 \ln \left( \frac{N_3}{\bar{N}_3} \right) \right\} \tag{5.1}\]

Differentiating $V$ with respect to $t$, we get

\[
\frac{dV}{dt} = \left( \frac{N_1 - \bar{N}_1}{N_1} \right) \frac{dN_1}{dt} + \left( \frac{N_2 - \bar{N}_2}{N_2} \right) \frac{dN_2}{dt} \tag{5.2}\]

\[
\frac{dV}{dt} = -\left( \frac{(\alpha_{12} - \alpha_{21} + \alpha_{31} + \alpha_{31})}{2} \right) \left[ N_1 - \bar{N}_1 \right]^2 - \left( \frac{(\alpha_{12} - \alpha_{21} + \alpha_{13} + \alpha_{31})}{2} \right) \left[ N_2 - \bar{N}_2 \right]^2 - \left( \frac{(\alpha_{32} + \alpha_{12} + \alpha_{13} + \alpha_{31})}{2} \right) \left[ N_3 - \bar{N}_3 \right]^2\]

Then,

\[
\frac{dV}{dt} < 0
\]

Therefore, the interior equilibrium point $E_5$ is globally asymptotically stable if $\alpha_{12} \geq \alpha_{21}$.

VI. NUMERICAL EXAMPLE

**Example 1.** Let

$a_1=1.5; a_2=3.45; a_3=2.65; a_4=1.01; a_5=0.2$;
$a_6=0.3; a_7=0.2; a_8=0.2; a_9=0.1; a_{10}=0.1.$
From the above graph N1, N2 & N3 converges with diminishing amplitude as time goes on.

Example 2. Let $a_1=1; a_2=1; a_3=1.5; a_{11}=0.2; a_{12}=0.1; a_{13}=0.1; a_{22}=0.2; a_{23}=0.1; a_{33}=0.2; a_{31}=0.1; a_{32}=0.2$
Fig 6.3.1: The Variation of $u_1$, $u_2$, & $u_3$ with respective Time (t) for system of Eq (4.2)

From the above graph $u_1$, $u_2$, & $u_3$ converges with diminishing amplitude. As time goes on increases the population sizes tends to equilibrium point.

Fig 6.3.2: The Phage portrait of $u_1$, $u_2$, & $u_3$ for system of Eq (4.2)

The above graphs show the variation with initial strengths 10, 8, 25 of prey, predator and competitor populations respectively.

VII. CONCLUSIONS

In this paper, we made an attempt to study a three species ecological model with a prey, predator and competitor to both the prey and predator. All possible equilibrium points are identified and stability of the interior equilibrium point is discussed using Routh-Hurwitz criteria for local stability and using Lyapunov function for global stability. The solutions of linearized equations for interior equilibrium point are determined and are represented graphically with a suitable example. Further the non-linear systems of equations are examined for global stability with the help of suitable examples by using Matlab. The analytical discussion and numerical examples clearly shows that the system is globally asymptotically stable.

REFERENCES