Some Properties of Fuzzy Hilbert Spaces
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Abstract—Some results on fuzzy inner product spaces are proved. We establish some fundamental results viz. Bessel's inequality, Riesz representation theorem etc. in fuzzy setting.

Index Terms—Fuzzy Hilbert space, orthonormal set, Bessel's inequality, Riesz representation theorem.

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I. INTRODUCTION


Section 2, provides some preliminary results which are used in this paper.

In section 3, we study some results on fuzzy inner product spaces.

Section 4, is devoted to establish Bessel’s inequality and Riesz representation theorem in fuzzy setting.

II. SOME PRELIMINARY RESULTS

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 2.1 [4]. Let $U$ be a linear space over a field $F$ (field of real / complex numbers).

A fuzzy subset $N$ of $U \times R$ ($R$ is the set of real numbers) is called a fuzzy norm on $U$ if $\forall x, u \in U$ and $c \in F$, the following conditions are satisfied:

(N1) $\forall t \in R \text{ with } t \leq 0, N(x, t) = 0$;

(N2) $\forall t \in R, t > 0, N(x, t) = 1$ if $x = 0$;

(N3) $\forall t \in R, t > 0, N(cx, t) = N(x, t)$ if $c \neq 0$;

Let $V$ be a linear space over the field $C$ of complex numbers. Let $\mu : V \times V \times C \rightarrow I = [0, 1]$ be a mapping such that the following holds

$N(x + u, s + t) \geq \min\{N(x, s), N(u, t)\}$

(N5) $N(x, \cdot)$ is a non-decreasing function of $R$ and $\lim_{t \to \infty} N(x, t) = 1$.

The pair $(U, N)$ will be referred to as a fuzzy normed linear space.

Theorem 2.1 [4]. Let $(U, N)$ be a fuzzy normed linear space.

Assume further that,

(N6) $\forall t > 0, N(x, t) > 0 \text{ implies } x = 0$.

Define $\|x\|_\alpha = \land\{t > 0 : N(x, t) \geq \alpha\}, \alpha \in (0, 1)$. Then $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on $U$ and they are called $\alpha$-norms on $U$ corresponding to the fuzzy norm $N$ on $U$.

Definition 2.2 [4]. Let $(U, N)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in $U$. Then $\{x_n\}$ is said to be convergent if $\exists x \in U$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1 \forall t > 0$. In that case $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n \to \infty} x_n$.

Definition 2.3 [9]. Let $(U, N_1)$ be a fuzzy normed linear space satisfying (N6). Let $T \in U^*$ and $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ be the family of $\alpha$-norms of $N_1$.

We define $\|T\|_\alpha^* = \bigvee_{x \in U, x \neq 0} \frac{|T(x)|}{\|x\|_\alpha}$ for $\alpha \in (0, 1)$.

Then $\{\|\cdot\|_\alpha^* : \alpha \in (0, 1)\}$ is an ascending family of norms on $U^*$.

Definition 2.4 [9]. Let $(U, N)$ be a fuzzy normed linear space and $\alpha \in (0, 1)$. A sequence $\{x_n\}$ in $U$ is said to be $\alpha$-convergent in $U$ if $\exists x \in U$ such that $\lim_{n \to \infty} N(x_n - x, t) > \alpha \forall t > 0$ and $x$ is called the limit of the sequence $\{x_n\}$.

Definition 2.5 [10]. Let $(U, N)$ be a fuzzy normed linear space satisfying (N6) and $F \subset U$. Then $F$ is $l$-fuzzy closed if for each $\alpha \in (0, 1)$ and for any sequence $\{x_n\}$ in $F$ and $x \in U$, $\lim_{n \to \infty} N(x_n - x, t) \geq \alpha \forall t > 0 \Rightarrow x \in F$.

Proposition 2.3 [10]. Let $(U, N)$ be a fuzzy normed linear space satisfying (N6) and $F \subset U$. Then $F$ is $l$-fuzzy closed if $F$ is closed w.r.t. $\|\cdot\|_\alpha$ ($\alpha$-norm of $N$) for each $\alpha \in (0, 1)$.

Definition 2.6 [7]. Let $V$ be a linear space over the field $C$ of complex numbers. Let $\mu : V \times V \times C \rightarrow I = [0, 1]$ be a mapping such that the following holds
(FIP1) For \( s, t \in C \), \( \mu(x + y, z, |t| + |s|) \geq \min\{\mu(x, z, |t|), \mu(y, z, |s|)\} \);
(FIP2) For \( s, t \in C \), \( \mu(x, y, |st|) \leq \min\{\mu(x, y, |s|^2), \mu(y, y, |t|^2)\} \);
(FIP3) For \( t \in C \), \( \mu(x, y, t) = \mu(y, x, t) \);
(FIP4) \( \mu(\alpha x, y, t) = \mu(x, y, t) \alpha, \alpha(\neq 0) \in C, t \in C \);
(FIP5) \( \mu(x, x, t) = 0 \) \( \forall t \in C \setminus R^+ \);
(FIP6) \( \mu(x, x, t) = 1 \) \( \forall t > 0 \) \iff \( x = 0 \);
(FIP7) \( \mu(x, x, \cdot) : R \rightarrow I[= [0, 1]] \) is a monotonic non-decreasing function of \( R \) and \( \lim_{t \rightarrow \infty} \mu(x, x, t) = 1 \).

We shall call \( \mu \) to be the fuzzy inner product (FIP in short) function on \( V \) and \( (V, \mu) \) is called a fuzzy inner product space (FIPS).

**Theorem 2.2 [7].** Let \( V \) be a linear space over \( C \). Let \( \mu \) be a FIP on \( V \). Then

\[
N(x, t) = \begin{cases} \mu(x, x, t^2), & t > 0 \\ 0, & t \leq 0 \end{cases}
\]

is a fuzzy norm on \( V \). Now if \( \mu \) satisfies the following conditions:

(FIP8) \( \mu(x, x, t^2) > 0 \), \( \forall t > 0 \) \implies \( x = 0 \) and

(FIP9) For all \( x, y \in V \) and \( p, q \in R \),

\[
\mu(x + y, x + y, 2t^2) \wedge \mu(x - y, x - y, 2t^2) \geq \mu(x, x, p^2) \wedge \mu(y, y, q^2)
\]

then \( ||x||_\alpha = \wedge \{t > 0 : N(x, t) \geq \alpha\} \alpha(\in (0, 1)) \) is an ordinary norm (\( \alpha \)-norm) on \( V \) satisfying parallelogram law.

Then using Polarization identity we can get ordinary inner product, called the \( \alpha \)-inner product, as follows:

\[
\langle x, y \rangle_\alpha = \frac{1}{2}(|x + y||x|^2 - |x - y||x|^2) + \frac{1}{2}(|x + iy||y|^2 - |x - iy||y|^2), \forall \alpha(\in (0, 1))
\]

**Definition 2.7 [7].** Let \( (V, \mu) \) be a FIP space satisfying (FIP8) and (FIP9). \( V \) is said to be level complete if for any \( \alpha(\in (0, 1)) \), every cauchy sequence converges in \( V \) w.r.t. \( ||\cdot||_\alpha \) (the \( \alpha \)-norm generated by the fuzzy norm \( N \) which is induced by fuzzy inner product \( \mu \)).

**Definition 2.8 [7].** Let \( (V, \mu) \) be a FIP space. \( V \) is said to be a fuzzy Hilbert space, if it is level complete.

**Definition 2.9 [7].** Let \( \alpha(\in (0, 1)) \) and \( (V, \mu) \) be a FIP space satisfying (FIP8) and (FIP9). Now if \( x, y \in V \) be such that \( \langle x, y \rangle_\alpha = 0 \) \( \forall \alpha(\in (0, 1)) \), then we say that \( x, y \) are \( \alpha \)-fuzzy orthogonal to each other and is denoted by \( x \perp_\alpha y \). Let \( M \) be a subset of \( V \) and \( x \in V \). Now if \( \langle x, y \rangle_\alpha = 0 \) \( \forall \alpha(\in M) \), then we say that \( x \) is \( \alpha \)-fuzzy orthogonal to \( M \) and is denoted by \( x \perp_\alpha M \).

**Definition 2.10 [7].** Let \( (V, \mu) \) be a FIP space satisfying (FIP8) and (FIP9). Now if \( x, y \in V \) be such that \( \langle x, y \rangle_\alpha = 0 \) \( \forall \alpha(\in (0, 1)) \), then we say that \( x, y \) are fuzzy orthogonal to each other and is denoted by \( x \perp y \).

Thus \( x \perp y \) iff \( x \perp_\alpha y \) \( \forall \alpha(\in (0, 1)) \).

III. SOME RESULTS ON FUZZY INNER PRODUCT SPACES

Here we study some results on fuzzy inner product spaces.

**Theorem 3.1.** Let \( (V, N) \) be a fuzzy normed linear space.

Assume that for \( x, y \in V \) and \( s, t \in C \),

\[
\min\{N(x, |st|), N(y, |st|)\} \geq \min\{N(x, |s|^2), N(y, |t|^2)\}
\]

Define \( \mu'(V \times V \times C \rightarrow [0, 1]) as \)

\[
\mu'(x, y, s + t) = 0
\]

if \( x = y \) and \( s + t \in C - R^+ \) and elsewhere as

\[
\mu'(x, y, s + t) = N(x, |s|) \vee N(y, |t|)
\]

Then \( \mu' \) is a fuzzy inner product on \( V \).

**Proof.** (FIP1) For \( s, t \in C \) and \( x, y, z \in V \) we have

\[
\mu'(x + y, z, |s| + |t|) = \mu'(x + y, z, |s| + |t| + 0) = N(x + y, |s| + |t|) \vee N(z, 0) = N(x + y, |s| + |t|) \geq \min\{N(x, |s|), N(y, |t|)\} = \min\{\mu'(x, z, s), \mu'(y, z, t)\}.
\]

(FIP2) \( \mu'(x, y, |st|) = N(x, |st|) = N(y, |st|) \leq \min\{\mu'(x, z, s), \mu'(y, z, t)\} \).

(FIP3) \( \mu'(x, y, t) = N(x, |t|) = N(y, |t|) = \mu'(y, x, t) \).

(FIP4) \( \mu'(\alpha x, y, t) = N(\alpha x, |t|) = N(x, |t|) = \mu'(x, y, t) \).

(FIP5) \( \mu'(x, x, t) = 0 \) \( \forall t \in C - R^+ \) [By definition].

(FIP6) \( \mu'(x, x, t) = 1 \) \( \forall t > 0 \) \iff \( x = 0 \).

(FIP7) Since \( \mu'(x, x, \cdot) = N(x, \cdot) \) and \( N(x, \cdot) \) is a monotonic nondecreasing function of \( R \) and \( \lim_{t \rightarrow \infty} N(x, t) = 1 \)

Thus \( \mu' \) has also the property. Thus \( \mu' \) is a fuzzy inner product on \( V \).

**Theorem 3.2.** Let \( (V, \mu) \) be a fuzzy inner product space satisfying (FIP8) and (FIP9) and \( \langle \cdot, \cdot \rangle_\alpha \) be its \( \alpha \)-inner product \( \forall \alpha(\in (0, 1)) \).

Define a function \( \mu'(V \times V \times C \rightarrow [0, 1]) as \)

\[
\mu'(x, y, t) = 0
\]

if \( x = y \) and \( \forall t \in C - R^+ \) and elsewhere as

\[
\mu'(x, y, t) = \sqrt{\{\alpha(\in (0, 1)) : \langle x, y \rangle_\alpha \leq |t|\}}
\]

Then \( \mu' \) is a fuzzy inner product on \( V \) if \( \langle \cdot, \cdot \rangle_\alpha \) is increasing function of \( R \).

**Proof.** (FIP1) For \( s, t \in C \) and \( x, y, z \in V \) we have to show that

\[
\mu'(x + y, z, |s| + |t|) \geq \min\{\mu'(x, z, |s|), \mu'(y, z, |t|)\}
\]

Let \( p = \mu'(x, z, |s|) \) and \( q = \mu'(y, z, |t|) \)

Without loss of generality assume that \( p \leq q \)

Let \( 0 < r < p \leq q \)

Then \( \exists \alpha > r \) such that \( \langle x, z \rangle_\alpha < |s| \) and

\[
\exists \beta > r \) such that \( \langle y, z \rangle_\beta < |t| \)

Let \( \gamma = \alpha \wedge \beta > r \)

Thus \( ||x, z||_\gamma < ||x, z||_\alpha < |s| \) and

\[
||y, z||_\gamma < ||y, z||_\beta < |t| \ [\text{Since } \langle \cdot, \cdot \rangle_\alpha \text{ is increasing}]
\]

Now \( ||x + y, z||_\gamma = ||x, z||_\gamma + ||y, z||_\gamma \)

\[
\leq ||x, z||_\gamma + ||y, z||_\gamma
\]

\[
||x, z||_\gamma + ||y, z||_\gamma
\]
Therefore \( \mu'(x + y, z, |s| + |t|) \geq \gamma > r \)
Since \( 0 < r < \gamma \) is arbitrary, thus \( \mu'(x + y, z, |s| + |t|) \geq \min(\mu'(x, z, |s|), \mu'(y, z, |t|)) \).

**(FIP2)** For \( s, t \in C \) and \( x, y, \in V \) we have to show that \( \mu'(x, y, \; |s|t|) \geq \min\{\mu'(x, y, |s|^2), \mu'(x, y, |t|^2)\} \)
Let \( p = \mu'(x, y, |s|^2) \) and \( q = \mu'(x, y, |t|^2) \)
Without loss of generality assume that \( p \leq q \)
Let \( 0 < r < p \leq q \)
Then \( \exists \alpha > r \) such that \( |\langle x, y \rangle \alpha| < |\langle x, y \rangle \beta| \leq |\langle x, y \rangle| \)
Let \( \gamma = \alpha \land \beta > r \)
Thus \( |\langle x, y \rangle \gamma| < |\langle x, y \rangle \alpha| \leq |\langle x, y \rangle| \)
Finally, \( |\langle x, y \rangle \beta| < |\langle x, y \rangle| \)

**(FIP3)** For \( t \in C \),
\( \mu'(x, y, t) = \mu'(x, y, t) = 0 \) if \( x = y \)
Now let \( t \in C \) and \( x \neq y \),
\( \mu'(x, y, t) = \mu'(x, y, t) = \mu'(x, y, t) = 1 \) for all \( t \in C \).

**(FIP4)** Fix \( c \in C \),
\( \mu'(x, y, t) = \mu'(x, y, t) = \mu'(x, y, t) = 1 \) for all \( t \).

**(FIP5)** \( \mu'(x, y, t) = \mu'(x, y, t) = \mu'(x, y, t) = 1 \) for all \( t \).

Now \( t_1 > t_2 \)
\( \Rightarrow t_1 > t_2 \)
\( \Rightarrow \{ \alpha \in (0, 1) : |\langle x, y \rangle_\alpha| \leq |\langle x, y \rangle| \} \)
\( \Rightarrow \{ \alpha \in (0, 1) : |\langle x, y \rangle_\alpha| \leq |\langle x, y \rangle| \} \)
\( \Rightarrow \{ \alpha \in (0, 1) : |\langle x, y \rangle_\alpha| \leq |\langle x, y \rangle| \} \)
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\( \Rightarrow \{ \alpha \in (0, 1) : |\langle x, y \rangle_\alpha| \leq |\langle x, y \rangle| \} \)

\( \mu'(x, y, t) = \mu'(x, y, t) = 1 \) for all \( t \).

Therefore \( \mu'(x, y, t) : R^+ \to [0, 1] \) is increasing and
\( \lim_{t \to \infty} \mu'(x, y, t) = 1 \).

Thus \( \mu' \) is a fuzzy inner product on \( V \).

**IV. Orthogonal set, Bessel’s inequality & Riesz representation theorem.**

In this section orthornormal set, sequence are defined and Bessel’s inequality, Riesz representation theorem are established in fuzzy Hilbert spaces.

**Definition 4.1.** Let \((V, \mu)\) be a fuzzy inner product space satisfying (FIP8) and (FIP9) and \( \alpha \in (0, 1) \). An \( \alpha \)-fuzzy orthogonal set \( M \) in \( V \) is said to be \( \alpha \)-fuzzy orthonormal if

\[ \langle x, y \rangle_\alpha = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \forall \alpha \in (0, 1) \]

where \( \langle \cdot, \cdot \rangle_\alpha \) is the induced inner product by \( \mu \).

**Definition 4.2.** Let \((V, \mu)\) be a fuzzy inner product space satisfying (FIP8) and (FIP9). A fuzzy orthogonal set \( M \) in \( V \) is said to be fuzzy orthonormal if the elements have \( \alpha \)-norm \( 1 \) for all \( \alpha \in (0, 1) \) that is \( \forall \alpha, \; x, y \in M \)

\[ \langle x, y \rangle_\alpha = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \forall \alpha \in (0, 1) \]

where \( \langle \cdot, \cdot \rangle_\alpha \) is induced inner product by \( \mu \).

**Proposition 4.1.** An \( \alpha \)-fuzzy orthonormal set and a fuzzy orthonormal set in a FIP space are linearly independent.

**Proof.** Proof is straightforward.

**Proposition 4.2.** Let \( \{e_k\} \) be a fuzzy orthonormal sequence in a fuzzy Hilbert space \((V, \mu)\) satisfying (FIP8) and (FIP9).
Then the series \( \sum_{k=1}^{\infty} \alpha_k e_k \) converges (w.r.t. \( \| \cdot \|_\alpha \) ; \( \alpha \in (0, 1) \))
where \( \| \cdot \|_\alpha \) are the \( \alpha \)-norms of \( N \) which is induced by \( \mu \) iff \( \sum_{k=1}^{\infty} |\alpha_k|^2 \) converges.

**Proof.** Let \( S_n = \sum_{k=1}^{n} \alpha_k e_k \) and \( \sigma_n = \sum_{k=1}^{n} |\alpha_k|^2 \).

Then

\[ \|S_n - S_m\|^2 = \sum_{k=1}^{\infty} |\alpha_k e_k|^2 = |\alpha_{m+1}|^2 + |\alpha_{m+2}|^2 + \ldots + |\alpha_n|^2 \]
i.e.

\[ \|S_n - S_m\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 \]
i.e.

\[ \|S_n - S_m\|^2 = \sigma_n - \sigma_m \quad \forall \alpha \in (0, 1) \]

Hence \( S_n \) is Cauchy w.r.t. \( \| \cdot \|_\alpha \) , \( \forall \alpha \in (0, 1) \) iff \( S_n \) is Cauchy in \( R \)(the set of real numbers).

Hence \( S_n \) is Cauchy iff \( S_n \) is Cauchy in \( R \).

Hence the proof.

**Note 4.1.** In case of \( \alpha \)-fuzzy orthonormal sequence the proposition holds.

**Proposition 4.3.** Let \((V, \mu)\) be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and \( \alpha \in (0, 1) \) and let \( \{e_k\} \) be an \( \alpha \)-fuzzy orthonormal sequence in \( V \). If the series \( \sum_{k=1}^{\infty} \beta_k e_k \) is \( \alpha \)-convergent w.r.t. \( \| \cdot \|_\alpha \) \( N \) induced by \( \mu \), then the coefficients \( \beta_k = \langle x, e_k \rangle_\alpha \) where \( x \) denotes the sum \( \sum_{k=1}^{\infty} \beta_k e_k \) and hence

\[ x = \sum_{k=1}^{\infty} \langle x, e_k \rangle_\alpha e_k \]

**Proof.** Since \( \sum_{k=1}^{\infty} \beta_k e_k \) is \( \alpha \)-convergent so it is convergent w.r.t. \( \| \cdot \|_\alpha \) (By Definition 2.4 & Proposition 2.2).

i.e. \( \| \sum_{k=1}^{\infty} \beta_k e_k - x \|_\alpha \to 0 \) as \( n \to \infty \).
Let $S_n = \sum_{k=1}^{n} \beta_k e_k$.

Taking innerproduct with $S_n$ and $e_j$ and using the definition of $\alpha$-fuzzy orthogonality we have

$\langle S_n, e_j \rangle_{\alpha} = \beta_j$ for $j = 1, 2, \ldots, k$.

Now $S_n \to x$ w.r.t. $\| \cdot \|_{\alpha}$

$\Rightarrow \langle S_n, e_j \rangle_{\alpha} \to \langle x, e_j \rangle_{\alpha} = \beta_j$

Therefore $x = \sum_{k=1}^{\infty} \beta_k e_k = \sum_{k=1}^{\infty} \langle x, e_k \rangle_{\alpha} e_k$.

**Theorem 4.1.** Let $(V, \mu)$ be a fuzzy Hilbert space satisfying (FIP9) and $(\{ e_k \})$ be a fuzzy orthonormal sequence in $V$. If the series $\sum_{k=1}^{\infty} \gamma_k e_k$ converges w.r.t. $N$ induced by $\mu$, then

$\gamma_k = \langle x, e_k \rangle_{\alpha} = \langle x, e_k \rangle_{\beta}$ $\forall \alpha, \beta \in (0, 1)$,

where $\langle \cdot, \cdot \rangle_{\alpha}$ denotes the $\alpha$-innerproduct induced by $\mu, x$ denotes the sum of $\sum_{k=1}^{\infty} \gamma_k e_k$.

Hence $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle_{\alpha} e_k = \sum_{k=1}^{\infty} \langle x, e_k \rangle_{\beta} e_k$ $\forall \alpha, \beta \in (0, 1)$.

**Proof.** Since $(e_k)$ is fuzzy orthonormal, it is orthonormal w.r.t. each $\langle \cdot, \cdot \rangle_{\alpha}$, $\alpha \in (0, 1)$.

Now $\sum_{k=1}^{\infty} \gamma_k e_k$ is convergent w.r.t. $N$ implies it is convergent w.r.t. $\| \cdot \|_{\alpha}$ (induced $\alpha$-norm of $N$) $\forall \alpha \in (0, 1)$ [by Proposition 2.1].

We have $\langle \sum_{k=1}^{n} \gamma_k e_k, e_j \rangle_{\alpha} = \gamma_j = \langle \sum_{k=1}^{n} \gamma_k e_k, e_j \rangle_{\beta}$ $j = 1, 2, \ldots, k$.

Taking limit as $n \to \infty$ we have

$\lim_{n \to \infty} \langle \sum_{k=1}^{n} \gamma_k e_k, e_j \rangle_{\alpha} = \gamma_j = \lim_{n \to \infty} \langle \sum_{k=1}^{n} \gamma_k e_k, e_j \rangle_{\beta}$

$\Rightarrow \langle \sum_{k=1}^{\infty} \gamma_k e_k, e_j \rangle_{\alpha} = \gamma_j = \langle \sum_{k=1}^{\infty} \gamma_k e_k, e_j \rangle_{\beta} \quad \text{[since $\langle \cdot, \cdot \rangle_{\alpha}$ is continuous $\forall \alpha \in (0, 1)$]}

$\Rightarrow \langle x, e_j \rangle_{\alpha} = \gamma_j = \langle x, e_j \rangle_{\beta} \quad \forall \alpha, \beta \in (0, 1), \quad j = 1, 2, 3, \ldots$.

**Definition 4.3.** Let $(V, \mu)$ be a fuzzy innerproduct space satisfying (FIP8) and (FIP9). A fuzzy orthonormal set $M \subset V$ is called fuzzy orthonormal set if there is no $\alpha$-fuzzy orthonormal set ($\alpha \in (0, 1)$) of which $M$ is a proper subset. If $M$ is countable then we call $M$ is a complete fuzzy orthonormal sequence.

**Theorem 4.2** (Bessel’s inequality) Let $(V, \mu)$ be a fuzzy Hilbert space satisfying (FIP9), (FIP9) and $(\{ e_k \})$ be an $\alpha$-fuzzy orthonormal sequence in $V$. Then for every $x \in V$, $\forall \alpha \in (0, 1)$

$\sum_{k=1}^{\infty} |\langle x, e_k \rangle_{\alpha}|^2 \leq \| x \|_{\alpha}^2$.

**Proof.** Since $\alpha$-fuzzy orthonormal sequence is orthonormal sequence in $(V, \langle \cdot, \cdot \rangle_{\alpha})$, so by Bessel’s inequality in crisp innerproduct we have

$\sum_{k=1}^{\infty} |\langle x, e_k \rangle_{\alpha}|^2 \leq \| x \|_{\alpha}^2$.

**Note 4.2** In case of fuzzy orthonormal sequence it will be

$\sum_{k=1}^{\infty} |\langle x, e_k \rangle_{\alpha}|^2 \leq \| x \|_{\alpha}^2$ $\forall \alpha \in (0, 1)$.

**Theorem 4.3.** Let $(V, \mu)$ be a Hilbert space satisfying (FIP9) and $(\{ e_i \})$ is a fuzzy orthonormal sequence in $V$. Then the following statements are equivalent.

(i) $(\{ e_i \})$ is complete fuzzy orthonormal.

(ii) $x \perp e_i$ for $i = 1, 2, \ldots$ $\Rightarrow x = 0$.

(iii) For every $x \in V$, $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle_{\alpha} e_i$ $\forall \alpha \in (0, 1)$ and hence $\langle x, e_k \rangle_{\alpha} = \langle x, e_k \rangle_{\beta} \quad \forall \alpha, \beta \in (0, 1)$ i.e. $x$ is independent on $\alpha$.

(iv) For every $x \in V$, $\| x \|_{\alpha}^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle_{\alpha}|^2$ $\forall \alpha \in (0, 1)$ and hence $\| x \|_{\alpha}^2 = \| x \|_{\beta}^2$ $\forall \alpha, \beta \in (0, 1)$.

**Proof.** (a) Suppose (i) holds.

Let $(\{ e_i \})$ be a complete fuzzy orthonormal sequence and $x \perp e_i$ for $i = 1, 2, \ldots$ $\Rightarrow x = 0$.

(b) Suppose (ii) holds.

Let $x \perp e_i$ for $i = 1, 2, \ldots$ implies $x = 0$.

(c) Suppose (iii) holds.

Let $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle_{\alpha} e_i$ $\forall \alpha \in (0, 1)$.

Now $\| x \|_{\alpha}^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle_{\alpha}|^2$.
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \langle x, e_i \rangle_{\alpha} = \lim_{n \to \infty} \sum_{i=1}^{n} \langle x, e_i \rangle_{\alpha} e_i
\]

So (iii) we have \( \langle x, e_i \rangle_{\alpha} = \langle x, e_i \rangle_{\beta} \quad \forall i = 1, 2, \ldots \) and \( \forall \alpha, \beta \in (0, 1) \).

Using this we get
\[
||x||_{\alpha}^2 = \sum_{i=1}^{\infty} ||\langle x, e_i \rangle_{\alpha}||^2 \quad \forall \alpha \in (0, 1)
\]

Now \( \alpha < \beta \) and \( \forall i = 1, 2, \ldots \)

\[
\Rightarrow \alpha = 0.
\]

Thus (iv) \( \Rightarrow \) (i).

**Theorem 4.4.(Riesz) Let (H, \mu) be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and \( f \in H^* \). Then for each \( \alpha \in (0, 1) \), \( \exists y_\alpha \in H \) such that \( f(x) = \langle x, y_\alpha \rangle_{\alpha} \) where \( y_\alpha \) depends on \( f \) and

\[
||f||_{\alpha}^* \geq ||y_\alpha||_{\alpha} \quad \text{when} \quad \alpha \geq \frac{1}{2} \quad \text{and} \quad ||f||_{1-\alpha} \leq ||y_\alpha||_{1-\alpha} \quad \text{when} \quad \alpha \leq \frac{1}{2}.
\]

Where \( (H^*, N^*) \) is the strong fuzzy dual space of \( H \). \( \{||.||_{\alpha} : \alpha \in (0, 1)\} \) are the \( \alpha \)-norms of \( N \) which is induced by the fuzzy inner product \( \langle.,.\rangle_{\alpha} \) and \( \langle.,.\rangle_{\alpha} \) are \( \alpha \)-inner product on \( H \) induced by ||.||_{\alpha}.

To prove this theorem we have to prove the following lemma:

**Lemma 4.1.** Let \( (H, \mu) \) be a fuzzy Hilbert space satisfying (FIP8) and (FIP9) and \( f \in H^* \). Then \( N(f) = \{x \in H : f(x) = 0\} \) is \( l \)-fuzzy closed subspace of \( H \).

**Proof.** Since \( f \in H^* \) so \( f \) is bounded w.r.t. \( ||.||^* \) of \( N^* \) i.e. \( f \) is continuous w.r.t. \( ||.||^* \) of \( N^* \).

Choose \( x_1, x_2 \in N(f) \) and \( k_1, k_2 \) be any two scalars

Then \( f(k_1x_1 + k_2x_2) = k_1f(x_1) + k_2f(x_2) = 0 \).

Therefore \( k_1x_1 + k_2x_2 \in N(f) \) and hence \( N(f) \) is a subspace of \( H \).

Now let \( \{x_n\} \) be a sequence in \( N(f) \) and \( 1 - \alpha \in (0, 1) \) be arbitrary such that \( ||x_n - x||_{1-\alpha} \to 0 \) as \( n \to \infty \).

\[
\Rightarrow ||f(x_n) - f(x)||_{\alpha}^* = ||f(x_n) - f(x)||_{1-\alpha} \to 0
\]

\[
\Rightarrow ||f||_{\alpha}^* = 0 \quad \text{as} \quad n \to \infty, \quad ||x_n - x||_{1-\alpha} \to 0 \quad \text{and} \quad f(x_n) = 0 \quad \forall n
\]

\[
\Rightarrow \exists x \in N(f)
\]

Thus \( N(f) \) is closed w.r.t. \( ||.||_{\alpha} \).

Now \( 1 - \alpha \in (0, 1) \) is arbitrary it follows that \( N(f) \) is a \( l \)-fuzzy closed in \( H \) [by Proposition 2.3].

Now we prove the main theorem.

**Proof.** Recall that

\[
N(x, t) = \begin{cases} 
\mu(x, x, t^2) & \forall t \in R, t > 0 \\
0 & \forall t \in R, t \leq 0
\end{cases}
\]

and \( ||x||_{\alpha} = \{t > 0 : \mu(x, x, t^2) \geq \alpha\} \), \( \alpha \in (0, 1) \).

Since \( f \in H^* \), so

\[
||f||_{\alpha}^* = \bigwedge_{x \in H \times \alpha} \frac{||f(x)||}{||x||_{1-\alpha}} \quad \forall \alpha \in (0, 1)
\]

and

\[
\{||.||_{\alpha}^* : \alpha \in (0, 1)\}
\]

is an ascending family of norms on \( H^* \) [by Definition 2.3].

Thus \( ||f(x)|| \leq ||f||_{\alpha}^* ||x||_{1-\alpha} \quad \forall x \in H \) and \( \forall \alpha \in (0, 1) \).

**Case-I.** If \( \alpha \) is the zero functional then we take \( y_\alpha = 0 \) \( \forall \alpha \in (0, 1) \). Then the theorem is proved in this case.

**Case-II.** If \( \alpha \neq 0 \), then \( y_\alpha \neq 0 \) \( \forall \alpha \in (0, 1) \), and \( \{|x \in H : f(x) = 0\} \neq H \). So \( N(f) \) is a proper and \( l \)-fuzzy closed subspace of \( H \).

Therefore for each \( \alpha \in (0, 1) \), \( N(f)^{\perp_\alpha} \neq \{0\} \) by projection theorem.

Hence for each \( \alpha \in (0, 1) \exists z_\alpha \in N(f)^{\perp_\alpha} \) with \( z_\alpha \neq 0 \).

Put \( v_\alpha = f(x)z_\alpha - f(z_\alpha)x \), where \( x \in H \) is arbitrary.

\[
\Rightarrow f(x)z_\alpha - f(z_\alpha)x = 0 \quad \forall \alpha \in (0, 1)
\]

\[
\Rightarrow f(x)z_\alpha = f(z_\alpha)x = 0 \quad \forall \alpha \in (0, 1)
\]

\[
\Rightarrow f(x) = f(z_\alpha) = 0 \quad \forall \alpha \in (0, 1)
\]

\[
\Rightarrow f(x) = f(z_\alpha) = 0 \quad \forall \alpha \in (0, 1)
\]

\[
\Rightarrow f(x) = f(z_\alpha) = 0 \quad \forall \alpha \in (0, 1)
\]

\[
\Rightarrow y_\alpha = y_\alpha.
\]

Clearly \( y_\alpha \) depends on \( f \).

For each \( \alpha \in (0, 1) \), \( z_\alpha \) is unique.

For, if possible suppose that, for some \( \alpha \in (0, 1) \), \( \exists y'_\alpha \) such that

\[
f(x) = \langle x, y'_\alpha \rangle_{\alpha} = \langle x, y_\alpha \rangle_{\alpha}
\]

\[
\Rightarrow \langle x, y_\alpha - y'_\alpha \rangle_{\alpha} = 0 \quad \forall x \in H
\]

\[
\Rightarrow y_\alpha - y'_\alpha = 0
\]

\[
\Rightarrow y_\alpha = y'_\alpha.
\]

Again, \( f(x) = \langle x, y'_\alpha \rangle_{\alpha} \quad \forall x \in H \)

\[
i.e. \quad f(y_\alpha) = \langle y_\alpha, y_\alpha \rangle_{\alpha} = ||y_\alpha||_{\alpha}^2
\]

\[
\Rightarrow ||y_\alpha||_{\alpha}^2 = f(y_\alpha) \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
\]

\[
\Rightarrow ||y_\alpha||_{\alpha}^2 = f(y_\alpha) \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
\]

\[
\Rightarrow ||y_\alpha||_{\alpha}^2 \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
\]

\[
\Rightarrow ||y_\alpha||_{\alpha} \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
\]

\[
\Rightarrow ||y_\alpha||_{\alpha} \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
\]

\[
\Rightarrow ||y_\alpha||_{\alpha} \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
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\[
\Rightarrow ||y_\alpha||_{\alpha} \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
\]

\[
\Rightarrow ||y_\alpha||_{\alpha} \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
\]

\[
\Rightarrow ||y_\alpha||_{\alpha} \leq ||f||_{\alpha}^* ||y_\alpha||_{1-\alpha}
\]

From (ii) and (iii) we have the required result.
V. Conclusion

In this paper, we consider fuzzy inner product space introduced by Pinaki Mazumder & S. K. Samanta. Some important concepts viz. $\alpha$- fuzzy orthonormal set, complete fuzzy orthonormal set etc. have been introduced. We establish Bessel’s inequality and Riesz representation theorem in fuzzy setting. We think that these results will be helpful for the researchers to develop fuzzy functional analysis specially for operator theory and spectral theory.

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References