Bayes Estimation for a Mixture of the Weibull Distributions

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***Abstract*—The present paper proposes some Bayes estimators for the parameters from the mixture of two Weibull distributions when item failure data are available. The Bayes estimators under the informative and non-informative priors have been obtained. The Bayes predictive intervals are also determined.**

***Index Terms*—Bayes estimator; Mixture distribution; Item failure censoring.**
***MSC 2010 Codes* – 62A15, 62F15**

I. Introduction

A Finite mixture of some suitable probability distribution is recommended to study a population that is supposed to comprise a number of subpopulations mixing in an unknown proportion. The predictive interval for the mixture of Exponential distributions has been obtained by [1]. The predictive intervals for the mixture of two Exponential distributions have discussed by [2]. The Bayesian analysis for the mixing function in a mixture of two Exponential distributions by [3].

Extensively in recent years, one distribution that has been used as a model to deal with problems for a product life is the Weibull distribution. The Weibull distribution is a versatile distribution that can take on the characteristics of other types of distributions, based on the value of its shape parameter. Its applications in life - testing problems and survival analysis have been widely advocated ([4] - [5]). It has been used as model with diverse types of items such as ball bearing [6], vacuum tube [7] and electrical isolation [8]. Reference [9] found that the Weibull distribution might be adequate statistical model for stock returns. The application of Weibull distribution in the modeling and analysis of the survival data have been studied by [10]. The probability density function of two-parameter Weibull distribution is given by

$$f\left(x;v,\theta\right)=\frac{v}{\theta}\,x^{v-1}e^{-\frac{x^{v}}{\theta}}\ \ ;x>0,v>0,\theta>0, \tag{1}$$

where the parameters v and θ are referred to as the shape and scale parameters of the distribution, respectively.

Reference [11 -[13] considered the estimation of the Weibull shape parameter in censored data. Recently, [14] presented the estimation of reliability and prediction intervals for the Weibull distribution under Bayesian viewpoint. References [15 - 16] and others discussed the prediction problems in the Weibull distribution. Recently, [17 - 20] are among others who studied predictive inference for the future observations.

The Bayes estimators corresponding to the informative and non-informative priors have been obtained for the parameters of the mixture of two Weibull distributions in the present paper. The present paper also predicts the nature of the future observation when sufficient information of the past and the present behaviour of an event or an observation are available. Statistical prediction limits have many applications in the field of quality control and reliability and determination of these limits has been extensively investigated.

The aim of the article is to discuss about the Bayes estimation for mixture of the Weibull model under the item failure censoring. The considered model and the prior distributions are discussed in Section 2. The Bayes estimators under the Informative prior are obtained in Section 3 and for Non-Informative prior in Section 4. Similarly, the Bayes prediction intervals are obtained in Section 5 for Informative prior and in Section 6 for Non-Informative prior. The Bayes estimators for the scale parameter are obtained here with known as well as unknown shape parameter.

II. The Considered Model And The Prior Distributions

Let $x_1.x_2,...,x_n$ be the life times of n items put to test under the model (1). In life testing, fatigue failures and other kinds of destructive test situations, the observations usually occurred in ordered manner such a way that weakest items failed first and then the second one and so on. Let these n items are put to test without replacement and the test terminates as soon as the first r^{th} items fails $(r\leq n)$. This censoring scheme is called as the item failure censoring.

Consider a population of electric devices that age with time i.e., the no memory property does not hold for them and suppose the population comprises two sub-populations each with a different life expectancy that may increase or decrease with age is long. For such models, a mixture of Weibull distributions is appropriate because of the nature of the devices that age with time.

The distribution function for the mixture of two Weibull distributions is defined as

$$F\left(t\right)=pF_1\left(t\right)\ +\ \left(1-p\right)\ F_2\left(t\right)\ ;\ 0<p<1\,; \tag{2}$$

where $F_i\left(t\right)\ =1-exp\left(-\frac{T^v}{\theta_i}\right)\ i=1,2.$

Similarly, the corresponding mixture density function is

$$f\left(t\right)=pf_1\left(t\right)\ +\ \left(1-p\right)\ f_2\left(t\right)\ ;\ 0<p<1\,; \tag{3}$$

Suppose n items from the above mixture model are employed to a test with termination at time T for the first r failure items. From the r items, the r_1 items be observed from first subpopulation and r_2 items be observed from second subpopulation with $r=r_1+r_2$ and $n-r$ items are still

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functioning when test terminates. The likelihood function is defined as

$$L(\theta_1, \theta_2, p|t) = \left\{ \prod_{j=1}^{r_1} p f_1(t_{1j}) \right\} \left\{ \prod_{j=1}^{r_2} (1-p) f_2(t_{2j}) \right\} \cdot (1 - F(T))^{n-r}.$$

Here, $1 - F(T)$ be the probability that an item has the lifetime in the interval (T, ∞) and t_{ij} be the failure time of the j^{th} unit belonging to the i^{th} sub-population, where $j = 1, 2, ..., r_i, i = 1, 2$. Hence, the likelihood function (when shape parameter v is known) is given as

$$L(\theta_1, \theta_2, p|t) \propto \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) p^{n-r_2-k} q^{k+r_2} \frac{e^{-\frac{T_1}{\theta_1}}}{\theta_1^{r_1}} \frac{e^{-\frac{T_2}{\theta_2}}}{\theta_2^{r_2}}; \tag{4}$$

where $T_1 = (n - r - k) T^v + \sum_{j=1}^{r_1} t_{1j}^v$, $T_2 = kT^v + \sum_{j=1}^{r_2} t_{2j}^v$ and $p + q = 1$.

Similarly, the likelihood function when shape parameter v is unknown, is obtained as

$$(\theta_1, \theta_2, v, p|t) = \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) C_0\, p^{n-r_2-k} \cdot q^{k+r_2} \frac{e^{-\frac{T_1}{\theta_1}}}{\theta_1^{r_1}} \frac{e^{-\frac{T_2}{\theta_2}}}{\theta_2^{r_2}}; \tag{5}$$

where $C_0 = v^{r_1+r_2} \left(\prod_{j=1}^{r_1} t_{1j}^{v-1}\right) \left(\prod_{j=1}^{r_2} t_{2j}^{v-1}\right)$.

When shape parameter v is known, the natural family of conjugate prior of θ_1 and θ_2 are taken as the inverted Gamma (with the parameters θ_1, θ_2 and p are independent) having the probability density function

$$g_1(\theta_i) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \theta_i^{-\alpha_i-1} e^{-\frac{\beta_i}{\theta_i}}\ ;\ \alpha_i, \beta_i > 0, i = 1, 2. \tag{6}$$

In the situation where the researchers have no prior information about the parameter $\theta (= \theta_1, \theta_2)$ one may use the non-informative prior (when v is known) such as

$$g_2(\theta_i) = \theta_i^{-1}\ ;\ \theta_i > 0\ ;\ i = 1, 2. \tag{7}$$

We further assume that p is uniform random variable between zero and one and is given by

$$g_3(p) = 1\ ;\ 0 < p < 1. \tag{8}$$

Reference [21] considered the uniform distribution for the shape parameter v when both shape and scale parameters considered being random variable and is defined as

$$g_4(v) = \vartheta^{-1}\ ;\ 0 < v < \vartheta, \vartheta > 0. \tag{9}$$

III. The Bayes Estimators Under The Informative Prior

A. Known Shape Parameter

The joint posterior density for the parameters θ_1, θ_2 and p is obtained as

$$h_1(\theta_1, \theta_2, p) = \frac{L(\theta_1, \theta_2, p|t)\, g_1(\theta_1)\, g_1(\theta_2)\, g_3(p)}{\int_{\theta_1} \int_{\theta_2} \int_p L(\theta_1, \theta_2, p|t)\, g_1(\theta_1)\, g_1(\theta_2)\, g_3(p)\, dp\, d\theta_2\, d\theta_1}$$

$$= \frac{\Gamma(n+2)\Delta_1}{\Gamma(r_1+\alpha_1)\Gamma(r_2+\alpha_2)} \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) p^{n-r_2-k} \cdot q^{k+r_2} \frac{e^{-\frac{T_3}{\theta_1}}}{\theta_1^{r_1+\alpha_1+1}} \frac{e^{-\frac{T_4}{\theta_2}}}{\theta_2^{r_2+\alpha_2+1}};$$

where $T_3 = T_1 + \beta_1$ and $T_4 = T_2 + \beta_2$.

The marginal posterior densities for the parameters θ_1, θ_2 and p are obtained respectively as

$$h_{11}(\theta_1) = \frac{\Delta_1}{\Gamma(r_1+\alpha_1)} \sum_{k=0}^{n-r} \Delta' \frac{e^{-\frac{T_3}{\theta_1}}}{T_4^{r_2+\alpha_2}\, \theta_1^{r_1+\alpha_1+1}}, \tag{10}$$

$$h_{12}(\theta_2) = \frac{\Delta_1}{\Gamma(r_2+\alpha_2)} \sum_{k=0}^{n-r} \Delta' \frac{e^{-\frac{T_4}{\theta_2}}}{T_3^{r_1+\alpha_1}\, \theta_1^{r_2+\alpha_2+1}} \tag{11}$$

and

$$h_{13}(p) = \Delta_1 \Gamma(n+2) \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) \frac{p^{n-r_2-k} q^{k+r_2}}{T_3^{r_1+\alpha_1}\, T_4^{r_2+\alpha_2}}; \tag{12}$$

where $\Delta' = \left({}^{n-r}C_k\right) \Gamma(n - r_2 - k + 1) \cdot \Gamma(k + r_2 + 1)$ and $\Delta_1 = \left(\sum_{k=0}^{n-r} \Delta' \frac{T_3^{-r_1-\alpha_1}}{T_4^{r_2+\alpha_2}} \right)^{-1}$.

The Bayes estimator for the parameter θ_1 with respect to the square error loss function (posterior mean) is obtained as

$$\hat{\theta}_{1B1} = E_P(\theta_1) = \frac{\Delta_1}{r_1 + \alpha_1 - 1} \sum_{k=0}^{n-r} \frac{\Delta'}{T_3^{r_1+\alpha_1-1} T_4^{r_2+\alpha_2}}. \tag{13}$$

Here, the suffix P indicates that the expectation is taken under the posterior density. Similarly, the Bayes estimators corresponding to the parameters θ_2 and p are obtained respectively as

$$\hat{\theta}_{2B1} = E_P(\theta_2) = \frac{\Delta_1}{r_2 + \alpha_2 - 1} \sum_{k=0}^{n-r} \frac{\Delta'}{T_3^{r_1+\alpha_1} T_4^{r_2+\alpha_2-1}} \tag{14}$$

and

$$\hat{p}_{B1} = E_P(p) = \frac{\Delta_1}{n+2} \sum_{k=0}^{n-r} \frac{\Delta'(n-r_2-k+1)}{T_3^{r_1+\alpha_1} T_4^{r_2+\alpha_2}}. \tag{15}$$

B. Unknown Shape Parameter

When the shape parameter v is unknown, the joint prior density $g(\theta_1, \theta_2, v, p)$ is defined as when θ_1, θ_2, v and p are independent

$$g(\theta_1, \theta_2, v, p) = g_1(\theta_1) \cdot g_1(\theta_2) \cdot g_3(p) \cdot g_4(v).$$

Under joint prior density $g(\theta_1, \theta_2, v, p)$ the joint posterior density for the parameters θ_1, θ_2, v and p is obtained as

$$h_2(\theta_1, \theta_2, v, p) = \frac{\Gamma(n+2)\Delta_2 C_0}{\Gamma(r_1+\alpha_1)\Gamma(r_2+\alpha_2)} \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) \cdot p^{n-r_2-k} q^{k+r_2} \frac{exp(-T_3/\theta_1)}{\theta_1^{r_1+\alpha_1+1}} \frac{exp(-T_4/\theta_2)}{\theta_2^{r_2+\alpha_2+1}};$$

where $\Delta_2 = \left(\int_{v=0}^{\infty} C_0 \sum_{k=0}^{n-r} \Delta' \frac{T_3^{-r_1-\alpha_1}}{T_4^{r_2+\alpha_2}} dv \right)^{-1}$.

The marginal posterior densities are obtained respectively for the parameters θ_1, θ_2 and p

$$h_{21}(\theta_1) = \frac{\Delta_2}{\Gamma(r_1+\alpha_1)} \int_{v=0}^{\infty} C_0 \sum_{k=0}^{n-r} \Delta' \frac{exp(-T_3/\theta_1)}{T_4^{r_2+\alpha_2}\,\theta_1^{r_1+\alpha_1+1}} dv, \tag{16}$$

$$h_{22}(\theta_2) = \frac{\Delta_2}{\Gamma(r_2+\alpha_2)} \int_{v=0}^{\infty} C_0 \sum_{k=0}^{n-r} \Delta' \frac{exp(-T_4/\theta_2)}{T_3^{r_1+\alpha_1}\,\theta_2^{r_2+\alpha_2+1}} dv \tag{17}$$

and

$$h_{23}(p) = \Delta_2 \Gamma(n+2) \int_{v=0}^{\infty} C_0 \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) \frac{p^{n-r_2-k}\, q^{k+r_2}}{T_3^{r_1+\alpha_1}\, T_4^{r_2+\alpha_2}} dv. \tag{18}$$

The Bayes estimators corresponding to the parameters θ_1, θ_2 and p are given respectively as

$$\hat{\theta}_{1B2} = \frac{\Delta_2 C_1}{r_1+\alpha_1-1}; C_1 = \sum_{k=0}^{n-r} \int_{v=0}^{\infty} C_0 \frac{\Delta'}{T_3^{r_1+\alpha_1-1}\, T_4^{r_2+\alpha_2}} dv, \tag{19}$$

$$\hat{\theta}_{2B2} = \frac{\Delta_2 C_2}{r_2+\alpha_2-1}; C_2 = \sum_{k=0}^{n-r} \int_{v=0}^{\infty} C_0 \frac{\Delta'}{T_3^{r_1+\alpha_1}\, T_4^{r_2+\alpha_2-1}} dv \tag{20}$$

and

$$\hat{p}_{B2} = \frac{\Delta_2 C_3}{n+2};\ C_3 = \sum_{k=0}^{n-r} \int_{v=0}^{\infty} C_0 \frac{\Delta'(n-r_2-k+1)}{T_3^{r_1+\alpha_1}\, T_4^{r_2+\alpha_2}} dv. \tag{21}$$

Remark:

The marginal posterior corresponding to the shape parameter v does not exist in a close form. Hence, the Bayes estimator of v has no closed form.

IV. Bayes Estimators Under The Non-Informative Prior

A. Known Shape Parameter

Using the non-informative prior (7), the joint posterior density for the parameters θ_1, θ_2 and p is obtained as

$$h_3(\theta_1, \theta_2, p) = \frac{\Gamma(n+2)\Delta_3}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) \cdot p^{n-r_2-k} q^{k+r_2} \frac{exp(-T_1/\theta_1)}{\theta_1^{r_1+1}} \frac{exp(-T_2/\theta_2)}{\theta_2^{r_2+1}}$$

with the marginal posterior densities for the parameters θ_1, θ_2 and p are

$$h_{31}(\theta_1) = \frac{\Delta_3}{\Gamma(r_1)} \sum_{k=0}^{n-r} \Delta' \frac{exp(-T_1/\theta_1)}{T_2^{r_2}\,\theta_1^{r_1+1}}, \tag{22}$$

$$h_{32}(\theta_2) = \frac{\Delta_3}{\Gamma(r_2)} \sum_{k=0}^{n-r} \Delta' \frac{exp(-T_2/\theta_2)}{T_1^{r_1}\,\theta_2^{r_2+1}} \tag{23}$$

and

$$h_{33}(p) = \Delta_3\, \Gamma(n+2) \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) \frac{p^{n-r_2-k} q^{k+r_2}}{T_1^{r_1}\, T_2^{r_2}}; \tag{24}$$

where

$$\Delta_3 = \left(\sum_{k=0}^{n-r} \Delta' T_1^{-r_1} T_2^{-r_2} \right)^{-1}.$$

On the similar line, the Bayes estimators for the parameters θ_1, θ_2 and p are respectively obtained as

$$\hat{\theta}_{1B3} = \frac{\Delta_3}{r_1-1} \sum_{k=0}^{n-r} \frac{\Delta'}{T_1^{r_1-1}\, T_2^{r_2}}, \tag{25}$$

$$\hat{\theta}_{2B3} = \frac{\Delta_3}{r_2-1} \sum_{k=0}^{n-r} \frac{\Delta'}{T_1^{r_1}\, T_2^{r_2-1}} \tag{26}$$

and

$$\hat{p}_{B3} = \frac{\Delta_3}{n+2} \sum_{k=0}^{n-r} \frac{\Delta'\ (n-r_2-k+1)}{T_1^{r_1}\, T_2^{r_2}}. \tag{27}$$

B. Unknown Shape Parameter

Under the non-informative prior for the scale parameter with unknown shape parameter v, the joint posterior density for the parameters θ_1, θ_2, v and p is given as

$$h_4(\theta_1, \theta_2, v, p) = \frac{\Gamma(n+2)\,\Delta_4}{\Gamma(r_1)\,\Gamma(r_2)}\, C_0 \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) \cdot p^{n-r_2-k}\, q^{k+r_2} \cdot \frac{exp(-T_1/\theta_1)}{\theta_1^{r_1+1}} \frac{exp(-T_2/\theta_2)}{\theta_2^{r_2+1}};$$

$$\Delta_4 = \left(\int_{v=0}^{\infty} C_0 \sum_{k=0}^{n-r} \Delta'\, T_1^{-r_1}\, T_2^{-r_2} dv\right)^{-1}.$$

In addition, the marginal posterior densities are obtained similarly as

$$h_{41}(\theta_1) = \frac{\Delta_4}{\Gamma(r_1)} \int_{v=0}^{\infty} C_0 \sum_{k=0}^{n-r} \Delta' \frac{exp(-T_1/\theta_1)}{T_2^{r_2}\, \theta_1^{r_1+1}}\, dv, \tag{28}$$

$$h_{42}(\theta_2) = \frac{\Delta_4}{\Gamma(r_2)} \int_{v=0}^{\infty} C_0 \sum_{k=0}^{n-r} \Delta' \frac{exp(-T_2/\theta_2)}{T_1^{r_1}\, \theta_1^{r_2+1}}\, dv \tag{29}$$

and

$$h_{43}(p) = \Delta_4 \Gamma(n+2) \int_{v=0}^{\infty} C_0 \sum_{k=0}^{n-r} \left({}^{n-r}C_k\right) \frac{p^{n-r_2-k}\, q^{k+r_2}}{T_1^{r_1}\, T_2^{r_2}} dv. \tag{30}$$

The Bayes estimators corresponding to the parameters θ_1, θ_2 and p are

$$\hat{\theta}_{1B4} = \frac{\Delta_3\, C_1}{r_1 - 1}\ ;\ C_1 = \sum_{k=0}^{n-r} \int_{v=0}^{\infty} C_0 \frac{\Delta'}{T_1^{r_1-1}\, T_2^{r_2}} dv, \tag{31}$$

$$\hat{\theta}_{2B4} = \frac{\Delta_4\, C_2}{r_2 - 1}\ ;\ C_2 = \sum_{k=0}^{n-r} \int_{v=0}^{\infty} C_0 \frac{\Delta'}{T_1^{r_1}\, T_2^{r_2-1}} dv \tag{32}$$

and

$$\hat{p}_{B4} = \frac{\Delta_4\, C_3}{n+2}\ ;\ C_3 = \sum_{k=0}^{n-r} \int_{v=0}^{\infty} C_0 \frac{\Delta'(n - r_2 - k + 1)}{T_1^{r_1}\, T_2^{r_2}} dv. \tag{33}$$

V. Bayes Prediction Limits Under The Informative Prior

A. Known Shape Parameter

In the context of prediction, we consider that l and u are prediction lower and upper limits for a future observation Y if

$$Pr(l \le Y \le u) = 1 - \varepsilon, \tag{34}$$

where $1 - \varepsilon$ is called the confidence prediction coefficient.

Further, the predictive distribution for the future observation Y is obtained from the model (3) as

$$h_1'(y|t) = \int_{\theta_1=0}^{\infty} \int_{\theta_2=0}^{\infty} \int_{p=0}^{1} f(y) \cdot h_1(\theta_1, \theta_2, p)\ dp\, d\theta_2\, d\theta_1$$
$$= vy^{v-1} \int_{\theta_1=0}^{\infty} \int_{\theta_2=0}^{\infty} \int_{p=0}^{1} \left\{ \frac{p}{\theta_1} e^{-\frac{y^v}{\theta_1}} + \frac{q}{\theta_2} e^{-\frac{y^v}{\theta_2}} \right\} \cdot h_1(\theta_1, \theta_2, p)\ dp\, d\theta_2\, d\theta_1$$

$$h_1'(y|t) = \frac{\Delta_1 v y^{v-1}}{n+2} \sum_{k=0}^{n-r} \Delta' \left\{ \frac{(r_1+\alpha_1)(n - r_2 - k + 1)}{(T_3 + y^v)^{r_1+\alpha_1+1}\, T_4^{r+\alpha_2}} + \frac{(r_2+\alpha_2)(k + r_2 + 1)}{(T_4 + y^v)^{r_2+\alpha_2+1}\, T_3^{r_1+\alpha_1}} \right\} \tag{35}$$

A $100(1-\varepsilon)\,\%$ equal tail prediction limits is obtained by solving

$$\int_0^l h_1'(y|t)\ dy\ =\ \int_u^{\infty} h_1'(y|t)\ dy\ =\ \frac{\varepsilon}{2}. \tag{36}$$

Using equations (35) & (36), the Bayes prediction lower limit is obtained by solving the given equation

$$\frac{\varepsilon(n+2)}{2\Delta_1} = \sum_{k=0}^{n-r} \Delta_5 I_1(l)\,; \tag{37}$$

where $I_1(\omega) = n + 2 - \frac{n-r_2-k+1}{(1+\omega^v/T_3)^{r_1+\alpha_1}} - \frac{k+r_2+1}{(1+\omega^v/T_4)^{r_2+\alpha_2}}$ and $\Delta_5 = \Delta'\, T_3^{-r_1-\alpha_1}\, T_4^{-r_2-\alpha_2}$.

Similarly, the Bayes prediction upper limit is obtained by solving the given equation

$$\frac{(2-\varepsilon)(n+2)}{2\Delta_1} = \sum_{k=0}^{n-r} \Delta_5 I_1(u)\,. \tag{38}$$

B. Unknown Shape Parameter

The predictive distribution of a future observation Y is obtained by simplifying the given equation when shape parameter v is considered as the random variable

$$h_2'(y|t) = \int_{\theta_1=0}^{\infty} \int_{\theta_2=0}^{\infty} \int_{p=0}^{1} \int_{v=0}^{\infty} f(y) \cdot h_2(\theta_1, \theta_2, p, v) \cdot dv\, dp\, d\theta_2\, d\theta_1$$
$$= \frac{\Delta_2}{n+2} \sum_{k=0}^{n-r} \Delta' \int_{v=0}^{\infty} vy^{v-1} C_0 \left\{ \frac{(r_1+\alpha_1)}{(T_3 + y^v)^{r_1+\alpha_1+1}} \cdot \frac{(n - r_2 - k + 1)}{T_4^{r_2+\alpha_2}} + \frac{(r_2+\alpha_2)\ (k + r_2 + 1)}{(T_4 + y^v)^{r_2+\alpha_2+1}\, T_3^{r_1+\alpha_1}} \right\} dv\,.$$

On similar line the Bayes prediction lower and upper limits are obtained by solving for l and u, respectively, the equations

$$\frac{\varepsilon(n+2)}{2\,\Delta_2} = \sum_{k=0}^{n-r} \Delta_5 \int_{v=0}^{\infty} C_0\, I_1(l)\, dv \tag{39}$$

and

$$\frac{(n+2)(2-\varepsilon)}{2\,\Delta_2} = \sum_{k=0}^{n-r} \Delta_5 \int_{v=0}^{\infty} C_0\, I_1(u)\, dv. \tag{40}$$

VI. Bayes Prediction Limits Under The Non-Informative Prior

A. Known Shape Parameter

On similar line the Bayes predictive distribution for the future observation Y is obtained as

$$h_3'\left(y|t\right) = \frac{\Delta_3\ v y^{v-1}}{n+2} \sum_{k=0}^{n-r} \Delta' \left\{ \frac{(n-r_2-k+1)}{\left(T_1+y^v\right)^{r_1+1}} \cdot \frac{r_1}{T_2^{r_2}} + \frac{r_2}{T_1^{r_1}} \frac{(k+r_2+1)}{\left(T_2+y^v\right)^{r_2+1}} \right\}.$$

The Bayes prediction lower and upper limits are obtained respectively by solving the equations

$$\frac{\varepsilon\left(n+2\right)}{2\ \Delta_3} = \sum_{k=0}^{n-r} \Delta_6 I_2\left(l\right)$$

and

$$\frac{\left(2-\varepsilon\right)\left(n+2\right)}{2\ \Delta_3} = \sum_{k=0}^{n-r} \Delta_6 I_2\left(u\right);$$

where $I_2\left(\omega\right) = n+2 - \frac{n-r_2-k+1}{(1+\omega^v/T_1)^{r_1}} - \frac{k+r_2+1}{(1+\omega^v/T_2)^{r_2}}$ and $\Delta_6 = \Delta'\, T_1^{-r_1}\, T_2^{-r_2}$.

B. Unknown Shape Parameter

Similarly, the Bayes predictive distribution of a future observation Y is given as

$$h_4'\left(y|t\right) = \frac{\Delta_3}{n+2} \sum_{k=0}^{n-r} \Delta' \int_{v=0}^{\infty} v y^{v-1}\, C_0 \left\{ \frac{(n-r_2-k+1)}{\left(T_1+y^v\right)^{r_1+1}} \cdot \frac{r_1}{T_2^{r_2}} + \frac{r_2}{T_1^{r_1}} \frac{(k+r_2+1)}{\left(T_2+y^v\right)^{r_2+1}} \right\} dv.$$

The Bayes prediction lower and upper limits are obtained respectively by simplifying the given equations

$$\frac{\varepsilon\left(n+2\right)}{2\ \Delta_3} = \sum_{k=0}^{n-r} \Delta_6 \int_{v=0}^{\infty} C_0\ I_2\left(l\right) dv$$

and

$$\frac{\left(2-\varepsilon\right)\left(n+2\right)}{2\ \Delta_3} = \sum_{k=0}^{n-r} \Delta_6 \int_{v=0}^{\infty} C_0\ I_2\left(u\right) dv.$$

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