Some Properties of Fuzzy Hilbert Spaces

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***Abstract*—Some results on fuzzy inner product spaces are proved. We establish some fundamental results viz. Bessel's inequality, Riesz representation theorem etc. in fuzzy setting.**

***Index Terms*—Fuzzy Hilbert space, orthonormal set, Bessel's inequality, Riesz representation theorem.**

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I. Introduction

In studying fuzzy topological vector spaces, Katsaras [1] in 1984, first introduced the notion of fuzzy norm on a linear space. Later on many author viz. Felbin [2], Cheng & Mordeson [3], Bag and Samanta [4] etc. have given different definitions of fuzzy normed linear spaces.
R. Biswas [5] and A. M. El-Abyed & H. M. El-Hamouly [6] first tried to give a meaningful definition of fuzzy inner product space and associated fuzzy norm function. Also those definitions are restricted to the real linear space only. Recently Pinaki Mazumder & S. K. Samanta [7] introduced a definition of fuzzy inner product space whose associated fuzzy norm is of Bag & Samanta [4] type. Where as A.Hasankhani, A. Nazari & M.Saheli [8] have introduced a definition of fuzzy inner product space whose associated norm is of Felbin [2] type.
Following the definition of fuzzy inner product space introduced by Pinaki Mazumder & S. K. Samanta [7], we study some results. Bessel's inequality and Riesz representation theorem are established in fuzzy setting.
The organization of the paper is as follows:
Section 2, provides some preliminary results which are used in this paper.
In section 3, we study some results on fuzzy inner product spaces.
Section 4, is devoted to establish Bessel's inequality and Riesz representation theorem in fuzzy setting.

II. Some preliminary results.

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 2.1 [4]. Let U be a linear space over a field F(field of real / complex numbers).
A fuzzy subset N of $U \times \mathbf{R}$ ($\mathbf{R}$ is the set of real numbers) is called a fuzzy norm on U if $\forall x, u \in U$ and $c \in F$, following conditions are satisfied:
(N1) $\forall t \in R$ with $t \leq 0,\ N(x\ ,\ t) = 0$;
(N2) ($\forall t \in R,\ t > 0,\ N(x\ ,\ t) = 1$) iff $x = \underline{0}$;
(N3) $\forall t \in R,\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;

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(N4) $\forall s, t \in R,\ x, u \in U$;

$$N(x+u\ ,\ s+t) \geq min\{N(x\ ,\ s)\ ,\ N(u\ ,\ t)\}$$

(N5) $N(x\ ,\ .)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x\ ,\ t) = 1$.
The pair $(U\ ,\ N)$ will be referred to as a fuzzy normed linear space.

Theorem 2.1 [4]. *Let $(U\ ,\ N)$ be a fuzzy normed linear space.*
Assume further that,
(N6) $\forall t > 0,\ N(x\ ,\ t) > 0$ *implies* $x = \underline{0}$.
Define $||x||_\alpha = \wedge\{t > 0 :\ N(x\ ,\ t) \geq \alpha\},\ \alpha \in (0\ ,\ 1)$. *Then* $\{||\ ||_\alpha :\ \alpha \in (0\ ,\ 1)\}$ *is an ascending family of norms on U and they are called α-norms on U corresponding to the fuzzy norm N on U.*

Definition 2.2 [4]. Let $(U\ ,\ N)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ is said to be convergent if $\exists x \in U$ such that $\lim_{n\to\infty} N(x_n - x\ ,\ t) = 1\ \forall t > 0$. In that case x is called the limit of the sequence $\{x_n\}$ and we denote it by $lim x_n$.

Proposition 2.1 [4]. *Let $(U\ ,\ N)$ be a fuzzy normed linear space satisfying (N6) and $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ converges to x iff $x_n \to x$ w.r.t. $||\ ||_\alpha$ (where $||\ ||_\alpha$ denotes the α-norm of N), $\forall \alpha \in (0\ ,\ 1)$.*

Definition 2.3 [9]. Let (U, N_1) be a fuzzy normed linear space satisfying (N6). Let $T \in U^*$ and $\{||\ ||^1_\alpha :\ \alpha \in (0,1)\}$ be the family of α-norms of N_1.
We define $$||T||^*_\alpha = \bigvee_{x\in U, x\neq \underline{0}} \frac{|T(x)|}{||x||^1_{1-\alpha}}\ \forall \alpha \in (0,1).$$
Then $\{||\ ||^*_\alpha :\ \alpha \in (0,1)\}$ is an ascending family of norms on U^*.

Definition 2.4 [9]. Let (U, N) be a fuzzy normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in U is said to be α-convergent in U if $\exists\ x \in U$ such that
$lim_{n\to\infty} N(x_n - x, t) > \alpha\ \forall\ t > 0$ and x is called the limit of $\{x_n\}$.

Proposition 2.2 [9]. *Let $(U\ ,\ N)$ be a fuzzy normed linear space satisfying (N6). If $\{x_n\}$ be an α-convergent sequence in $(U\ ,\ N)$. Then $||x_n - x||_\alpha \to 0$ as $n \to \infty$ ($||\ ||_\alpha$ denotes the α-norm of N).*

Definition 2.5 [10]. Let $(U\ ,\ N)$ be a fuzzy normed linear space. A subset F of U is said to be l-fuzzy closed if for each $\alpha \in (0\ ,\ 1)$ and for any sequence $\{x_n\}$ in F and $x \in U, (\underline{lim}_{n\to\infty} N(x_n - x\ ,\ t) \geq \alpha\ \forall t > 0) \Rightarrow x \in F$.

Proposition 2.3 [10]. *Let $(U\ ,\ N)$ be a fuzzy normed linear space satisfying (N6) and $F \subset U$. Then F is l-fuzzy closed iff F is closed w.r.t. $||\ ||_\alpha$ ($\alpha-norm\ of\ N$) for each $\alpha \in (0\ ,\ 1)$.*

Definition 2.6 [7]. Let V be a linear space over the field C of complex numbers. Let $\mu : V \times V \times C \to I = [0,1]$ be a mapping such that the following holds

(FIP1) For $s, t \in C$, $\mu(x+y, z, |t|+|s|) \geq min\{\mu(x,z,|t|), \mu(y,z,|s|)\}$;
(FIP2) For $s, t \in C$, $\mu(x,y,|st|) \leq min\{\mu(x,x,|s|^2), \mu(y,y,|t|^2)\}$;
(FIP3) For $t \in C$, $\mu(x,y,t) = \mu(y,x,\bar{t})$;
(FIP4) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|})$, $\alpha(\neq 0) \in C, t \in C$;
(FIP5) $\mu(x,x,t) = 0 \ \ \forall t \in C \setminus R^+$;
(FIP6) $(\mu(x,x,t) = 1 \ \forall t > 0) \ iff \ x = \underline{0}$;
(FIP7) $\mu(x,x,\cdot) : \mathbf{R} \to I(=[0,1])$ is a monotonic non-decreasing function of $\mathbf{R}$ and $\lim_{t\to\infty} \mu(x,x,t) = 1$.
We shall call μ to be the fuzzy inner product (FIP in short) function on V and (V, μ) is called a fuzzy inner product space(FIP space).

Theorem 2.2 [7]. *Let V be a linear space over C. Let μ be a FIP on V. Then*

$$N(x,t) = \begin{cases} \mu(x,x,t^2), \ t \in \mathbf{R}, & if \ t > 0 \\ 0 & if \ t \leq 0 \end{cases}$$

is a fuzzy norm on V. Now if μ satisfies the following conditions:
*(**FIP8**) $(\mu(x,x,t^2) > 0 \ , \ \forall t > 0) \ \Rightarrow x = \underline{0}$ and*
*(**FIP9**) For all $x, y \in V$ and $p, q \in \mathbf{R}$,*
$\mu(x+y, x+y, 2q^2) \bigwedge \mu(x-y, x-y, 2p^2) \geq \mu(x,x,p^2) \bigwedge \mu(x,x,q^2)$
then $||x||_\alpha = \wedge\{t > 0 : N(x, t) \geq \alpha\}$ $\alpha \in (0, 1)$, is an ordinary norm (α-norm) on V satisfying parallelogram law.
Then using Polarization identity we can get ordinary inner product, called the α-inner product,as follows:
$\langle x, y\rangle_\alpha = \frac{1}{4}(||x+y||_\alpha{}^2 - ||x-y||_\alpha{}^2) + \frac{1}{4}i(||x+iy||_\alpha{}^2 - ||x-iy||_\alpha{}^2)$, $\forall \alpha \in (0,1)$.

Definition 2.7 [7]. Let (V, μ) be a FIP space satisfying (FIP8). V is said to be level complete if for any $\alpha \in (0, 1)$, every cauchy sequence converges in V w.r.t. $|| \ ||_\alpha$ (the α-norm generated by the fuzzy norm N which is induced by fuzzy inner product μ).

Definition 2.8 [7]. Let (V, μ) be a FIP space. V is said to be a fuzzy Hilbert space, if it is level complete.

Definition 2.9 [7]. Let $\alpha \in (0,1)$ and (V, μ) be a FIP space satisfying (FIP8) and (FIP9). Now if $x, y \in V$ be such that $\langle x, y\rangle_\alpha = 0$, then we say that x, y are α-fuzzy orthogonal to each other and is denoted by $x \perp_\alpha y$. Let M be a subset of V and $x \in V$. Now if $\langle x, y\rangle_\alpha = 0 \ \ \forall y \in M$, then we say that x is α-fuzzy orthogonal to M and is denoted by $x \perp_\alpha M$.

Definition 2.10 [7]. Let (V, μ) be a FIP space satisfying (FIP8) and (FIP9). Now if $x, y \in V$ be such that $\langle x, y\rangle_\alpha = 0 \ \ \forall \alpha \in (0, 1)$, then we say that x, y are fuzzy orthogonal to each other and is denoted by $x \perp y$.
Thus $x \perp y$ iff $x \perp_\alpha y \ \ \forall \alpha \in (0, 1)$.

III. Some results on fuzzy inner product spaces.

Here we study some results on fuzzy inner product spaces.

Theorem 3.1. *Let (V, N) be a fuzzy normed linear space. Assume that for $x, y \in V$ and $s, t \in C$,*
$min\{N(x, |st|), N(y, |st|)\} \geq min\{N(x, |s|^2), N(y, |t|^2)\}$.
Define $\mu' : V \times V \times C \to [0, 1]$ as

$$\mu'(x, y, s+t) = 0$$

if $x = y$ and $s + t \in C - R^+$ and elsewhere as

$$\mu'(x, y, s+t) = N(x, |s|) \bigvee N(y, |t|)$$

Then μ' is a fuzzy inner product on V.

Proof. (FIP1) For $s, t \in C$ and $x, y, z \in V$ we have
$\mu'(x+y, z, |s|+|t|)$
$= \mu'(x+y, z, |s|+|t|+0)$
$= N(x+y, |s|+|t|) \bigvee N(z, 0)$
$= N(x+y, |s|+|t|) \geq min\{N(x, |s|), N(y, |t|)\}$
$= min\{\mu'(x, z, |s|), \mu'(y, z, |t|)\}$.
(FIP2) $\mu'(x, y, |st|) = N(x, |st|) = N(y, |st|)$
$= min\{N(x, |st|), N(y, |st|)\}$
$\geq min\{N(x, |s|^2), N(y, |t|^2)\}$ [By condition]
$min\{\mu'(x, x, |s|^2), \mu'(y, y, |t|^2)\}$.
(FIP3) $\mu'(x, y, t) = N(x, |t|) = N(x, |\bar{t}|)$
$= \mu'(x, y, \bar{t}) = N(y, |\bar{t}|)$
$= \mu'(y, x, \bar{t})$.
(FIP4) $\mu'(\alpha x, y, t) = N(\alpha x, |t|)$
$= N(x, \frac{|t|}{|\alpha|})[\alpha \neq 0]$
$= \mu'(x, y, \frac{t}{|\alpha|})$.
(FIP5) $\mu'(x, x, t) = 0 \ \ \forall \ t \in C - R^+$ [By definition].
(FIP6) $\mu'(x, x, t) = 1 \ \ \forall \ t > 0$
$\Leftrightarrow N(x, t) = 1 \forall \ t > 0$
$\Leftrightarrow x = 0$.
(FIP7) Since $\mu'(x, x, .) = N(x, .)$
and $N(x, .)$ is a monotonic nondecreasing function of R and $\lim_{t\to\infty} N(x, t) = 1$
$\Rightarrow \mu'$ has also the property. Thus μ' is a fuzzy inner product on V.

Theorem 3.2. *Let (V, μ) be a fuzzy innerprduct space satisfying (FIP8) and (FIP9) and $\langle \ , \ \rangle_\alpha$ be its α-inner product $\forall \ \alpha \in (0, 1)$.*
Define a function $\mu' : V \times V \times C \to [0, 1]$ as

$$\mu'(x, y, t) = 0$$

if $x = y$ and $\forall \ t \in C - R^+$ and elsewhere as

$$\mu'(x, y, t) = \bigvee\{\alpha \in (0, 1) : |\langle x, y\rangle_\alpha| \leq |t|\}$$

Then μ' is a fuzzy inner product on V if $|\langle \ , \ \rangle_\alpha|$ is increasing function of R.

Proof. (FIP1) For $s, t \in C$ and $x, y, z \in V$ we have to show that
$\mu'(x+y, z, |s|+|t|) \geq min\{\mu'(x, z, |s|), \mu'(y, z, |t|)\}$
Let $p = \mu'(x, z, |s|)$ and $q = \mu'(y, z, |t|)$
Without loss of generality assume that $p \leq q$
Let $0 < r < p \leq q$
Then $\exists \ \alpha > r$ such that $|\langle x, z\rangle_\alpha| < |s|$ and
$\exists \ \beta > r$ such that $|\langle y, z\rangle_\beta| < |t|$
Let $\gamma = \alpha \wedge \beta > r$
Thus $|\langle x, z\rangle_\gamma| < |\langle x, z\rangle_\alpha| < |s|$ and
$|\langle y, z\rangle_\gamma| < |\langle y, z\rangle_\beta| < |t|$ [Since $|\langle \ , \ \rangle_\alpha|$ is increasing]
Now $|\langle x+y, z\rangle_\gamma| = |\langle x, z\rangle_\gamma + \langle y, z\rangle_\gamma|$
$\leq |\langle x, z\rangle_\gamma| + |\langle y, z\rangle_\gamma|$

$< |s| + |t|$
Therefore $\mu'(x+y,\ z,\ |s|+|t|) \geq \gamma > r$
Since $0 < r < \gamma$ is arbitrary, thus $\mu'(x+y,\ z,\ |s|+|t|) \geq min\{\mu'(x,\ z,\ |s|),\ \mu'(y,\ z,\ |t|)\}$.
(FIP2) For $s,\ t \in C$ and $x,\ y, \in V$ we have to show that $\mu'(x,\ y,\ |st|) \geq min\{\mu'(x,\ y,\ |s|^2),\ \mu'(x,\ y,\ |t|^2)\}$
Let $p = \mu'(x,\ y,\ |s|^2)$ and $q = \mu'(x,\ y,\ |t|^2)$
Without loss of generality assume that $p \leq q$
Let $0 < r < p \leq q$
Then $\exists\ \alpha > r$ such that $|\langle x,\ y\rangle_\alpha| < |s|^2$ and
$\exists\ \beta > r$ such that $|\langle x,\ y\rangle_\beta| < |t|^2$
Let $\gamma = \alpha \wedge \beta > r$
Thus $|\langle x,\ y\rangle_\gamma| < |\langle x,\ y\rangle_\alpha| < |s|^2$ and
$|\langle x,\ y\rangle_\gamma| < |\langle x,\ y\rangle_\beta| < |t|^2$ [Since $|\langle\ ,\ \rangle_\alpha|$ is increasing]
Therefore $|\langle x,\ y\rangle_\gamma|^2 < |s|^2|t|^2$
$\Rightarrow |\langle x,\ y\rangle_\gamma| < |st|$
$\Rightarrow \mu'(x,\ y,\ |st|) \geq \gamma > r$
Since $0 < r < \gamma$ is arbitrary, thus
$\mu'(x,\ y,\ |st|) \geq min\{\mu'(x,\ y,\ |s|^2),\ \mu'(x,\ y,\ |t|^2)\}$.
(FIP3) For $t \in C$,
$\mu'(x,\ y,\ t) = \mu'(x,\ y,\ \bar{t}) = 0$ if $x = y$ and $\forall\ t \in C - R^+$
Now let $t \in C$ and $x \neq y$ then
$\mu'(x,\ y,\ t) = \bigvee\{\alpha \in (0,\ 1) : |\langle x,\ y\rangle_\alpha| \leq |t|\}$
$= \bigvee\{\alpha \in (0,\ 1) : |\langle x,\ y\rangle_\alpha| \leq |\bar{t}|\} = \mu'(x,\ y,\ \bar{t})$.
(FIP4)For $c \in C$,
$\mu'(cx,\ y,\ t) = \bigvee\{\alpha \in (0,\ 1) : |\langle cx,\ y\rangle_\alpha| \leq |t|\} = \bigvee\{\alpha \in (0,\ 1) : |c||\langle x,\ y\rangle_\alpha| \leq |t|\}$
$= \bigvee\{\alpha \in (0,\ 1) : |c||\langle x,\ y\rangle_\alpha| \leq \frac{|t|}{|c|}\}$
$= \mu'(cx,\ y,\ \frac{t}{|c|})$.
(FIP5) $\mu'(x,\ x,\ t) = 0 \forall\ t \in C - R^+$ [By definition].
(FIP6) $\mu'(x,\ x,\ t) = 1\ \ \forall\ t > 0$
$\Leftrightarrow \bigvee\{\alpha \in (0,\ 1) : |\langle x,\ x\rangle_\alpha| \leq |t|\} = 1\ \ \forall\ t > 0$
$\Leftrightarrow \langle x,\ x\rangle_\alpha = 0\ \ \forall\ \alpha \in (0,\ 1)$
$\Leftrightarrow x = \underline{0}$.
(FIP7) $\mu'(x,\ x,\ t) = \bigvee\{\alpha \in (0,\ 1) : |\langle x,\ x\rangle_\alpha| \leq |t|\}$
$= \bigvee\{\alpha \in (0,\ 1) : ||x||_\alpha^2 \leq t\}\ \ \forall\ t > 0$
$= \bigvee\{\alpha \in (0,\ 1) : ||x||_\alpha \leq \sqrt{t}\}$
Now $t_1 > t_2$
$\Rightarrow \sqrt{t_1} > \sqrt{t_2}$
$\Rightarrow \{\alpha \in (0,\ 1) : ||x||_\alpha \leq \sqrt{t_1}\} \supset \{\alpha \in (0,\ 1) : ||x||_\alpha \leq \sqrt{t_2}\}$
$\Rightarrow \bigvee\{\alpha \in (0,\ 1) : ||x||_\alpha \leq \sqrt{t_1}\} \geq \bigvee\{\alpha \in (0,\ 1) : ||x||_\alpha \leq \sqrt{t_2}\}\ \ \forall\ \alpha \in (0,\ 1)$
$\Rightarrow \mu'(x,\ x,\ t_1) \geq \mu'(x,\ x,\ t_2)$
Therefore $\mu'(x,\ x,\ .) : R^+ \rightarrow [0,\ 1]$ is increasing and $\lim_{t\to\infty} \mu'(x,\ x,\ t) = 1$.
Thus μ' is a fuzzy inner product on V.

IV. Orthonormal set, Bessel's inequality & Reisz representation theorem.

In this section orthonormal set, sequence are defined and Bessel's inequality, Riesz representation theorem are established in fuzzy Hilbert spaces.

Definition 4.1. Let $(V,\ \mu)$ be a fuzzy inner product space satisfying (FIP8) and (FIP9) and $\alpha \in (0,\ 1)$. An α-fuzzy orthogonal set M in V is said to be α-fuzzy orthonormal if the elements have α-norm 1 that is $\forall x,\ y \in M$,

$$\langle x,\ y\rangle_\alpha = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

where $\langle\ ,\ \ \rangle_\alpha$ is the induced inner product by μ.

Definition 4.2. Let $(V,\ \mu)$ be a fuzzy inner product space satisfying (FIP8) and (FIP9) . A fuzzy orthogonal set M in V is said to be fuzzy orthonormal if the elements have α-norm 1 $\forall \alpha \in (0,\ 1)$ that is $\forall x,\ y \in M$

$$\langle x,\ y\rangle_\alpha = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \qquad \forall \alpha \in (0,\ 1)$$

where $\langle\ ,\ \rangle_\alpha$ is induced inner product by μ.

Proposition 4.1. *An α-fuzzy orthonormal set and a fuzzy orthonormal set in a FIP space are linearly independent.*

Proof. Proof is straightforward.

Proposition 4.2. *Let $\{e_k\}$ be a fuzzy orthonormal sequence in a fuzzy Hilbert space $(V,\ \mu)$ satisfying (FIP8) and (FIP9). Then the series $\sum_k \alpha_k e_k$ converges (w.r.t. $||\ ||_\alpha$; $\alpha \in (0,\ 1)$, where $||\ ||_\alpha$ are the α-norms of N which is induced by μ) iff $\sum_k |\alpha_k|^2$ converges.*

Proof. Let $S_n = \sum_{k=1}^{n} \alpha_k e_k$ and $\sigma_n = \sum_{k=1}^{n} |\alpha_k|^2$.
Then

$$||S_n - S_m||_\alpha^2 = \langle \sum_{k=1}^{n} \alpha_k e_k - \sum_{k=1}^{m} \alpha_k e_k,\ \sum_{k=1}^{n} \alpha_k e_k - \sum_{k=1}^{m} \alpha_k e_k \rangle_\alpha$$

i.e.

$$||S_n - S_m||_\alpha^2 = |\alpha_{m+1}|^2 + |\alpha_{m+2}|^2 + ... + |\alpha_n|^2$$

i.e.

$$||S_n - S_m||_\alpha^2 = \sigma_n - \sigma_m\ \ \forall \alpha \in (0,\ 1).$$

Hence S_n is Cauchy w.r.t $||\ ||_\alpha,\ \ \forall \alpha \in (0,\ 1)$ iff σ_n is Cauchy in R(the set of real numbers).
Hence S_n is Cauchy iff σ_n is Cauchy in R.
Hence the proof.

Note 4.1. In case of α-fuzzy orthonormal sequence the Proposition holds.

Proposition 4.3. *Let $(V,\ \mu)$ be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and $\alpha \in (0,\ 1)$ and let $\{e_k\}$ be an α-fuzzy orthonormal sequence in V. If the series $\sum_{k=1}^{\infty} \beta_k e_k$ is α-convergent w.r.t. N induced by μ, then the coefficients $\beta_k = \langle x,\ e_k\rangle_\alpha$ where x denotes the sum $\sum_{k=1}^{\infty} \beta_k e_k$ and hence $x = \sum_{k=1}^{\infty} \langle x,\ e_k\rangle_\alpha e_k$.*

Proof. Since $\sum_{k=1}^{\infty} \beta_k e_k$ is α-convergent so it is convergent w.r.t. $||\ ||_\alpha$ [By Definition 2.4 & Proposition 2.2].
i.e. $||\sum_{k=1}^{n} \beta_k e_k - x||_\alpha \rightarrow 0$ as $n \rightarrow \infty$.

Let $S_n = \sum_{k=1}^{n} \beta_k e_k$.
Taking innerproduct with S_n and e_j and using the definition of α-fuzzy orthogonality we have
$\langle S_n,\ e_j \rangle_\alpha = \beta_j$ for $j = 1, 2, ...k$
Now $S_n \to x$ w.r.t. $||\ ||_\alpha$
$\Rightarrow \langle S_n,\ e_j \rangle_\alpha \to \langle x,\ e_j \rangle_\alpha = \beta_j$
Therefore $x = \sum_{k=1}^{\infty} \beta_k e_k = \sum_{k=1}^{\infty} \langle x,\ e_k \rangle_\alpha e_k$.

Theorem 4.1 *Let $(V,\ \mu)$ be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and $\{e_k\}$ be a fuzzy orthonormal sequence in V. If the series $\sum_{k=1}^{\infty} \gamma_k e_k$ converges w.r.t N induced by μ, then*
$\gamma_k = \langle x,\ e_k \rangle_\alpha = \langle x,\ e_k \rangle_\beta \quad \forall \alpha,\ \beta \in (0,\ 1)$,
where $\langle\ ,\ \rangle_\alpha$ denotes the α-innerproduct induced by μ, x denotes the sum of $\sum_{k=1}^{\infty} \gamma_k e_k$.
Hence $x = \sum_{k=1}^{\infty} \langle x,\ e_k \rangle_\alpha e_k = \sum_{k=1}^{\infty} \langle x,\ e_k \rangle_\beta e_k \quad \forall \alpha,\ \beta \in (0,\ 1)$.

Proof. Since $\{e_k\}$ is fuzzy orthonormal, it is orthonormal w.r.t. each $\langle\ ,\ \rangle_\alpha$, $\alpha \in (0,\ 1)$.
Now $\sum_{k=1}^{\infty} \gamma_k e_k$ is convergent w.r.t. N implies it is convergent w.r.t. $||\ ||_\alpha$ (induced α-norm of N) $\forall \alpha \in (0,\ 1)$ [by Proposition 2.1].
We have $\langle \sum_{k=1}^{n} \gamma_k e_k,\ e_j \rangle_\alpha = \gamma_j = \langle \sum_{k=1}^{n} \gamma_k e_k,\ e_j \rangle_\beta$, $j = 1, 2, ...k$
Taking limit as $n \to \infty$ we have
$\lim_{n\to\infty} \langle \sum_{k=1}^{n} \gamma_k e_k,\ e_j \rangle_\alpha = \lim_{n\to\infty} \gamma_j = \lim_{n\to\infty} \langle \sum_{k=1}^{n} \gamma_k e_k,\ e_j \rangle_\beta$
$\Rightarrow \langle \sum_{k=1}^{\infty} \gamma_k e_k,\ e_j \rangle_\alpha = \gamma_j = \langle \sum_{k=1}^{\infty} \gamma_k e_k,\ e_j \rangle_\beta$ [since $\langle\ ,\ \rangle_\alpha$ is continous $\forall \alpha \in (0,\ 1)$]
$\Rightarrow \langle x,\ e_j \rangle_\alpha = \gamma_j = \langle x,\ e_j \rangle_\beta \quad \forall \alpha,\ \beta \in (0,\ 1), \quad j = 1,\ 2,\ 3....$.

Definition 4.3. Let $(V,\ \mu)$ be a fuzzy innerproduct space satisfying (FIP8) and (FIP9). A fuzzy orthonormal set $M \subset V$ is called complete fuzzy orthonormal set if there is no α-fuzzy orthonormal set ($\alpha \in (0,\ 1)$) of which M is a proper subset. If M is countable then we call M is a complete fuzzy orthonormal sequence.

Theorem 4.2 (Bessel's inequality) *Let $(V,\ \mu)$ be a fuzzy Hilbert space satisfying (FIP8), (FIP9) and $\alpha \in (0,\ 1)$ and $\{e_k\}$ be an α-fuzzy orthonormal sequence in V. Then for every $x \in V$,*
$\sum_{k=1}^{\infty} |\langle x,\ e_k \rangle_\alpha|^2 \leq ||x||_\alpha^2$.

Proof. Since α-fuzzy orthonormal sequence is orthonormal sequence in $(V,\ \langle\ ,\ \rangle_\alpha)$, so by Bessel's inequality in crisp innerproduct we have
$\sum_{k=1}^{\infty} |\langle x,\ e_k \rangle_\alpha|^2 \leq ||x||_\alpha^2$.

Note 4.2 In case of fuzzy orthonormal sequence it will be $\sum_{k=1}^{\infty} |\langle x,\ e_k \rangle_\alpha|^2 \leq ||x_\alpha||^2 \quad \forall \alpha \in (0,\ 1)$.

Theorem 4.3. *Let $(V,\ \mu)$ be a Hilbert space satisfying (FIP9) and $\{e_i\}$ is fuzzy orthonormal sequence in V. Then the following statements are equivalent.*
(i) $\{e_i\}$ is complete fuzzy orthonormal.
(ii) $x \perp e_i$ for $i = 1, 2, ...\quad \Rightarrow x = \underline{0}$.
(iii) For every $x \in V$, $x = \sum_{k=1}^{\infty} \langle x,\ e_i \rangle_\alpha e_i \quad \forall \alpha \in (0,\ 1)$ and hence
$\langle x,\ e_k \rangle_\alpha = \langle x,\ e_k \rangle_\beta \quad \forall \alpha,\ \beta \in (0,\ 1)$ i.e. x is independent on α.
(iv) For every $x \in V$, $||x||_\alpha^2 = \sum_{k=1}^{\infty} |\langle x,\ e_i \rangle_\alpha| \quad \forall \alpha \in (0,\ 1)$ and hence
$||x||_\alpha^2 = ||x||_\beta^2 \quad \forall \alpha,\ \beta \in (0,\ 1)$.

Proof. (a) Suppose (i) holds.
Let $\{e_i\}$ be a complete fuzzy orthonormal sequence and $x \perp e_i$ for $i = 1, 2, ..$
$\Rightarrow x \perp_\alpha e_i \quad \forall \alpha \in (0,\ 1)$ and $i = 1, 2, ..$
$\Rightarrow \langle x,\ e_i \rangle_\alpha = 0 \quad \forall \alpha \in (0,\ 1)$ and $i = 1, 2, ..$
Set for a fixed α_0, $\quad e^{\alpha_0} = \frac{x}{||x||_{\alpha_0}}$
Then $||e^{\alpha_0}||_{\alpha_0}^2 = \langle e^{\alpha_0},\ e^{\alpha_0} \rangle_{\alpha_0} = 1$ and
$\langle e^{\alpha_0},\ e_i \rangle_{\alpha_0} = 0$ for $i = 1, 2, ..$
Threfore we get an α_0-fuzzy orthonormal sequence $\{e^{\alpha_0},\ e_1,\ e_2, ...\}$ of which $\{e_1,\ e_2, ...\}$ is proper subset-a contraction to completeness.
Therefore $e^{\alpha_0} = \underline{0}$.
$\Rightarrow x = 0$.
So (i) $\Rightarrow$ (ii).

(b) Suppose (ii) holds.
Let $x \perp e_i$ for $i = 1, 2, ..$ implies $x = 0$.
$\Rightarrow x - \sum_{i=1}^{\infty} \langle x,\ e_i \rangle_\alpha e_i \perp_\alpha e_j \quad j = 1, 2, ..$ and $\forall \alpha \in (0,\ 1)$
$\Rightarrow x - \sum_{i=1}^{\infty} \langle x,\ e_i \rangle_\alpha e_i = 0 \quad \forall \alpha \in (0,\ 1)$
$\Rightarrow x = \sum_{i=1}^{\infty} \langle x,\ e_i \rangle_\alpha e_i = \sum_{i=1}^{\infty} \langle x,\ e_i \rangle_\beta e_i \quad \forall \alpha,\ \beta \in (0,\ 1)$
$\Rightarrow \sum_{i=1}^{\infty} (\langle x,\ e_i \rangle_\alpha - \langle x,\ e_i \rangle_\beta) e_i = 0 \quad \forall \alpha,\ \beta \in (0,\ 1)$
Since $\{e_i\}$ is linearly independent, therefore
$\langle x,\ e_i \rangle_\alpha - \langle x,\ e_i \rangle_\beta = 0 \quad \forall\ i = 1, 2, ..$ and $\forall \alpha,\ \beta \in (0,\ 1)$
$\Rightarrow \langle x,\ e_i \rangle_\alpha = \langle x,\ e_i \rangle_\beta \quad \forall\ i = 1, 2, ..$ and $\forall \alpha,\ \beta \in (0,\ 1)$
Thus (ii) $\Rightarrow$ (iii).

(c) Suppose (iii) holds
Let $x = \sum_{i=1}^{\infty} \langle x,\ e_i \rangle_\alpha e_i \quad \forall \alpha \in (0,\ 1)$
Now $||x||_\alpha^2 = \langle x,\ x \rangle_\alpha$
$= \langle \sum_{i=1}^{\infty} \langle x,\ e_i \rangle_\alpha e_i,\ \sum_{i=1}^{\infty} \langle x,\ e_i \rangle_\alpha e_i \rangle_\alpha$

$= \langle \lim_{n\to\infty} \sum_{i=1}^{n} \langle x,\ e_i \rangle_\alpha e_i,\ \ \lim_{n\to\infty} \sum_{i=1}^{n} \langle x,\ e_i \rangle_\alpha e_i \rangle_\alpha$

$= \lim_{n\to\infty} \langle \sum_{i=1}^{n} \langle x,\ e_i \rangle_\alpha e_i,\ \ \sum_{j=1}^{n} \langle x,\ e_j \rangle_\alpha e_j \rangle_\alpha$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \langle x,\ e_i \rangle_\alpha\ \overline{\langle x,\ e_i \rangle_\alpha}$

$= \sum_{i=1}^{\infty} |\langle x,\ e_i \rangle_\alpha|^2 \quad \forall \alpha \in (0,\ 1).$

Now from (iii) we have $\langle x,\ e_i \rangle_\alpha = \langle x,\ e_i \rangle_\beta \ \ \forall\ i = 1, 2, ..$ and $\forall \alpha,\ \beta \in (0,\ 1)$

Using this we get

$||x||_\alpha^2 = \sum_{i=1}^{\infty} |\langle x,\ e_i \rangle_\alpha|^2 = \sum_{i=1}^{\infty} |\langle x,\ e_i \rangle_\beta|^2 = ||x||_\beta^2 \ \ \forall \alpha,\ \beta \in (0,\ 1)$

So (iii) $\Rightarrow$ (iv).

(d) Suppose (iv) holds and $\{e_i\}$ is not complete. Then we get for an $\alpha \in (0,\ 1)$

$\{e^\alpha,\ e_1,\ e_2, ...\}$ of which $\{e_1,\ e_2, ...\}$ is a proper subset and $||e^\alpha||_\alpha = 1$ and $\langle e^\alpha,\ e_i \rangle_\alpha = 0 \ \ \forall\ i = 1, 2, ..$

Now $||e^\alpha||_\alpha^2 = \sum_{i=1}^{\infty} |\langle e^\alpha,\ e_i \rangle_\alpha|^2 = 0$

$\Rightarrow e^\alpha = \bar{0}$.

Thus (iv) $\Rightarrow$ (i)

Theorem 4.4.(Riesz) *Let $(H,\ \mu)$ be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and $f \in H^*$. Then for each $\alpha \in (0,\ 1)$, $\exists\ y_\alpha \in H$ such that $f(x) = \langle x,\ y_\alpha \rangle_\alpha$ where y_α depends on f and*

$||f||_\alpha^* \geq ||y_\alpha||_\alpha$ when $\alpha \geq \frac{1}{2}$ and $||f||_{1-\alpha}^* \leq ||y_\alpha||_{1-\alpha}$ when $\alpha < \frac{1}{2}$.

Where $(H^*,\ N^*)$ is the strong fuzzy dual space of H, $\{||\ ||_\alpha^* : \alpha \in (0,\ 1)\}$ is an ascending family of norm in H^*, $||\ ||_\alpha$ are the α-norms of N which is induced by the fuzzy inner product μ and $\langle\ ,\ \rangle_\alpha$ are α-innerproduct on H induced by $||\ ||_\alpha$.

To prove this theorem we have to prove the following lemma:

Lemma 4.1. *Let $(H,\ \mu)$ be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and $f \in H^*$. Then $N(f) = \{x \in H : f(x) = 0\}$ is l-fuzzy closed subspace of H.*

Proof. Since $f \in H^*$ so f is bounded w.r.t. $||\ ||^*$ of N^* i.e. f is continous w.r.t. $||\ ||^*$ of N^*.

Choose $x_1,\ x_2 \in N(f)$ and $k_1,\ k_2$ be any two scalars

Then $f(x_1) = f(x_2) = 0$ and

$f(k_1x_1 + k_2x_2) = k_1 f(x_1) + k_2 f(x_2) = 0.$

Therfore $k_1x_1 + k_2x_2 \in N(f)$ and hence $N(f)$ is a subspace of H.

Now let $\{x_n\}$ be a sequence in $N(f)$ and $1 - \alpha \in (0,\ 1)$ be arbitrary such that $||x_n - x||_{1-\alpha} \to 0$ as $n \to \infty$.

$||f(x_n) - f(x)||_\alpha^* = |f(x_n) - f(x)| \leq ||f||_\alpha^* ||x_n - x||_{1-\alpha}$

$\Rightarrow ||f(x)||_\alpha^* = 0$ (as $n \to \infty$, $||x_n - x||_{1-\alpha} \to 0$ and $f(x_n) = 0 \ \ \forall n$)

$\Rightarrow f(x) = 0.$

$\Rightarrow x \in N(f)$

Thus $N(f)$ is closed w.r.t. $||\ ||_\alpha$

Since $1 - \alpha \in (0,\ 1)$ is arbitrary it follws that $N(f)$ is a l-fuzzy closed in H [by Proposition 2.3].

Now we prove the main theorem.

Proof. Recall that

$$N(x,\ t) = \begin{cases} \mu(x,\ x,\ t^2) & \forall\ t \in R,\ t > 0 \\ 0 & \forall\ t \in R,\ t \leq 0 \end{cases}$$

and $||x||_\alpha = \wedge\{t > 0 : \mu(x,\ x,\ t^2) \geq \alpha\}, \quad \alpha \in (0,\ 1).$

Since $f \in H^*$, so

$$||f||_\alpha^* = \bigwedge_{x \in H_{x\neq 0}} \frac{|f(x)|}{||x||_{1-\alpha}} \quad \forall \alpha \in (0,\ 1)$$

and

$$\{||\ ||_\alpha^* : \alpha \in (0,\ 1)\}$$

is an ascending family of norms on H^* [by Definition 2.3].

Thus $|f(x)| \leq ||f||_\alpha^* ||x||_{1-\alpha} \ \ \forall\ \ x \in H$ and $\forall\ \alpha \in (0,\ 1)$.

Case-I. If f is the zero functional then we take $y_\alpha = \underline{0} \ \ \forall\ \alpha \in (0,\ 1)$. Then the theorem is proved in this case.

Case-II. If $f \neq 0$, then $y_\alpha \neq \underline{0} \ \ \forall\ \alpha \in (0,\ 1)$.

and $N(f) = \{x \in H : f(x) = 0\} \neq H$. So $N(f)$ is a proper and l-fuzzy closed subspace of H.

Therefore for each $\alpha \in (0,\ 1), N(f)^{\perp_\alpha} \neq \{0\}$ by projection theorem.

Hence for each $\alpha \in (0,\ 1) \exists z_\alpha \in N(f)^{\perp_\alpha}$ with $z_\alpha \neq \underline{0}$.

Put $v_\alpha = f(x)z_\alpha - f(z_\alpha)x$, where $x \in H$ is arbitrary.

Now $f(v_\alpha) = f(x)f(z_\alpha) - f(z_\alpha)f(x) = 0.$

So $v_\alpha \in N(f) \ \ \forall\ \alpha \in (0,\ 1)$.

We have $0 = \langle v_\alpha,\ z_\alpha \rangle_\alpha \quad \forall\ \alpha \in (0,\ 1)$

$\Rightarrow \langle f(x)z_\alpha - f(z_\alpha)x,\ z_\alpha \rangle_\alpha = 0 \quad \forall\ \alpha \in (0,\ 1)$

$\Rightarrow f(x)\langle z_\alpha,\ z_\alpha \rangle_\alpha - f(z_\alpha)\langle x,\ z_\alpha \rangle_\alpha = 0 \quad \forall\ \alpha \in (0,\ 1)$

$\Rightarrow f(x) = \frac{f(z_\alpha)}{||z_\alpha||_\alpha^2} \langle x,\ z_\alpha \rangle_\alpha \quad \forall\ \alpha \in (0,\ 1)$

$\Rightarrow f(x) = \langle x,\ \overline{\frac{f(z_\alpha)}{||z_\alpha||_\alpha^2}} z_\alpha \rangle_\alpha \quad \forall\ \alpha \in (0,\ 1)$

$\Rightarrow f(x) = \langle x,\ y_\alpha \rangle_\alpha \quad \forall\ \alpha \in (0,\ 1)$

where $y_\alpha = \overline{\frac{f(z_\alpha)}{||z_\alpha||_\alpha^2}} z_\alpha$.

Clearly y_α depends on f.

For each $\alpha \in (0,\ 1)$, z_α is unique.

For, if possible suppose that, for some $\alpha \in (0,\ 1)$, $\exists y'_\alpha$ such that

$f(x) = \langle x,\ y_\alpha \rangle_\alpha = \langle x,\ y'_\alpha \rangle_\alpha$

$\Rightarrow \langle x,\ y_\alpha - y'_\alpha \rangle_\alpha = 0 \ \ \forall\ x \in H$

$\Rightarrow y_\alpha - y'_\alpha = 0$

$\Rightarrow y_\alpha = y'_\alpha$.

Again, $f(x) = \langle x,\ y_\alpha \rangle_\alpha \ \ \forall\ x \in H$

i.e. $f(y_\alpha) = \langle y_\alpha,\ y_\alpha \rangle_\alpha = ||y_\alpha||_\alpha^2$

$\Rightarrow ||y_\alpha||_\alpha^2 = f(y_\alpha) \leq ||f||_\alpha^* ||y_\alpha||_{1-\alpha}$——(i)

Case-I. When $\alpha \geq \frac{1}{2}$ then $1 - \alpha \leq \alpha$.

From (i) we get $||y_\alpha||^2 \leq ||f||_\alpha^* ||y_\alpha||_{1-\alpha} \leq ||f||_\alpha^* ||y_\alpha||_\alpha$

i.e. $||y_\alpha||_\alpha \leq ||f||_\alpha^*$——(ii).

Case-II. When $\alpha < \frac{1}{2}$ then $1 - \alpha > \alpha$.

$\Rightarrow ||y_\alpha||_{1-\alpha} \geq ||y_\alpha||_\alpha$

Thus we have

$$|f(x)| = |\langle x,\ y_\alpha \rangle_\alpha| \leq ||x||_\alpha ||y_\alpha||_\alpha \leq ||x||_\alpha ||y_\alpha||_{1-\alpha}$$

$\Rightarrow \bigvee_{x \in X_{x\neq 0}} \frac{|f(x)|}{||x||_{1-(1-\alpha)}} \leq ||y_\alpha||_{1-\alpha}$

$\Rightarrow ||f||_{1-\alpha}^* \leq ||y_\alpha||_{1-\alpha}$——(iii).

From (ii) and (iii) we have the required result.

V. Conclusion

In this paper, we consider fuzzy inner product space introduced by Pinaki mazumder & S. K. Samanta. Some important concepts viz. α- fuzzy orthonormal set, complete fuzzy orthonormal set etc. have been introduced. We establish Bessel's inequality and Riesz representation theorem in fuzzy setting. We think that these results will be helpful for the researchers to develop fuzzy functional analysis specially for operator theory and spectral theory.

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