Some Fundamental Theorems in Felbin's Type Fuzzy Normed Linear Spaces

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Abstract—**In this paper, some fundamental theorems of functional analysis viz. the Open mapping theorem, the Closed graph theorem and the Uniform boundedness principle theorem are established in fuzzy setting.**

Index Terms—**Fuzzy continuous, α-convergent, α-Cauchy, α-complete.**
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I. Introduction

In the field of fuzzy functional analysis, the concept of a fuzzy norm on a linear space is of comparatively recent origin. It was Katsaras [1], who while studying fuzzy topological vector spaces, was the first to introduce in 1984 the idea of fuzzy norm on a linear space. Following his pioneering work, Felbin [2] offered in 1992 an alternative definition of a fuzzy norm on a linear space with an associated metric of the Kaleva & Seikkala type [3]. A further development along this line of inquiry took place when in 1994, S.C.Cheng & J.N.Mordeson [4] evolved the definition of a further type of fuzzy norm having a corresponding metric of the Kramosil & Michalek type[5]. It is in the context of such an evolution of the concept of a fuzzy norm on a linear space that Bag & Samanta[6] undertook their joint study of the issues involved with a view to exploring the possibilities of arriving at yet another definition of the norm that might prove to be capable of more effective application in appropriate fields. The novelty of this definition is the validity of a decomposition theorem for this type a fuzzy norm on a linear space that Bag & Samanta undertook their joint study of the issues involved with a view to exploring the possibilities of arriving at yet another definition of the norm that might prove to be capable of more effective application in appropriate fields. The novelty of this definition is the validity of a decomposition theorem for this type of fuzzy norm and using this decomposition theorem it has been possible to establish many important results of fuzzy functional analysis (for references please see [7], [8], [9], [10].

In this context it is worth mentioning the work done by J.Xiao & X.Zhu[11] who have considered Felbin-type fuzzy norm in its general form and studied various properties of fuzzy normed linear spaces.
In [12], Felbin introduced an idea of fuzzy bounded linear operators over fuzzy normed linear spaces and defined "fuzzy norm" for such an operator which seems to be erroneous as shown in Example 3.1 in [10].

In [10], Bag & Samanta have tried to introduce a correct definition of a fuzzy bounded linear operator and "fuzzy norm" for such an operator. Firstly Felbin's definition of "fuzzy normed linear space" is slightly modified in the sense that;

(i) the value of the fuzzy norm is taken to be a fuzzy real number in the sense of Xiao and Zhu [11];
(ii) the conditions (A) and (B) of Felbin[12] (please see Definition 2.1) is relaxed by the condition (A').

With this definition of fuzzy normed linear space, it has been possible to introduce a notion of fuzzy bounded linear operator over fuzzy normed linear spaces and to define "fuzzy norm" for such an operator.

In the present paper modified definition of Felbin's type fuzzy normed linear space is taken and establish three fundamental theorems viz. the Open maqpping theorem, the Closed graph theorem and the Uniform boundedness principle theorem in fuzzy setting.

The organization of this paper is as follows:

Section 2 comprises some useful definitions, notations and preliminary results. Section 3 is devoted to introduce some notions viz. α-convergent, α-Cauchy sequence etc. The Open mapping theorem is established in Section 4. In Section 5, the Closed graph theorem is proved. The Uniform boundedness principle theorem is established in the last Section 6.

II. Some Preliminary Results.

According to Mizumoto & Tanaka [13], a fuzzy real number is a mapping $\eta : R \to [0 , 1]$ over the set R of all reals.

η is called **convex** if $\eta(t) \geq min\ (\eta(s) , \eta(r))$ where $s \leq t \leq r$.

If there exists a $t_0 \in R$ such that $\eta(t_0) = 1$, then η is called **normal**.

For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy real number η (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha , b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible.

When $a_\alpha = -\infty$, for instance, then $[a_\alpha , b_\alpha]$ means the interval $(-\infty , b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
η is called non-negative if $\eta(t) = 0, \forall t < 0$.
For any real number r, $\bar{r}$ is defined by $\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$.
Kaleva & Seikkala [3] (Felbin [2]) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by E (R(I)) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

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As α-level sets of a convex fuzzy number is an interval, there is a debate in the nomenclature of *fuzzy numbers / fuzzy real numbers* . In [14], D. Dubois & H. Prade suggested to call this as *fuzzy interval* . They developed a different notion of a fuzzy real number by considering it as a fuzzy element of the real line, each α-cut of which a real number. From now on *"fuzzy real numbers"* are renamed as *"fuzzy intervals"*. While refering to previous results involving *fuzzy real number* , the term *fuzzy interval* is written within brackets after *fuzzy real number* to avoid any confusion; otherwise the new nomenclature i.e. *fuzzy interval* is used.

In this paper definitions and results of Kaleva & Seikkala [3], Felbin[2] and Xiao & Zhu [11] are quoted maintaining their respective notations as far as possible.

A partial ordering " $\preceq$ " in E is defined by $\eta \preceq \delta$ if and only if $a_\alpha^1 \leq a_\alpha^2$ and $b_\alpha^1 \leq b_\alpha^2$ for all $\alpha \in (0\ ,\ 1]$ where $[\eta]_\alpha = [a_\alpha^1\ ,\ b_\alpha^1]$ and $[\delta]_\alpha = [a_\alpha^2\ ,\ b_\alpha^2]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a_\alpha^1 < a_\alpha^2$ and $b_\alpha^1 < b_\alpha^2$ for each $\alpha \in (0\ ,\ 1]$.

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below:

Definition 2.1 [2] Let X be a vector space over R. Let $||\ || : X \to R^*(I)$ and let the mappings

$$L, U : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$$

be symmetric, nondecreasing in both arguments and satisfying $L(0\ ,\ 0) = 0$ and $U(1\ ,\ 1) = 1$.

Write

$$[||x||]_\alpha = [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$$

for $x \in X,\ 0 < \alpha \leq 1$ and suppose for all $x \in X,\ x \neq \underline{0}$, there exists $\alpha_0 \in (0\ ,\ 1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $||x||_\alpha^2 < \infty$,

(B) $\inf ||x||_\alpha^1 > 0$.

The quadruple $(X\ ,\ ||\ ||,\ L\ ,\ U)$ is called a **fuzzy normed linear space** and $||\ ||$ is a **fuzzy norm** if

(i) $||x|| = \bar{0}$ if and only if $x = \underline{0}$ (the null vector),

(ii)$||rx|| = |r|||x||,\ x \in X,\ r \in R$,

(iii) for all $x, y \in X$,

(a) whenever $s \leq ||x||_1^1,\ t \leq ||y||_1^1$ and $s + t \leq ||x + y||_1^1$, $||x + y||(s + t) \geq L(||x||(s)\ ,\ ||y||(t))$.

(b) whenever $s \geq ||x||_1^1,\ t \geq ||y||_1^1$ and $s + t \geq ||x + y||_1^1$, $||x + y||(s + t) \leq U(||x||(s)\ ,\ ||y||(t))$.

Remark 2.1. Felbin[2] proved that if $L = \bigwedge$(Min) and $U = \bigvee$(Max) then the triangle inequality (iii) in the Definition 2.1 is equivalent to $||x + y|| \preceq ||x|| \bigoplus ||y||$.

Further $||\ ||_\alpha^i;\ i = 1, 2$ are crisp norms on X for each $\alpha \in (0\ ,\ 1]$.

In this paper we consider the concept of fuzzy real numbers (*fuzzy intervals*) in the sense of Xiao and Zhu[11] which is defined below:

Definition 2.2[11]. A mapping $\eta : R \to [0\ ,\ 1]$ is called a **fuzzy real number** (**fuzzy interval**), whose α-level set is denoted by $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:

(N1) There exists $t_0 \in R$ such that $\eta(t_0) = 1$.

(N2) For each $\alpha \in (0\ ,\ 1]$; $[\eta]_\alpha = [\eta_\alpha^-\ ,\ \eta_\alpha^+]$, $where -\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathcal{F}$.

Since to each $r \in R$, one can consider $\bar{r} \in \mathcal{F}$ defined by $\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$, R can be embedded in $\mathcal{F}$.

Lemma 2.1[11]. *$\eta \in \mathcal{F}$ if and only if $\eta : R \to [0\ ,\ 1]$ satisfies :*

1. *η is normal, convex, and upper semicontinuous.*

2. $\lim_{t \to -\infty} \eta(t) = \lim_{t \to +\infty} \eta(t) = 0$.

Remark 2.2[10]. It is clear that $F \subset E$ and an element η of E belongs to F iff η satisfies condition (2) of Lemma 2.1.

Definition 2.3[11]. Let $\eta \in \mathcal{F}$. Then η is called a positive fuzzy real number (fuzzy interval) if $\eta(t) = 0\ \forall t < 0$. The set of all positive fuzzy real numbers (fuzzy intervals) is denoted by $\mathcal{F}^+$.

Proposition 2.1[3]. *Let $\eta\ ,\ \delta \in \mathcal{F}$ and $[\eta]_\alpha = [a_\alpha^1\ ,\ b_\alpha^1]$, $[\delta]_\alpha = [a_\alpha^2\ ,\ b_\alpha^2],\ \alpha \in (0\ ,\ 1]$.*

Then

$[\eta \bigoplus \delta]_\alpha = [a_\alpha^1 + a_\alpha^2\ ,\ b_\alpha^1 + b_\alpha^2]$,

$[\eta \ominus \delta]_\alpha = [a_\alpha^1 - b_\alpha^2\ ,\ b_\alpha^1 - a_\alpha^2]$,

$[\eta \odot \delta]_\alpha = [a_\alpha^1 a_\alpha^2\ ,\ b_\alpha^1 b_\alpha^2]$,

$[\bar{1} \oslash \delta]_\alpha = [\frac{1}{b_\alpha^2}\ ,\ \frac{1}{a_\alpha^2}],\ a_\alpha^2 > 0,\ \forall\ \alpha \in (0\ ,\ 1]$.

For the case when $U = \bigvee(max)$ and $L = \bigwedge(min)$, we slightly change the conditions (A) and (B) of Felbin[2] to define a fuzzy norm on a linear space as given below:

Definition 2.4[10]. Let X be a linear space over R. Let $||\ || : X \to \mathcal{F}^+$ be a mapping satisfying:

(i) $||x|| = \bar{0}$ iff $x = \underline{0}$,

(ii) $||rx|| = |r|||x||,\ x \in X,\ r \in R$,

(iii) for all $x,\ y \in X$,
$||x + y|| \preceq ||x|| \oplus ||y||$

and

$(A') : \quad x \neq \underline{0} \Rightarrow ||x||(t) = 0,\ \forall t \leq 0$.

Then $(X\ ,\ ||\ ||)$ is called a fuzzy normed linear space and $||\ ||$ is called a fuzzy norm on X.

Remark 2.3[10]. (i) Condition (A') in Definition 2.4 is equivalent to the condition

$(A'') :$ for all $x(\neq \underline{0}) \in X, ||x||_\alpha^1 > 0,\ \forall \alpha \in (0\ ,\ 1]$ where $[||x||]_\alpha = [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$

and (ii) $||\ ||_\alpha^i : i = 1, 2$ are crisp norms on X.

Definition 2.5[2]. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to **converge to** $x \in X$ denoted by $\lim_{n \to \infty} x_n = x$ if and only if $\lim_{n \to \infty} ||x_n - x|| = \bar{0}$ i.e. $\lim_{n \to \infty} ||x_n - x||_\alpha^1 = \lim_{n \to \infty} ||x_n - x||_\alpha^2 = 0,\ \forall \alpha \in (0\ ,\ 1]$.

Definition 2.6[10]. Let $(X\ ,\ ||\ ||)$ and $(Y\ ,\ ||\ ||^*)$ be two fuzzy normed linear spaces and $T : X \to Y$ be a linear operator. T is said to be **strongly fuzzy bounded** if there exists a real number $k > 0$ such that $||Tx||^* \oslash ||x|| \preceq \bar{k}\ \ \forall x(\neq \underline{0}) \in X$.

Remark 2.4[10]. T is a strongly fuzzy bounded linear operator from $(X\ ,\ ||\ ||)$ to $(Y\ ,\ ||\ ||^*)$, iff T is a bounded linear operator from $(X\ ,\ ||\ ||_\alpha^1)$ to $(Y\ ,\ ||\ ||_\alpha^{*2})$ and from $(X\ ,\ ||\ ||_\alpha^2)$ to $(Y\ ,\ ||\ ||_\alpha^{*1})$ uniformly for all $\alpha \in (0\ ,\ 1]$.

Definition 2.7[10]. Let $(X\ ,\ ||\ ||)$ and $(Y\ ,\ ||\ ||^*)$ be two fuzzy normed linear spaces and $T : X \to Y$ be a linear operator. T is said to be weakly fuzzy bounded if there exists a fuzzy real number $\eta \in \mathcal{F}^+$, $\eta \succ \bar{0}$ such that $||Tx||^* \oslash ||x|| \preceq \eta \ \forall x (\neq \underline{0}) \in X$.

Remark 2.5[10]. T is weakly fuzzy bounded linear operator from $(X , || ||)$ to $(Y , || ||^*)$, iff T is a bounded linear operator from $(X , || ||^1_\alpha)$ to $(Y , || ||^{*2}_\alpha)$ and from $(X , || ||^2_\alpha)$ to $(Y , || ||^{*1}_\alpha) \ \forall \alpha \in (0 , 1]$.

Proposition 2.2[9]. *Let* $\{[a_\alpha , b_\alpha]; \alpha \in (0 , 1]\}$ *be a family of nested bounded closed intervals.*
Let $\eta : R \to [0 , 1]$ *be a function defined by*
$\eta(t) = \bigvee\{\alpha \in (0 , 1] : t \in [a_\alpha , b_\alpha]\}$.
Then η *is a fuzzy real number (fuzzy interval).*
α*-level sets of* η *are denoted by* $[\eta]_\alpha = [\eta^-_\alpha , \eta^+_\alpha]$, $\alpha \in (0 , 1]$.

Proposition 2.3[9]. *If* η_i *are the fuzzy real numbers (fuzzy intervals) generated by the family of nested bounded closed intervals* $\{[a^i_\alpha , b^i_\alpha], \alpha \in (0 , 1]\}$ $i = 1, 2$ *and if* $a^1_\alpha \leq a^2_\alpha$, $b^1_\alpha \leq b^2_\alpha$, $\forall \alpha \in (0 , 1]$ *then* $\eta_1 \preceq \eta_2$.

Proposition 2.4[9]. *Let* $T : (X , || ||) \to (Y , || ||^\sim)$ *be a strongly fuzzy bounded linear operator. Define a function* $||T||^* : R \to [0 , 1]$ *by*
$||T||^*(t) = \bigvee\{\alpha \in (0 , 1] : t \in [||T||^{*1}_\alpha , ||T||^{*2}_\alpha]\}$.
Then $||T||^*$ *is a fuzzy interval (fuzzy real number) and it is the fuzzy norm of T. Where*

$$||T||^{*1}_\alpha = Sup_{x \in X, x \neq \underline{0}} \frac{||Tx||^{\sim 1}_\alpha}{||x||^2_\alpha} \ (\leq k),$$

$$||T||^{*2}_\alpha = Sup_{x \in X, x \neq \underline{0}} \frac{||Tx||^{\sim 2}_\alpha}{||x||^1_\alpha} \ (\leq k)$$

and
$[||x||]_\alpha = [||x||^1_\alpha , ||x||^2_\alpha]$, $[||Tx||^\sim]_\alpha = [||Tx||^{\sim 1}_\alpha , ||Tx||^{\sim 2}_\alpha]$

Definition 2.8[3] A sequence $\{x_n\}$ in R(I) converges to x denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} a^n_\alpha = a_\alpha$ and $\lim_{n\to\infty} b^n_\alpha = b_\alpha \ \forall \alpha \in (0 , 1]$ where $[x_n]_\alpha = [a^n_\alpha , b^n_\alpha]$ and $[x]_\alpha = [a_\alpha , b_\alpha]$.

Definition 2.9[3] A sequence $\{x_n\}$ in R(I) is said to be Cauchy if both $\{a^n_\alpha\}$ and $\{b^n_\alpha\}$ are Cauchy $\forall \alpha \in (0 , 1]$.

III. Some Results on Fuzzy Norms

In this section we introduce the notion of α-convergent sequence, α-Cauchy sequence, α-complete fuzzy normed linear space and study some of their properties.

Definition 3.1 Let $\{x_n\}$ be a sequence in R(I) and $\alpha \in (0 , 1]$. Then $\{x_n\}$ is said to be α-**convergent** if $\exists x \in R(I)$ such that $\{a^n_\alpha\}$ and $\{b^n_\alpha\}$ converges to a_α and b_α respectively where $[x_n]_\alpha = [a^n_\alpha , b^n_\alpha]$ and $[x]_\alpha = [a_\alpha , b_\alpha]$.
$\{x_n\}$ is said to be α-Cauchy if $\{a^n_\alpha\}$ and $\{b^n_\alpha\}$ are Cauchy sequences in R.

Definition 3.2 Let $(X , || ||)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X and $\alpha \in (0 , 1]$. Then $\{x_n\}$ is said to be α-**Cauchy** if $\{||x_n||\}$ is α-Cauchy in R(I) and $\{x_n\}$ is said to be α-convergent if $\{||x_n||\}$ is α-convergent in R(I).

Proposition 3.1 *Let* $(X , || ||)$ *be a fuzzy normed linear space and* $\alpha \in (0 , 1]$. *Then a sequence* $\{x_n\}$ *in X is* α-*convergent if it is convergent but converse is not necessarily true.*

Proof Suppose $\{x_n\}$ is convergent and converges to x.
So $||x_n||^1_\alpha \to ||x||^1_\alpha$ and $||x_n||^2_\alpha \to ||x||^2_\alpha \ \forall \alpha \in (0 , 1]$, where $[||x_n||]_\alpha = [||x_n||^1_\alpha , ||x_n||^2_\alpha]$ and $[||x||]_\alpha = [||x||^1_\alpha , ||x||^2_\alpha] \ \alpha \in (0 , 1]$.
From above it follows that $\{x_n\}$ is α-convergent.

To show that the converse result does not hold , we consider the following example:

Example 3.1 Let $U = l^\infty$, be the sequence space.
Define,

$$||x||' = Sup_n\{\frac{|x_n|}{n}\}$$

$$||x||'' = Sup_n\{|x_n|\}$$

where $x = (x_1, x_2, \ldots, x_n, \ldots)$. Then $|| ||'$ and $|| ||''$ are norms on X.
We now define a function $|| || : R \to [0, 1]$ as follows:
For $t \neq 0$,

$$||x||(t) = \begin{cases} 1 & \text{if } t \geq 2||x||'' \\ \frac{||x||''}{t} & \text{if } ||x||'' \leq t < 2||x||'' \\ \frac{||x||'}{2t} & \text{if } ||x||' \leq t < ||x||'' \\ 0 & \text{Otherwise} \end{cases}.$$

For $x = \underline{0}$, $t = 0$;
$||x||(t) = 1$.
Then $(X , || ||)$ is a fuzzy normed linear space and

$$[||x||]_\alpha = [||x||'' , \tfrac{||x||''}{\alpha}] \text{ when } \tfrac{1}{2} < \alpha \leq 1$$

$$= [||x||' , \tfrac{||x||'}{2\alpha}] \text{ when } 0 < \alpha \leq \tfrac{1}{2}.$$

Now we choose a sequence $\{e_n\}$ in X where

$e_1 = (1, 0, 0, \ldots)$
$e_2 = (0, 1, 0, \ldots)$
$- - - - - - - - - -$
$e_n = (0, 0, \ldots, \underbrace{1}_{nth\ term}, 0, \ldots)$

and $e = (0, 0, 0, \ldots)$.
We have $||e_n - e||' = Sup_n\{\frac{|x_n|}{n}\} = \frac{1}{n}$.
So $\lim_{n\to\infty} ||e_n - e||' = 0$.
Now $||e_n - e||'' = Sup_n\{ |x_n|\} = 1$.
So $\lim_{n\to\infty} ||e_n - e||'' = 1$.
Thus $\lim_{n\to\infty} ||e_n - e||^1_\alpha = \lim_{n\to\infty} ||e_n - e||^2_\alpha = 0 \ \forall \alpha \leq \frac{1}{2}$- - - - - - - $-(i)$.
Again,
$||e_n - e||'' = Sup_n\{ |x_n|\} = 1$.
So $\lim_{n\to\infty} ||e_n - e||'' = 1$.
i.e. $\lim_{n\to\infty} ||e_n - e||^1_\alpha = \lim_{n\to\infty} ||e_n - e||^2_\alpha = 1 \ \forall \alpha > \frac{1}{2}$- - - - - - - $-(ii)$.
From (i) and (ii), it follows that $\{e_n\}$ is α-convergent (for $\alpha \leq \frac{1}{2}$) but not convergent.

Proposition 3.2 *Let* $(X , || ||)$ *be a fuzzy normed linear space and* $\alpha \in (0 , 1]$. *Then every* α-*convergent sequence in* $(X , || ||)$ *is an* α-*Cauchy sequence in* $(X , || ||)$.

Proof Let $\{x_n\}$ be an α-convergent sequence in X. Thus $\exists x \in X$ such that $||x_n||^1_\alpha \to ||x||^1_\alpha$ and $||x_n||^2_\alpha \to ||x||^2_\alpha$ where $[||x_n||]_\alpha = [||x_n||^1_\alpha , ||x_n||^2_\alpha]$ and $[||x||]_\alpha = [||x||^1_\alpha , ||x||^2_\alpha]$.
Now,
$||x_n - x_m||^2_\alpha = ||x_n - x + x - x_m||^2_\alpha \leq ||x_n - x||^2_\alpha + ||x - x_m||^2_\alpha$.

i.e. $||x_n - x_m||^2_\alpha \to 0$ as $m, n \to \infty$.
Thus $\{x_n\}$ is an α-Cauchy sequence in $(X \ , \ || \ ||)$.

Proposition 3.3 *Every constant sequence in a fuzzy normed linear space is an α-Cauchy sequence for each $\alpha \in (0 \ , \ 1]$.*

Proof The proof is straightforward.

Definition 3.3 Let $(X \ , \ || \ ||)$ be a fuzzy normed linear space and $\alpha \in (0 \ , \ 1]$. X is said to be α-complete if it is complete w.r.t. $|| \ ||^1_\alpha$ where $[||x||]_\alpha = [||x||^1_\alpha \ , \ ||x||^2_\alpha]$.

Remark 3.1 It is easily followed that if X is α-complete then it is also complete w.r.t. $|| \ ||^2_\alpha$.

Definition 3.4 Let $(X \ , \ || \ ||)$ be a fuzzy normed linear space and $\alpha \in (0 \ , \ 1]$. A subset F of X is said to be α-closed if every α-convergent sequence in F converges to some point in F.

Proposition 3.4 *Let $(X \ , \ || \ ||)$ be a fuzzy normed linear space and $A \subset X$ and $\alpha \in (0 \ , \ 1]$. Then A is α-closed iff A is closed w.r.t. $|| \ ||^2_\alpha$ where $[||x||]_\alpha = [||x||^1_\alpha \ , \ ||x||^2_\alpha]$.*

Proof The proof is straightforward.

IV. Open Mapping Theorem

In this section we establish the Open mapping theorem in fuzzy setting.

Theorem 4.1 *Let $(X \ , \ || \ ||)$ be a fuzzy normed linear space. Define,*
$\tau_1 = \{G \subset X : \ G \text{ is open in } (X \ , \ || \ ||^1_\alpha), \ \forall \alpha \in (0 \ , \ 1]\}$,
$\tau_2 = \{H \subset X : \ H \text{ is open in } (X \ , \ || \ ||^2_\alpha), \ \forall \alpha \in (0 \ , \ 1]\}$
where $[||x||]_\alpha = [||x||^1_\alpha \ , \ ||x||^2_\alpha]$, $x \in X$ *and* $\alpha \in (0 \ , \ 1]$.
Then τ_1 and τ_2 are topologies on X.

Proof First we show that τ_1 is a topology on X.
Clearly,
(i) $\phi \in \tau_1$.
(ii) $X \in \tau_1$.
(iii) Let $G = \bigcap_{1 \le i \le k} G_i$ where $G_i \in \tau_1$ for $i = 1, 2, \ldots, k$.
Take $x \in G$, so $x \in G_i$ for $i = 1, 2, \ldots, k$.
Since $G_i \in \tau_1$ for $i = 1, 2, \ldots, k$ it follows that for each $\alpha \in (0 \ , \ 1]$, $\exists r_{i,\alpha}(\text{ say }) > 0$ such that $B^{1i}_\alpha(x \ , \ r_{i,\alpha}) \subset \tau_1$ where $B^{1i}_\alpha(x \ , \ r_{i,\alpha})$ is an open ball in G_i $(1 \le i \le k)$ with centre at x and radius $r_{i,\alpha}$ w.r.t. $|| \ ||^1_\alpha$ $\forall \alpha \in (0 \ , \ 1]$.
Choose $r^1_\alpha = min_{1 \le i \le k}\{r_{i,\alpha}\}$.
Then $B^1_\alpha(x \ , \ r_\alpha) \subset B^{1i}_\alpha(x \ , \ r_{i,\alpha})$ for $1 \le i \le k$.
i.e. $B^1_\alpha(x \ , \ r_\alpha) \subset \bigcap_{1 \le i \le k} = G$ where $B^1_\alpha(x \ , \ r_\alpha)$ is an open ball with centre at x and radius r_α w.r.t. $|| \ ||^1_\alpha$, $\alpha \in (0 \ , \ 1]$.
Hence G is open in $(X \ , \ || \ ||^1_\alpha)$ $\forall \alpha \in (0 \ , \ 1]$.
So $G \in \tau_1$.
(iv) Let $G = \bigcup_i G_i$ where $G_i \in \tau_1$ for $i = 1, 2, \ldots$.
Take $x \in G$. Then $x \in G_i$ for some i say i_0.
So $x \in G_{i_0}$. Since $G_{i_0} \in \tau_1$, thus for each $\alpha \in (0 \ , \ 1]$, $\exists r_{i_0,\alpha}(\text{ say }) > 0$ such that $B^{1i_0}_\alpha(x \ , \ r_{i_0,\alpha}) \subset G_{i_0}$ where $B^{1i_0}_\alpha(x \ , \ r_{i_0 \ , \ \alpha})$ is an open ball with centre at x and radius $r_{i_0,\alpha}$ w.r.t. $|| \ ||^1_\alpha$.
Thus $B^{1i_0}_\alpha(x \ , \ r_{i_0 \ , \ \alpha}) \subset G$.
i.e. G is open in $(X \ , \ || \ ||^1_\alpha)$ $\forall \alpha \in (0 \ , \ 1]$ and hence $G \in \tau_1$.
Thus τ_1 is a topology on X.
In a similar way we can verify that τ_2 is also a topology on X.
Let $\tau = \tau_1 \bigcap \tau_2$. Then τ is a topology on X.

Definition 4.1 Let $(X \ , \ || \ ||)$ be a fuzzy normed linear space and $O \subset X$. O is said to be open in X iff $O \in \tau$ where $\tau = \tau_1 \bigcap \tau_2$.

Theorem 4.2 (**Open Mapping Theorem**) *Let $(X \ , \ || \ ||)$ and $(Y \ , \ || \ ||^*)$ be two α-complete fuzzy normed linear spaces for each $\alpha \in (0 \ , \ 1]$. Let $T : \ (X \ , \ || \ ||) \to (Y \ , \ || \ ||^*)$ be a strongly fuzzy bounded linear operator. If T is onto then T is an open mapping.*

Proof Since $(X \ , \ || \ ||)$ and $(Y \ , \ || \ ||^*)$ are α-complete fuzzy normed linear spaces for each $\alpha \in (0 \ , \ 1]$, thus $(X \ , \ || \ ||^1_\alpha)$, $(X \ , \ || \ ||^2_\alpha)$, $(Y \ , \ || \ ||^{*1}_\alpha)$ and $(Y \ , \ || \ ||^{*2}_\alpha)$ are Banach spaces for each $\alpha \in (0 \ , \ 1]$.
Again since $T : \ (X \ , \ || \ ||) \to (Y \ , \ || \ ||^*)$ is a strongly fuzzy bounded linear operator, there exists a real number $k > 0$ such that $||Tx||^* \oslash ||x|| \ \preceq \bar{k} \ \ \forall x (\neq \underline{0}) \in X$.
i.e. $||Tx||^{*1}_\alpha \ \le k|| \ ||^2_\alpha$ and $||Tx||^{*2}_\alpha \ \le k|| \ ||^1_\alpha \ \ \forall \alpha \in (0 \ , \ 1]$.
So, $T : \ (X \ , \ || \ ||^1_\alpha) \to (Y \ , \ || \ ||^{*2}_\alpha)$
and
$T : \ (X \ , \ || \ ||^2_\alpha) \to (Y \ , \ || \ ||^{*1}_\alpha)$ is a bounded linear operator for each $\alpha \in (0 \ , \ 1]$.
Let O be an open subset of X. From Definition 4.1, it follows that O is open w.r.t. $|| \ ||^1_\alpha$ and $|| \ ||^2_\alpha$ $\forall \alpha \in (0 \ , \ 1]$.
Since T is onto, by applying open mapping theorem it follows that T(O) is open w.r.t. $|| \ ||^{*1}_\alpha$ and $|| \ ||^{*2}_\alpha$ $\forall \alpha \in (0 \ , \ 1]$.
Thus by Theorem 4.1 and Definition 4.1, it follows that T(O) is open in $(Y \ , \ || \ ||^*)$.
Hence T is an open mapping.

Note 4.1 If T is weakly fuzzy bounded linear operator then the Theorem 4.2 is also true.

V. Closed graph theorem.

In this section we establish the Closed graph theorem in fuzzy setting.

Let $(X \ , \ || \ ||)$ and $(Y \ , \ || \ ||^*)$ be two α-complete fuzzy normed linear spaces for each $\alpha \in (0 \ , \ 1]$.
Define
$(x \ , \ y) + (x' \ , \ y') = (x + x' \ , \ y + y')$ and $c(x \ , \ y) = (cx \ , \ cy)$ where $(x \ , \ y), \ (x' \ , \ y') \in X \times Y$ and c is a scalar.
Since $(X \ , \ || \ ||)$ and $(Y \ , \ || \ ||^*)$ are α-complete fuzzy normed linear spaces for each $\alpha \in (0 \ , \ 1]$ thus $(X \ , \ || \ ||^i_\alpha)$ and $(Y \ , \ || \ ||^i_\alpha)$ are Banach spaces for $i = 1, \ 2$ and for each $\alpha \in (0 \ , \ 1]$.
Now we define functions $|| \ ||'_\alpha$ and $|| \ ||''_\alpha$ from $X \times Y$ to R^+ by
$||(x \ , \ y)||'_\alpha = ||x||^1_\alpha \ + \ ||y||^{*2}_\alpha$
and
$||(x \ , \ y)||''_\alpha = ||x||^2_\alpha \ + \ ||y||^{*1}_\alpha, \ \ \alpha \in (0 \ , \ 1]$.
Then it can be verified that $(X \times Y \ , \ || \ ||'_\alpha)$ and $(X \times Y \ , \ || \ ||''_\alpha)$ are normed linear spaces for each $\alpha \in (0 \ , \ 1]$.
Also $(X \times Y \ , \ || \ ||'_\alpha)$ and $(X \times Y \ , \ || \ ||''_\alpha)$ are Banach spaces for each $\alpha \in (0 \ , \ 1]$.

We recall that,
if X and Y are linear spaces and $T \ : X \to Y$ is a linear operator, then the set given by $G(T) = \{(x \ , \ Tx) : \ x \in X\} \subset X \times Y$ is called the graph of T.

Definition 5.1 Let $(X\ ,\ ||\ ||)$ and $(Y\ ,\ ||\ ||^*)$ be two fuzzy normed linear spaces and $T\ :(X\ ,\ ||\ ||)\rightarrow(Y\ ,\ ||\ ||^*)$ be a linear operator and $\alpha\in(0\ ,\ 1]$.
Then $G(T)\ =\{(x\ ,\ Tx):\ x\in X\}\subset X\times Y$ is said to be α-**closed** if for every sequence $\{x_n\}$ in X,
$||x_n-x||_\alpha^2\rightarrow 0$ and $||Tx_n-y||_\alpha^{*2}\rightarrow 0$ as $n\rightarrow\infty$ implies $x\in X$ and $y=Tx$ where
$[||x||]_\alpha = [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$ and $[||y||]_\alpha = [||y||_\alpha^{*1}\ ,\ ||y||_\alpha^{*2}],\ \ \alpha\in(0\ ,\ 1]$.
If G(T) is α-closed then T is called an α-closed linear operator.

Lemma 5.1 *Let* $(X\ ,\ ||\ ||)$ *and* $(Y\ ,\ ||\ ||^*)$ *be two fuzzy normed linear spaces and* $\alpha\in(0\ ,\ 1]$. *Then* $T\ :(X\ ,\ ||\ ||)\rightarrow(Y\ ,\ ||\ ||^*)$ *is an* α*-closed linear operator iff*
$T\ :(X\ ,\ ||\ ||_\alpha^1)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*2})$ *and* $T\ :(X\ ,\ ||\ ||_\alpha^2)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*1})$ *are closed linear operators.*
Where $[||x||]_\alpha = [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$ *and* $[||y||]_\alpha = [||y||_\alpha^{*1}\ ,\ ||y||_\alpha^{*2}],\ \ \alpha\in(0\ ,\ 1]$.

Proof Suppose $T\ :(X\ ,\ ||\ ||)\rightarrow(Y\ ,\ ||\ ||^*)$ is an α-closed linear operator for some $\alpha\in(0\ ,\ 1]$.
Let $\{x_n\}$ be a sequence in X such that
$(||x_n-x||_\alpha^2\rightarrow 0,\ ||Tx_n-y||_\alpha^{*1}\rightarrow 0$ as $n\rightarrow\infty)$ and $(||x_n-x'||_\alpha^1\rightarrow 0,\ ||Tx_n-y'||_\alpha^{*2}\rightarrow 0$ as $n\rightarrow\infty)$.
Now,
$||x-x'||_\alpha^1\leq||x-x_n||_\alpha^1+||x_n-x'||_\alpha^1\leq||x-x_n||_\alpha^2+||x_n-x'||_\alpha^1$
$\Rightarrow||x-x'||_\alpha^1\ =0$
$\Rightarrow x=x'$.
Similarly $y=y'$.
Thus $(||x_n-x||_\alpha^2\rightarrow 0,\ ||Tx_n-y||_\alpha^{*1}\rightarrow 0$ as $n\rightarrow\infty)$ and $(||x_n-x'||_\alpha^1\rightarrow 0,\ ||Tx_n-y'||_\alpha^{*2}\rightarrow 0$ as $n\rightarrow\infty)$.
$\Rightarrow$ $(||x_n-x||_\alpha^2\rightarrow 0,\ ||Tx_n-y||_\alpha^{*1}\rightarrow 0$ as $n\rightarrow\infty)$ and $(||x_n-x||_\alpha^1\rightarrow 0,\ ||Tx_n-y||_\alpha^{*2}\rightarrow 0$ as $n\rightarrow\infty)$
$\Rightarrow||x_n-x||_\alpha^2\rightarrow 0$ and $||Tx_n-y||_\alpha^{*2}\rightarrow 0$ as $n\rightarrow\infty$.
Since T is α-closed, it follows that $x\in X$ and $y=Tx$.
Hence $T\ :(X\ ,\ ||\ ||_\alpha^1)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*2})$ and $T\ :(X\ ,\ ||\ ||_\alpha^2)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*1})$ are closed linear operators.
Conversely suppose that for some $\alpha\in(0\ ,\ 1]$,
$T\ :(X\ ,\ ||\ ||_\alpha^1)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*2})$ and $T\ :(X\ ,\ ||\ ||_\alpha^2)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*1})$ are closed linear operators.
Let $\{x_n\}$ be a sequence in X such that $||x_n-x||_\alpha^2\rightarrow 0$ and $||Tx_n-y||_\alpha^{*2}\rightarrow 0$ as $n\rightarrow\infty$.
This is equivalent to
$(||x_n-x||_\alpha^2\rightarrow 0,\ ||Tx_n-y||_\alpha^{*1}\rightarrow 0$ as $n\rightarrow\infty)$,
and
$(||x_n-x||_\alpha^1\rightarrow 0,\ ||Tx_n-y||_\alpha^{*2}\rightarrow 0$ as $n\rightarrow\infty)$.
Since T is a closed linear operator w.r.t. $(||\ ||_\alpha^1,\ ||\ ||_\alpha^{*2})$ and $(||\ ||_\alpha^2,\ ||\ ||_\alpha^{*1})$, it follows that $x\in X$ and $y\ =Tx$.
i.e. T is an α-closed linear operator.

Theorem 5.1 (**Closed Graph Theorem**) *Let* $(X\ ,\ ||\ ||)$ *and* $(Y\ ,\ ||\ ||^*)$ *be* α*-complete fuzzy normed linear spaces for each* $\alpha\in(0\ ,\ 1]$ *and* $T:\ X\rightarrow Y$ *be a linear operator. Then T is weakly fuzzy bounded if T is* α*-closed for each* $\alpha\in(0\ ,\ 1]$.

Proof Since $(X\ ,\ ||\ ||)$ and $(Y\ ,\ ||\ ||^*)$ are α-complete fuzzy normed linear spaces for each $\alpha\in(0\ ,\ 1]$, it follows that $(X\ ,\ ||\ ||_\alpha^i)$ and $(Y\ ,\ ||\ ||_\alpha^{*i})$ are Banach spaces (Remark 3.1) for each $\alpha\in(0\ ,\ 1]$ and for $i=1,\ 2$ where
$[||x||]_\alpha\ =\ [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$ and $[||y||^*]_\alpha\ =\ [||y||_\alpha^{*1}\ ,\ ||y||_\alpha^{*2}]$ for $x\in X,\ y\in Y$.
Now $T:\ (X\ ,\ ||\ ||_\alpha^1)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*2})$ is a linear operator and $(X\times Y\ ,\ ||\ ||'_\alpha)$ is a normed linear space w.r.t. linear operations
$(x\ ,\ y)+(x'\ ,\ y')\ =(x+x'\ ,\ y+y')$ and $c(x\ ,\ y)\ =(cx\ ,\ cy)$ where c is any scalar and $||\ ||'_\alpha$ is given by
$||(x\ ,\ y)||'_\alpha\ =||x||_\alpha^1+||y||_\alpha^{*2}\ \ \forall\alpha\in(0\ ,\ 1]$.
Also $(X\times Y\ ,\ ||\ ||'_\alpha)$ is a Banach space since $(X\ ,\ ||\ ||_\alpha^1)$ and $(Y\ ,\ ||\ ||_\alpha^{*2})$ are so.
Again since T is an α-closed linear operator for each $\alpha\in(0\ ,\ 1]$, thus by Lemma 5.1, it follows that $T:\ (X\ ,\ ||\ ||_\alpha^1)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*2})$ is a closed linear mapping $\forall\alpha\in(0\ ,\ 1]$.
Now by applying Closed graph theorem, it follows that $T:\ (X\ ,\ ||\ ||_\alpha^1)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*2})$ is bounded $\forall\alpha\in(0\ ,\ 1]$.
So $||Tx||_\alpha^{*2}\ \leq||T||_\alpha^2||x||_\alpha^1\ \ \forall\alpha\in(0\ ,\ 1]--------(i)$
where $||T||_\alpha^2\ =\bigvee_{x\in X,x\neq\underline{0}}\dfrac{||Tx||_\alpha^{*2}}{||x||_\alpha^1}$.
By similar argument it can be shown that
$T:\ (X\ ,\ ||\ ||_\alpha^2)\rightarrow(Y\ ,\ ||\ ||_\alpha^{*1})$ is bounded $\forall\alpha\in(0\ ,\ 1]$.
So $||Tx||_\alpha^{*1}\ \leq||T||_\alpha^1||x||_\alpha^2\ \ \forall\alpha\in(0\ ,\ 1]--------(ii)$
where $||T||_\alpha^1\ =\bigvee_{x\in X,x\neq\underline{0}}\dfrac{||Tx||_\alpha^{*1}}{||x||_\alpha^2}$.
Now $\{[||T||_\alpha^1\ ,\ ||T||_\alpha^2]:\ \ \alpha\in(0\ ,\ 1]\}$ is a family of closed, bounded and nested interval of real numbers and thus generate (Proposition 2.2) a fuzzy real number $||T||$ which is the fuzzy norm of T.
Now from (i) and (ii) we have (by Proposition 2.3)
$||Tx||^*\oslash||x||\ \ \preceq||T||$.
Hence T is weakly fuzzy bounded.

VI. UNIFORM BOUNDEDNESS PRINCIPLE THEOREM

In this section Uniform boundedness principle theorem is established in fuzzy setting.

Theorem 6.1 (**Uniform Boundedness Principle Theorem**). *Let* $\{T_n\}\ \in B(X\ ,\ Y)$ *such that for each* $x\in X,\ \{T_nx\}$ *is bounded in Y, i.e.* $\exists$ *a fuzzy number* η_x *(say) such that* $||T_nx||^*\preceq\eta_x\ \ \forall n$. *Then* $\exists$ *a fuzzy real number* δ *(say) such that* $||T_n||^*\preceq\delta\ \ \forall n$ *where* $(X\ ,\ ||\ ||)$ *is an* α*-complete fuzzy normed linear space for each* $\alpha\in(0\ ,\ 1]$ *and* $(Y\ ,\ ||\ ||^*)$ *is a fuzzy normed linear space.* $B(X\ ,\ Y)$ *denotes the set of all strongly fuzzy bounded linear operators defined from X to Y.*

Proof. Let

$$[||T_nx||^*]_\alpha\ =\ [||T_nx||_\alpha^{*1}\ ,\ ||T_nx||_\alpha^{*2}]$$
$$[||x||]_\alpha\ =\ [||x||_\alpha^1\ ,\ ||x||_\alpha^2]$$

and

$$[\eta_x]_\alpha\ =\ [\eta_{x,\alpha}^1\ ,\ \eta_{x,\alpha}^2],\ \ \ \alpha\in(0\ ,\ 1].$$

Since

$$||T_nx||^*\preceq\eta_x\ \ \ \forall n$$

we have

$$||T_nx||_\alpha^{*1}\ \ \leq\ \ \eta_{x,\alpha}^1$$

and

$$||T_nx||_\alpha^{*2}\leq\ \ \eta_{x,\alpha}^2\ \ \ \forall n,\ \forall\alpha\in(0\ ,\ 1].$$

Again, since X is an α-complete fuzzy normed linear space for each $\alpha \in (0\ ,\ 1]$ thus $(X\ ,\ ||\ ||^1_\alpha)$ and $(X\ ,\ ||\ ||^2_\alpha)$ are Banach spaces for each $\alpha \in (0\ ,\ 1]$.
Now $\{T_n\}\ \in B(X\ ,\ Y)$.

So,

$$T_n\ :(X\ ,\ ||\ ||^1_\alpha) \to (Y\ ,\ ||\ ||^{*2}_\alpha)$$

and

$$T_n\ :(X\ ,\ ||\ ||^2_\alpha) \to (Y\ ,\ ||\ ||^{*1}_\alpha)$$

are sequences of bounded linear operators $\forall \alpha \in (0\ ,\ 1]$.

Now by Uniform boundedness principle theorem, it follows that for each $\alpha \in (0\ ,\ 1]$, $\exists$ constants $C^1_\alpha,\ C^2_\alpha$ (say) such that

$\sup_n ||T_n||^1_\alpha\ = C^1_\alpha$ and $Sup_n||T_n||^2_\alpha\ = C^2_\alpha$

where $||T_n||^1_\alpha\ = \sup_{x\in X, x\neq \underline{0}} \frac{||T_n x||^{*1}_\alpha}{||x||^2_\alpha}$ and $||T_n||^2_\alpha\ = \sup_{x\in X, x\neq \underline{0}} \frac{||T_n x||^{*2}_\alpha}{||x||^1_\alpha}$.

We have

$$||T_n||^1_\alpha\ \leq C^1_\alpha \text{ and } ||T_n||^2_\alpha\ \leq C^2_\alpha\ \forall n\ \forall \alpha \in (0\ ,\ 1] \qquad (i)$$

It can be easily verified that $\{[C^1_\alpha\ ,\ C^2_\alpha]\},\ \alpha \in (0\ ,\ 1]$ is a family of nested, bounded and closed interval of real numbers and thus it generates a fuzzy real number say δ (by Proposition 2.2).

Also $\{[||T_n||^1_\alpha\ ,\ ||T_n||^2_\alpha]\},\ \alpha \in (0\ ,\ 1]$ generates a fuzzy real number $||T_n||$ which is the fuzzy norm of T_n (by Proposition 2.4) for each $n\ = 1,\ 2,\ \ldots$.

Hence by Proposition 2.3, from (i) we have $||T_n||\ \preceq \delta\ \ \forall n$.

Remark 6.1 If T is weakly fuzzy bounded, then Theorem 6.1 is also true.

VII. Conclusion

In this paper, we consider the fuzzy norm in the sense of Felbin. Some basic concepts viz. α-convergent sequence, α-complete fuzzy normed linear space etc. have been introduced. Three fundamental theorems viz. Open mapping theorem, Closed graph theorem and Uniform boundedness principle theorem have been established in fuzzy setting.

Since these theorems have many applications in functional analysis, I think that the results of this paper will be helpful for the researchers to develop fuzzy functional analysis.

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