Common Fixed Points of $(\psi, \varphi)-$Weak Quasi Contractions with Property (E.A.)

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***Abstract*—In this paper, we introduce $(\psi, \varphi)-$weak quasi contraction for four selfmaps and establish common fixed point theorems using property (E.A.)/common property(E.A.). Examples are provided in support of the results.**

***Index Terms*—Selfmap, point of coincidence, common fixed point, occasionally weakly compatible maps, property (E.A.), common property (E.A.).**

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I. Introduction

Throughout this paper, (X, d) is a metric space which we write it by X. We write $R^+ = [0, \infty)$.

In 1997, Alber and Gurre-Delabrierre [1] defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for a selfmap on a Hilbert space. Rhoades [2] extended this concept to metric spaces.

A mapping $T : X \to X$ is said to be a *weak contraction* if there exists a function $\varphi : R^+ \to R^+,\ \varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))\ \forall\ x,\ y \in X.$$

As weak contractions are defined through φ, these are referred as *φ-weak contractions* [3, 4, 5].

Rhoades [2] established that every φ-weak contraction has a unique fixed point in complete metric spaces when φ is continuous. After words, Dutta and Choudhury [6] generalized the concept of weak contraction and proved the following theorem.

Theorem 1.1: [6] Let (X, d) be a complete metric space and let T be a selfmap on X satisfying

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$$

for each $x,\ y \in X$ where $\psi,\ \varphi : R^+ \to R^+$ are both continuous and non decreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in X.

Throughout this paper, we denote

$$\Psi_1 = \Big\{\psi : R^+ \to R^+ : \text{(i) } \psi \text{ is continuous (ii) } \psi \text{ is non-decreasing and (iii) } \psi(t) = 0 \text{ if and only if } t = 0\Big\}.$$

$$\Psi = \Big\{\psi : R^+ \to R^+ : \text{(i) } \psi \text{ is continuous (ii) } \psi(t) > 0 \text{ for } t > 0\Big\}.$$

$$\Phi = \Big\{\varphi : R^+ \to R^+\text{: (i) } \varphi \text{ is lower semi continuous and (ii) } \varphi(t) = 0 \text{ if and only if } t = 0\Big\}.$$

Definition 1.2: Let f and g be selfmaps of a nonempty set X. Then

(i) a point $x \in X$ is said to be a *coincidence point* of f and g if $f(x) = g(x)$. The set of coincidence points of f and g is denoted by $C(f, g)$. If $x \in C(f, g)$, then $f(x) = g(x) = w$ (say) and w is called a *point of coincidence* of f and g.

(ii) a pair of maps (f, g) is said to be *weakly compatible* if $fgx = gfx$ whenever $x \in C(f, g)$.

(iii) a pair of maps (f, g) is said to be *occasionally weakly compatible* (*owc*) if there exists $x \in C(f, g)$ such that $fgx = gfx$.

Every pair of weakly compatible maps is occasionally weakly compatible, but its converse need not be true [7].

Doric [3] introduced *generalized $(\psi, \varphi)-$weak contractions* for a pair of selfmaps as follows:

Definition 1.3: [3] Let (X, d) be a metric space. Let S and T be selfmaps of X. If there exist $\psi \in \Psi_1$ and $\varphi \in \Phi$ such that

$$\psi(d(Tx, Sy)) \leq \psi(M(x, y)) - \varphi(M(x, y))$$

for each $x, y \in X$ where

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Sy, y), \tfrac{1}{2}[d(y, Tx) + d(x, Sy)]\}$$

then we say that S and T satisfy '*generalized $(\psi, \phi) -$ weak contraction condition*'.

Theorem 1.4: [3] Let (X, d) be a metric space. Let S and T be selfmaps of X satisfying generalized $(\psi, \varphi)-$weak contraction condition. Then S and T have a unique common fixed point in X.

In 2007, Abbas and Doric [8] extended the concept of 'generalized $(\psi, \varphi)-$weak contraction' for a pair of selfmaps to four selfmaps in the following way.

Definition 1.5: [8] Let (X, d) be a metric space. Let f, g, S and T be selfmaps of X. If there exist $\psi \in \Psi_1$ and $\varphi \in \Phi$ such that

$$\psi(d(fx, gy)) \leq \psi(m(x, y)) - \varphi(m(x, y)) \qquad (1.5.1)$$

for each $x, y \in X$ where

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$$m(x,y) = \max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \tfrac{1}{2}[d(Sx,gy) + d(fx,Ty)]\}$$

then we say that f, g, S and T satisfy '*generalized* $(\psi,\varphi)-$*weak contraction condition*'.

Theorem 1.6: [8] Let (X,d) be a complete metric space. Let f, g, S and T be selfmaps of X satisfying generalized (ψ,φ) − weak contraction condition. Suppose that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and that the pairs (f,S) and (g,T) are weakly compatible. Then f, g, S and T have a unique common fixed point in X provided one of the range spaces $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is closed in X.

Choudhury, Konar, Rhoades, Metiya [9] defined 'generalized weakly contractive map' on a metric space X and proved that every generalized weakly contractive map has a fixed point in a complete metric space X.

A selfmap $T : X \to X$ is said to be a *generalized weakly contractive map* if for all $x,\ y \in X$,

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \varphi(\max\{d(x,y), d(y,Ty)\}),$$

where

$$m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \tfrac{1}{2}[d(x,Tx) + d(y,Tx)]\},$$

$\psi \in \Psi_1$ and $\varphi : R^+ \to R^+$ is continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

In this paper, we define $(\psi,\varphi)-$*weak quasi contraction* for four selfmaps in the following way:

Definition 1.7: Let (X,d) be a metric space. Let f, g, S and T be selfmaps of X. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(d(fx,gy)) \leq \psi(M(x,y)) - \varphi(M(x,y)) \qquad (1.7.1)$$

for each $x,y \in X$ where

$$M(x,y) = \max\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), d(Sx,gy), d(fx,Ty)\}$$

then f, g, S and T are said to satisfy '(ψ,φ) −*weak quasi contraction*'.

Definition 1.8: Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a *quasi contraction* if there exists a number $k,\ 0 \leq k < 1$ such that

$$d(Tx,Ty) \leq k\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

for every $x,y \in X$.

By choosing $f = g$ and $S = T = I_X$, the identity map on X, $\psi(t) = t$ and $\varphi(t) = (1-k)t,\ 0 \leq k < 1,\ t \geq 0$ in (1.7.1) we get

$$d(fx,fy) \leq k\max\{d(x,y), d(fx,x), d(fy,y), d(fx,y), d(x,fy)\}$$

which is a quasi contraction. Hence every quasi contraction is a special case of '(ψ,φ) − weak quasi contraction'. Ćirić [10] established that every quasi contraction in a (an orbitally) complete metric space has a unique fixed point.

If ψ and φ in Definition 1.5 satisfy '$\psi - \varphi$ is non-decreasing' then inequality (1.5.1) implies the inequality (1.7.1). Thus, in this case every generalized (ψ,φ) − weak contraction is a $(\psi,\varphi)-$weak quasi contraction for four selfmaps. But its converse need not be true. The following example shows that there exist maps f, g, S and T which are of '$(\psi,\varphi)-$weak quasi contraction', but they do not satisfy 'generalized $(\psi,\varphi)-$weak contraction'.

Example 1.9: Let $X = [0,\frac{3}{2}]$ with the usual metric. We define selfmaps f,g,S and T on X by

$$fx = \begin{cases} 1, & if\ 0 \leq x < \frac{1}{2} \\ \frac{x}{2}, & if\ \frac{1}{2} \leq x \leq \frac{3}{2}, \end{cases}$$

$$gx = \begin{cases} \frac{1}{2}, & if\ 0 \leq x < \frac{1}{2} \\ 0, & if\ \frac{1}{2} \leq x \leq \frac{3}{2}, \end{cases}$$

$$Sx = \begin{cases} \frac{3}{2}, & if\ 0 \leq x < \frac{1}{2} \\ x, & if\ \frac{1}{2} \leq x \leq \frac{3}{2} \end{cases} \quad \text{and}$$

$$Tx = \begin{cases} \frac{x}{2}, & if\ 0 \leq x < \frac{1}{2} \\ 1, & if\ \frac{1}{2} \leq x \leq \frac{3}{2}. \end{cases}$$

We define $\psi,\ \varphi : R^+ \to R^+$ by $\psi(t) = 1 + \frac{t}{2},\ t \geq 0$ and

$$\varphi(t) = \begin{cases} \frac{t}{16}, & if\ 0 \leq t \leq \frac{3}{4} \\ \frac{t}{32}, & if\ t > \frac{3}{4}. \end{cases}$$

With these ψ and φ it is easy to see that f, g, S and T satisfy $(\psi,\varphi)-$weak quasi contraction. Now, for $0 \leq x < \frac{1}{2}$ and $\frac{1}{2} \leq y \leq \frac{3}{2}$, we have $d(fx,gy) = 1$ and $m(x,y) = 1$. Thus $\psi(d(fx,gy)) = \psi(1)$ and $\psi(m(x,y)) - \varphi(m(x,y)) = \psi(1) - \varphi(1)$ and $\varphi \in \Phi$, so that f,g,S and T do not satisfy (1.5.1) for any $\psi \in \Psi_1$ and $\varphi \in \Phi$.

In 2002, Aamri and Moutawakil [11] introduced the notion of property (E.A.), after which many common fixed point theorems are established for multi selfmaps by relaxing the containment condition of range spaces.

Definition 1.10: [11] A pair of selfmaps (f,g) of a metric space X is said to satisfy *property* (E.A.) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X.

Definition 1.11: [12] Let f, g, S and T be selfmaps of a metric space X. The pairs (f,S) and (g,T) are said to satisfy *common property* (E.A.) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t$, for some t in X.

Pathak, Rodrigues-Lopez and Verma [13] observed that 'weak compatibility' and 'property (E.A.)' are independent of each other. Also it was shown by Babu and Alemayehu [14] that 'occasionally weak compatibility' and 'property (E.A)' are independent to each other. Babu and Alemayehu [14] proved the following theorem for two selfmaps satisfying a generalized weakly contractive condition using property (E.A).

Theorem 1.12: [14] Let f and T be selfmaps of a metric space X satisfying property (E.A.). Suppose that there exists a map $\varphi : R^+ \to R^+,\ \varphi$ is continuous, $\varphi(t) > 0$ for $t > 0$

such that

$$d(Tx,Ty) \leq M(x,y) - \varphi(M(x,y)) \qquad (1.12.1)$$

for each $x, y \in X$ where

$$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \tfrac{1}{2}[d(fx,Ty) + d(fy,Tx)]\}.$$

If $f(X)$ is closed and (f,T) is *owc* then f and T have a unique common fixed point in X.

In this paper, we prove common fixed point theorems for four selfmaps satisfying (ψ,φ) − weak quasi contraction using property (E.A.)/common property (E.A.). Examples are provided in support of our results.

II. Point of coincidence and common fixed points

Lemma 2.1: [15] Let X be a non empty set, f, g be *owc* selfmaps of X. If f and g have a unique point of coincidence v, then v is the unique common fixed point of f and g.

In this section, we extend Lemma 2.1 to four selfmaps.

Lemma 2.2: Let f, g, S and T be selfmaps of a metric space X satisfying $(\psi,\varphi)-$weak quasi contraction. If f, g, S and T have a point of coincidence then it is unique.

Proof. Suppose v_1 and v_2 are two points of coincidences of f, g, S and T. Therefore there exists u_1 and u_2 in X such that $fu_1 = Su_1 = gu_1 = Tu_1 = v_1$ and $fu_2 = Su_2 = gu_2 = Tu_2 = v_2$. (2.2.1)

Suppose $v_1 \neq v_2$. Now,

$$\psi(d(v_1,v_2)) = \psi(d(fu_1,gu_2)) \leq \psi(M(u_1,u_2)) - \varphi(M(u_1,u_2)) \qquad (2.2.2)$$

where

$$M(u_1,u_2) = \max\{d(Su_1,Tu_2), d(fu_1,Su_1), d(gu_2,Tu_2), d(fu_1,Tu_2), d(Su_1,gu_2)\}$$
$$= \max\{d(v_1,v_2), d(v_1,v_1), d(v_2,v_2), d(v_1,v_2), d(v_1,v_2)\}$$
$$= d(v_1,v_2).$$

Hence, from (2.2.2), we get

$$\psi(d(v_1,v_2)) \leq \psi(d(v_1,v_2)) - \varphi(d(v_1,v_2)) < \psi(d(v_1,v_2)),$$

which is a contradiction. Hence we must have $v_1 = v_2$.

Lemma 2.3: Let f, g, S and T be selfmaps of a metric space X satisfying $(\psi,\varphi)-$weak quasi contraction. If each of the pairs (f,S) and (g,T) is owc then each of the pairs (f,S) and (g,T) is weakly compatible provided $C(f,S) \neq \emptyset$ and $C(g,T) \neq \emptyset$.

Proof. If $C(f,S)$ is singleton and $C(g,T)$ is singleton then the conclusion of the lemma follows clearly. Hence with out loss of generality we assume that $C(f,S)$ and $C(g,T)$ contain more than one element. By hypotheses there exist $u \in C(f,S)$ and $v \in C(g,T)$ such that

$$fSu = Sfu \;\text{ and }\; gTv = Tgv. \qquad (2.3.1)$$

We write $fu = Su = u_1$ (say) and $gv = Tv = u_2$. (2.3.2)

We first show that $u_1 = u_2$. Suppose $u_1 \neq u_2$. Now

$$\psi(d(u_1,u_2)) = \psi(d(fu,gv)) \leq \psi(M(u,v)) - \varphi(M(u,v)) \qquad (2.3.3)$$

where

$$M(u,v) = \max\{d(Su,Tv), d(fu,Su), d(gv,Tv), d(fu,Tv), d(Su,gv)\}.$$
$$= \max\{d(u_1,u_2), 0, 0, d(u_1,u_2), d(u_1,u_2)\}.$$
$$= d(u_1,u_2).$$

Hence, from (2.3.3), we get

$$\psi(d(u_1,u_2)) \leq \psi(d(u_1,u_2)) - \varphi(d(u_1,u_2)),$$

which is a contradiction. Hence $u_1 = u_2$. Therefore from (2.3.2), we get

$$fu = Su = gv = Tv = w \text{ (say)}. \qquad (2.3.4)$$

We now show that each pair (f,S) and (g,T) is weakly compatible.

Let $u' \in C(f,S)$ and $v' \in C(g,T)$.

Therefore,

$$fu' = Su' = w_1 \text{ (say) and } gv' = Tv' = w_2 \text{ (say)}. \qquad (2.3.5)$$

We show that $w_1 = w = w_2$. Suppose $w_1 \neq w_2$. Now

$$\psi(d(w_1,w)) = \psi)(d(fu',gv) \leq \psi(M(u',v)) - \varphi(M(u',v)) \qquad (2.3.6)$$

where

$$M(u',\ x) = \max\{d(Su',Tv), d(fu',Su'), d(gx,Tv), d(fu',Tv), d(Su',gv)\}$$
$$= \max\{d(w_1,w), 0, 0,\ d(w_1,w),\ d(w_1,w)\}.$$
$$= d(w_1,w).$$

Hence from (2.3.6) we get

$$\psi(d(w_1,w)) \leq \psi(d(w_1,w)) - \varphi(d(w_1,w)),$$

a contradiction. Hence $w_1 = w$. Similarly, we can prove that $w_2 = w$. Hence we have $w_1 = w = w_2$.

Hence from (2.3.5) we get,

$$fu' = Su' = gv' = Tv' = w. \qquad (2.3.7)$$

Now using (2.3.7), (2.3.4) and (2.3.1) we get

$fSu' = fw = fSu = Sfu = Sw = Sfu'$ and

$gTv' = gw = gTw = Tgw = Tw = Tgv'$.

Hence the pairs of maps (f,S) and (g,T) are weakly compatible.

The conclusion of the Lemma 2.3 fails to hold if we relax the condition 'f, g, S and T satisfy (ψ,φ) weak quasi contraction.

Example 2.4: Let $X = [0,1]$ with the usual metric. We define f, g, S and T on X by $f(x) = \frac{x}{2}$, $g(x) = \frac{x}{3}$,

$$Sx = \begin{cases} x, & if\ 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & if\ \frac{1}{2} \leq x < 1 \end{cases} \quad \text{and}$$

$$Tx = \begin{cases} x, & if\ 0 \leq x < \frac{1}{2} \\ \frac{1}{3}, & if\ \frac{1}{2} \leq x < 1. \end{cases}$$

Clearly each of the pair (f,S) and (g,T) is *owc*. f, g, S and T fails to satisfy '$(\psi,\varphi)-$weak quasi contraction'. For, at $x = 0,\ y = 1$ we have $d(fx,gy) = \frac{1}{3}$ and $M(x,y) = \frac{1}{3}$ so that $\psi(d(fx,Ty)) = \psi(\frac{1}{3})$ and $\psi(M(x,y)) - \varphi(M(x,y)) = \psi(\frac{1}{3}) - \varphi(\frac{1}{3})$. Hence f, g, S and T fails to satisfy '$(\psi,\varphi)-$weak quasi contraction' for any $\psi \in \Psi$ and $\varphi \in \Phi$. We observe that the pairs (f,S) and $(g,T)$are not weakly compatible because, we have $f1 = S1 = \frac{1}{2}$ but $Sf1 \neq fS1$ and $g1 = T1 = \frac{1}{3}$ but $gT1 \neq Tg1$.

Since every pair of weakly compatible maps is *owc* and in view of Lemma 2.3, we can state the following proposition.

Proposition 2.5: Let f, g, S and T be selfmaps of a metric space X satisfying $(\psi,\varphi)-$weak quasi contraction. Assume that $C(f,S) \neq \emptyset$ and $C(g,T) \neq \emptyset$. Then the pairs (f,S) and (g,T) are weakly compatible if and only if they are *owc*.

Lemma 2.6: Let f, g, S and T be selfmaps of a metric space X satisfying $(\psi,\varphi)-$weak quasi contraction. If f, g, S and T have a point of coincidence v and if each of the pairs (f,S) and (g,T) is *owc* then v is a unique common fixed point of f, g, S and T.

Proof. Since v is a point of coincidence of $f,\ g,\ S$ and T there exists u in X such that $fu = Su = gu = Tu = v$. This v is unique by Lemma 2.2.

From Proposition 2.5 each of the pairs (f,S) and (g,T) are weakly compatible. Hence

$fv = fSu = Sfu = Sv = u_1$(say) and $gv = gTu = Tgu = Tv = u_2$ (say). (2.6.1)

We show that $u_1 = v = u_2$. Suppose $u_1 \neq v$. Then

$$\begin{aligned} \psi(d(u_1,v)) &= \psi(d(fv,gu)) \\ &\leq \psi(M(v,u)) - \varphi(M(v,u)) \qquad (2.6.2) \end{aligned}$$

where

$$\begin{aligned} M(v,u) &= \max\{d(Sv,gu), d(fv,Sv), d(gu,Tu), d(fv,Tu), d(Sv,gu)\}. \\ &= \max\{d(u_1,v), 0, 0, d(u_1,v), d(u_1,v)\}. \\ &= d(u_1,v). \end{aligned}$$

Hence, from, (2.6.2) we get

$$\begin{aligned} \psi(d(u_1,v)) &\leq \psi(d(u_1,v)) - \varphi(d(u_1,v)) \\ &< \psi(d(u_1,v)), \end{aligned}$$

which is a contradiction. Hence $u_1 = u_2$. Similarly we can prove that $u_2 = v$. Thus $u_1 = u_2 = v$. Therefore from (2.6.1) we get $fv = Sv = gv = Tv = v$, so that v is a common fixed point of f, g, S and T. Uniqueness of fixed point follows from Lemma 2.2.

Remark 2.7: Theorem 2.6 extends Lemma 2.1 to four selfmaps. Infact, in Lemma 2.1, the maps need not satisfy any contraction inequality to obtain common fixed points. But the same is not true for four selfmaps, unless these maps satisfy '$(\psi,\varphi)-$weak quasi contraction'. Thus, if we relax the condition 'f, g, S and T satisfy $(\psi,\varphi)-$weak quasi contraction' in Theorem 2.6 then f, g, S and T may not have a common fixed point.

Example 2.8: Let $X = [0,1]$ with the usual metric. We define selfmaps f, g, S and T on X by

$$fx = \begin{cases} 2x, & if\ 0 \leq x < \frac{1}{2} \\ \frac{x}{2}, & if\ \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$gx = \begin{cases} x^2, & if\ 0 \leq x < \frac{1}{2} \\ 1, & if\ \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$Sx = \begin{cases} \frac{x}{2}, & if\ 0 \leq x < \frac{1}{2} \\ 1, & if\ \frac{1}{2} \leq x \leq 1 \end{cases} \qquad \text{and}$$

$$Tx = \begin{cases} x + \frac{1}{2}, & if\ 0 \leq x < \frac{1}{2} \\ x, & if\ \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly each of the pairs (f,S) and (g,T) are *owc*. Since $f\frac{1}{2} = S\frac{1}{2} = g\frac{1}{2} = T\frac{1}{2} = 1$, 1 is a point of coincidence of f, g, S and T. f, g, S and T fails to satisfy the inequality (1.7.1) at $x = 0,\ y = \frac{1}{2}$ for any $\psi \in \Psi,\ \varphi \in \Phi$, for at $x = 0,\ y = 1$ we have $d(fx,gy) = 1$ and $M(x,y) = 1$. Thus f, g, S and T satisfy all the hypotheses of Theorem 2.6 except the inequality (1.7.1), and we observe that f, g, S and T have no common fixed points.

III. Main Results

Theorem 3.1: Let $f,\ g,\ S$ and T be selfmaps of a metric space X satisfying $(\psi,\varphi)-$weak quasi contraction', where φ satisfies the condition 'for any sequence $\{t_n\}$ of real numbers, $\lim_{n\to\infty} \varphi(t_n) = 0$ implies $\lim_{n\to\infty} t_n = 0$'. Suppose that either

(i) the pair (f,S) satisfies property *(E.A)*, $(f(X) \subseteq T(X))$ / $(S(X) \subseteq T(X))$ and $S(X)$ is closed

or

(ii) the pair (g,T) satisfies property *(E.A)*, $(g(X) \subseteq S(X))$/ $(T(X) \subseteq S(X))$ and $T(X)$ is closed

holds. If both the pairs (f,S) and (g,T) are *owc* maps on X then f, g, S and T have a unique common fixed point in X.

Proof. We first suppose that (i) holds, with $f(X) \subseteq T(X)$. Since (f,S) satisfies property (E.A), there exist sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z,\ z \in X.$

Since $f(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $fx_n = Ty_n$ for each $n = 1, 2, 3, \ldots$.

Hence, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = t$. We show that $\lim_{n\to\infty} gy_n = z$.

By taking $x = x_n$ and $y = y_n$ for each $n = 1,\ 2,\ 3, \ldots$ in (1.7.1), we get

$\psi(d(fx_n,\ gy_n)) \leq \psi(M(x_n,\ y_n)) - \varphi(M(x_n,\ y_n))$ (3.1.1)

where

$$\begin{aligned} M(x_n,y_n) = \max\{&d(Sx_n,Ty_n),\ d(fx_n,Sx_n),\ d(gy_n,Ty_n), \\ &d(fx_n,Ty_n),\ d(Sx_n,gy_n)\}. \end{aligned}$$

Thus,

$$\begin{aligned} \liminf_{n\to\infty} M(x_n,y_n) &= \max\{\liminf_{n\to\infty} d(Sx_n,Ty_n),\ \liminf_{n\to\infty} d(fx_n,Sx_n), \\ &\quad \liminf_{n\to\infty} d(gy_n,\ Ty_n), \liminf_{n\to\infty} d(fx_n,Ty_n), \\ &\quad \liminf_{n\to\infty} d(Sx_n,gy_n)\} \\ &= \liminf_{n\to\infty} d(z,\ gy_n). \end{aligned}$$

Now taking limit infimum as $n \to \infty$ in (3.1.1), we get

$\liminf_{n\to\infty} \psi(d(fx_n,gy_n))$

$$\leq \liminf_{n\to\infty} \psi(M(x_n, y_n)) - \limsup_{n\to\infty} \varphi(M(x_n, y_n)).$$

ie., $\liminf_{n\to\infty} \psi(d(z, gy_n))$

$$\leq \liminf_{n\to\infty} \psi(d(z, gy_n)) - \limsup_{n\to\infty} \varphi(M(x_n, y_n))$$

Hence $\limsup_{n\to\infty} \phi(M(x_n,\ y_n)) = 0$.

Therefore

$$\limsup_{n\to\infty} \phi(M(x_n,\ y_n)) = \liminf_{n\to\infty} \varphi(M(x_n,\ y_n)) = 0,$$

which shows that $\lim_{n\to\infty} \varphi(M(x_n, y_n))$ exists and $\lim_{n\to\infty} \varphi(M(x_n, y_n)) = 0$.

Hence $\varphi(\lim_{n\to\infty} M(x_n, y_n)) = 0$ and so $\lim_{n\to\infty} d(z, gy_n) = 0$.

Therefore, $\lim_{n\to\infty} gy_n = z$. Hence

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = z$.

Now, since $S(X)$ is closed, there exists u in X such that $z = Su$.

We now show that $fu = z$. Suppose $fu \neq z$.

Now by taking $x = u$ and $y = y_n$ for each $n = 1, 2, 3, ...$ in (1.7.1), we get

$$\psi(d(fu, gy_n)) \;\leq\; \psi(M(u, y_n)) - \varphi(M(u,_n y)) \qquad (3.1.2)$$

where,

$$M(u, y_n) = \max\{d(Su, Ty_n), d(fu, Su), d(gy_n, Ty_n), d(fu, Ty_n), d(Su, gy_n)\}.$$

Hence,

$$\limsup_{n\to\infty} M(u, y_n) = \max\{\limsup_{n\to\infty} d(Su, Ty_n), d(fu, Su), \limsup_{n\to\infty} d(gy_n, Ty_n), \limsup_{n\to\infty} d(fu, Ty_n), \limsup_{n\to\infty} d(Su, gy_n)\}$$
$$= \max\{0, d(fu, z), 0, d(fu, z), 0\}.$$
$$= d(fu, z).$$

Now on taking limit supremum as $n \to \infty$ in (3.1.2), we get

$$\limsup_{n\to\infty} \psi(d(fu, gy_n)) \leq \limsup_{n\to\infty} \psi(M(u, y_n))) - \liminf_{n\to\infty} \varphi(M(u, y_n)),$$
$$\psi(d(fu, z)) \;\leq\; \psi(d(fu, z)) - \varphi(d(fu, z)),$$

a contradiction. Therefore, $fu = z$.

Since $f(X) \subseteq T(X)$, there exists v in X such that $fu = Tv = z$.

We now show that $gv = z$. Suppose $gv \neq z$. Now,

$$\psi(d(fx_n, gv)) \;\leq\; \psi(M(x_n, v)) - \varphi(M(x_n, v)) \qquad (3.1.3)$$

where

$$M(x_n, v) = \max\{d(Sx_n, Tv), d(fx_n, Sx_n), d(gv, Tv), d(fx_n, Tv), d(Sx_n, gv)\}.$$

Now,

$$\limsup_{n\to\infty} M(x_n, v) = \max\{\limsup_{n\to\infty} d(Sx_n, Tv), \limsup_{n\to\infty} d(fx_n, Sx_n), d(gv, Tv), \limsup_{n\to\infty} d(fx_n, Tv), \limsup_{n\to\infty} d(Sx_n, gv)\}$$
$$= \max\{0, 0, d(gv, z), 0, d(z, gv)\}.$$
$$= d(z, gv).$$

On taking limit supremum as $n \to \infty$ in (3.1.3), we get

$$\limsup_{n\to\infty} \psi(d(fx_n, gv)) \leq \limsup_{n\to\infty} \psi(M(x_n, v)) - \liminf_{n\to\infty} \varphi(M(x_n, v)),$$

which implies,

$$\psi(d(z, gv)) \;\leq\; \psi(d(z, gv)) - \varphi(d(z, gv)), < \psi(d(z, gv))$$

a contradiction. Therefore, $gv = z$.

Therefore $fu = Su = gv = Tv = z$. (3.1.4)

Hence $C(f, S) \neq \emptyset$ and $C(g, T) \neq \emptyset$. (3.1.5)

Since each pair (f, S) and (g, T) is *owc*, there exists $u' \in C(f, S)$ and $v' \in C(g, T)$ such that

$fSu' = Sfu' = w_1$ (say) and $gTv' = Tgv' = w_2$ (say). (3.1.6)

Since, $u' \in C(f, S)$ and $v' \in C(g, T)$, we have

$fu' = Su' = u_1$ (say) and $gv' = Tv' = u_2$ (say). (3.1.7)

We now show that $w_1 = z = w_2$. Suppose $w_1 \neq z$. Now using (3.1.4), (3.1.6) and (3.1.7) we get

$$\psi(d(w_1, z)) = \psi(d(fu_1, gu)) \leq \psi(M(u_1, u)) - \varphi(M(u_1, u)), \qquad (3.1.8)$$

where

$$M(u_1, u) = \max\{d(Su_1, Tu), d(fu_1, Su_1), d(gu, Tu), d(fu_1, Tu), d(Su_1, gu)\}.$$
$$= \max\{d(w_1, z), 0, 0, d(w_1, z), d(w_1, z)\}.$$
$$= d(w_1, z).$$

Thus from (3.1.8) we get

$$\psi(d(w_1, z)) \;\leq\; \psi(d(w_1, z)) - \varphi(d(w_1, z)),$$

a contradiction. Hence $w_1 = z$. Similarly we can show that $w_2 = z$. Hence $w_1 = z = w_2$. Therefore from (3.1.6) and (3.1.7), we get $fu_1 = Su_1 = gu_2 = Tu_2 = z$. Finally we show that $u_1 = u_2$. Suppose $u_1 \neq u_2$. Now,

$$\psi(d(u_1, u_2)) = \psi(d(fu', gv')) \leq \psi(M(u', v')) - \varphi(M(u', v')) \qquad (3.1.9)$$

where

$$M(u', v') = \max\{d(Su', Tv'), d(fu', Su'), d(gv', Tv'), d(fu', Tv'), d(Su', gv')\}.$$
$$= \max\{d(u_1, u_2), 0, 0, d(u_1, u_2), d(u_1, u_2)\}.$$
$$= d(u_1, u_2).$$

Hence from (3.1.9) we get

$$\psi(d(u_1, u_2)) \;\leq\; \psi(d(u_1, u_2)) - \varphi(d(u_1, u_2)), < \psi(d(u_1, u_2))$$

which is a contradiction. Hence $u_1 = u_2$ so that $fu_1 = Su_1 = gu_1 = Tu_1 = z$.

Hence z is a point of coincidence of f, g, S and T. Hence from Theorem 2.6, z is a unique common fixed point of f, g, S and T.

Now if we assume that $S(X) \subseteq T(X)$ then on the similar lines of the above proof we obtain the conclusion of the theorem. The proof is similar when (ii) holds.

Theorem 3.2: Let $f,\ g,\ S$ and T be selfmaps of a metric space X satisfying $(\psi,\varphi)-$weak quasi contraction'. Suppose that

(i) (f,S) and (g,T) satisfy common property (E.A.).

(ii) (f,S) and (g,T) are owc.

If $f(X)$ and $g(X)$ are closed in X then f,g,S and T have a unique common fixed point in X.

Proof. Since the pairs (f,S) and (g,T) satisfy common property (E.A), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X and t in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t.$$

Since $S(X)$ and $T(X)$ are closed, there exist u,v in X such that $Su = Tv = z$.

We now show that $fu = gv = z$. Suppose $fu \neq z$.

By taking $x = u$ and $y = y_n$ for each $n = 1,\ 2,\ 3,...$ in (1.7.1), we get

$$\psi(d(fu,gy_n)) \ \leq\ \psi(M(u,y_n)) - \varphi(M(u,y_n)) \qquad (3.2.1)$$

where

$$M(u,y_n) = \max\{d(Su,Ty_n), d(fu,Su), d(gy_n,Ty_n), d(fu,Ty_n), d(Su,gy_n)\}.$$

Now,

$$\begin{aligned}\limsup_{n\to\infty} M(u,y_n) &= \max\{\limsup_{n\to\infty} d(Su,Ty_n), d(fu,Su), \limsup_{n\to\infty} d(gy_n,Ty_n), \limsup_{n\to\infty} d(fu,Ty_n), \limsup_{n\to\infty} d(Su,gy_n)\}\\ &= \max\{d(Su,z), d(Su,z), 0, d(fu,z), d(Su,z)\}.\\ &= d(Su,z).\end{aligned}$$

Now, by taking limit supremum as $n\to\infty$ in (3.2.1) we get

$$\limsup_{n\to\infty} d(fu,gy_n) \leq \limsup_{n\to\infty} M(u,y_n) - \liminf_{n\to\infty} \varphi(M(u,y_n)),$$

which implies

$$\psi(d(fu,z)) \ \leq\ \psi(d(fu,z)) - \varphi(d(fu,z)),$$

a contradiction. Therefore, $fu = z$. Now suppose $gv \neq z$.

We now consider

$$\psi(d(fx_n,gv)) \ \leq\ \psi(M(x_n,v)) - \varphi(M(x_n,v)) \quad (3.2.2)$$

where

$$M(x_n,v) = \max\{d(Sx_n,Tv), d(fx_n,Sx_n), d(gv,Tv), d(fx_n,Tv), d(Sx_n,gv)\}.$$

Now,

$$\begin{aligned}\limsup_{n\to\infty} M(x_n,v) &= \max\{\limsup_{n\to\infty} d(fx_n,gv), \limsup_{n\to\infty} d(fx_n,Sx_n), d(gv,Tv), \limsup_{n\to\infty} d(fx_n,Tv), \limsup_{n\to\infty} d(Sx_n,gv)\}\\ &= \max\{d(z,Tv), 0, 0, d(z,Tv), d(z,gv)\}.\\ &= d(z,gv).\end{aligned}$$

Hence, from (3.2.2), we get

$$\begin{aligned}\limsup_{n\to\infty} d(fx_n,gv) &\leq \limsup_{n\to\infty} M(x_n,v) - \liminf_{n\to\infty} \varphi(M(x_n,v)),\\ &\leq \psi(d(z,gv)) - \varphi(d(z,gv)).\end{aligned}$$

Hence,

$$\begin{aligned}\psi(d(z,gv)) &\leq \psi(d(z,gv)) - \varphi(d(z,gv)),\\ &< \psi(d(z,gv)),\end{aligned}$$

a contradiction. Hence, $gv = z$. Therefore $fu = Su = gv = Tv = z$. Hence $C(f,S) \neq \emptyset$ and $C(g,T) \neq \emptyset$. Hence (3.1.8) of Theorem 3.1 holds and the remaining proof is same as that of Theorem 3.1 and the conclusion of the theorem follows.

Corollary 3.3: Let f,g,S and T be selfmaps of a metric space X satisfying generalized $(\psi,\varphi)-$weak quasi contraction, such that $\psi-\varphi$ is non-decreasing. Suppose that either

(i) the pair (f,S) satisfies property (E.A.), $(f(X) \subseteq T(X))$ / $(S(X) \subseteq T(X))$ and $S(X)$ is closed

or

(ii) the pair (g,T) satisfies property (E.A.), $(g(X) \subseteq S(X))$/ $(T(X) \subseteq S(X))$ and $T(X)$ is closed

holds. If both the pairs (f,S) and (g,T) are *owc* maps on X then f,g,S and T have a unique common fixed point in X.

Proof. Since $\psi-\varphi$ is non-decreasing and f,g,S and T satisfy 'generalized $(\psi,\varphi)-$weak quasi contraction', f,g,S and T satisfy '$(\psi,\varphi)-$weak quasi contraction'. Hence the conclusion of the corollary follows from Theorem 3.1.

Corollary 3.4: Let f,g,S and T be selfmaps of a metric space (X,d). Suppose that there exists $\varphi \in \Phi$ such that

$$d(fx,gy) \leq M(x,y) - \varphi(M(x,y))$$

for each $x,y \in X$. Suppose that either

(i) the pair (f,S) satisfies property (E.A.), $(f(X) \subseteq T(X))$ / $(S(X) \subseteq T(X))$ and $S(X)$ is closed

or

(ii) the pair (g,T) satisfies property (E.A.), $(g(X) \subseteq S(X))$/ $(T(X) \subseteq S(X))$ and $T(X)$ is closed

holds. If both the pairs (f,S) and (g,T) are *owc* maps on X then f,g,S and T have a unique common fixed point in X.

Proof. Follows from Theorem 3.1 by choosing $\psi(t) = t$, for each $t \geq 0$.

Corollary 3.5: Let (X,d) be a metric space. Suppose that f,g,S and T be selfmaps of X satisfying

$$d(fx,gy) \leq a_1 d(Sx,Ty) + a_2 d(fx,Sx) + a_3 d(gy,Ty) + a_4 d(fx,Ty) + a_5 d(Sx,gy) \qquad (3.5.1)$$

for all x,y in X with $a_1+a_2+a_3+a_4+a_5 < 1$. Suppose that either

(i) the pair (f,S) satisfies property *(E.A)*, $(f(X) \subseteq T(X))$ / $(S(X) \subseteq T(X))$ and $S(X)$ is closed

or

(ii) the pair (g,T) satisfies property *(E.A)*, $(g(X) \subseteq S(X))$/ $(T(X) \subseteq S(X))$ and $T(X)$ is closed

holds. If both the pairs (f,S) and (g,T) are owc maps on X then f,g,S and T have a unique common fixed point in X.

Proof. From (3.5.1), we have

$$\begin{aligned} d(fx, gy) &\le a_1 d(Sx, Ty) + a_2 d(fx, Sx) + a_3 d(gy, Ty) \\ &\quad + a_4 d(fx, Ty) + a_5 d(Sx, gy) \\ &\le (a_1 + a_2 + a_3 + a_4 + a_5) \max\{d(Sx, Ty), d(fx, Sx), \\ &\qquad d(gy, Ty), d(fx, Ty), d(Sx, gy)\} \\ &= hM(x, y), \end{aligned}$$

where $h = a_1 + a_2 + a_3 + a_4 + a_5 < 1$. (3.5.2)

We define $\varphi : R^+ \to R^+$ by $\varphi(t) = (1 - h)t,\ t \ge 0$. Then $\varphi \in \Phi$.

Then from (3.5.2) we get

$$d(fx, gy) \le M(x, y) - \varphi(M(x, y)).$$

Now, the conclusion of the theorem follows from Corollary 3.4.

Corollary 3.6: Let f and g be selfmaps of a metric space X. Assume that there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi(d(fx, fy)) \le \psi(M(x, y)) - \varphi(M(x, y)) \qquad (3.6.1)$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}.$$

Suppose that the pair (f, g) satisfies property (E.A.). If $g(X)$ is closed and f and g are *owc* maps then f and g have a unique common fixed point in X.

Proof. Follows from Theorem 3.1 by choosing $g = f$ and $S = T = g$.

By choosing $\psi(t) = t,\ t \ge 0$ in (3.6.1) we get the following corollary from Corollary 3.6.

Corollary 3.7: Let f and g be selfmaps of a metric space X. Assume that there exists $\varphi \in \Phi$ such that

$$d(fx, fy) \le M(x, y) - \varphi(M(x, y)) \qquad (3.7.1)$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}.$$

Suppose that the pair (f, g) satisfies property (E.A.). If $g(X)$ is closed and f and g are *owc* maps then f and g have a unique common fixed point in X.

Remark 3.8: If $t - \varphi(t)$ is non-decreasing then Theorem 1.12 follows as a corollary to Corollary 3.7, since the inequality (1.12.1) implies the inequality (3.7.1).

The following is an example in support of Theorem 3.1.

Example 3.9: Let $X = [1, 5)$ with the usual metric. We define selfmaps f, g, S and T on X by

$$fx = \begin{cases} \frac{3}{2}, & if\ 1 \le x < 2 \\ \frac{3}{2} + \frac{x}{4}, & if\ \ 2 \le x < 5, \end{cases}$$

$$gx = \begin{cases} \frac{5}{2}, & if\ 1 \le x < 2 \\ 1 + \frac{x}{2}, & if\ 2 \le x < 5. \end{cases}$$

$$Sx = \begin{cases} 4, & if\ 1 \le x < 2 \\ 1 + \frac{x}{2}, & if\ 2 \le x < 5, \end{cases} \quad \text{and}$$

$$Tx = \begin{cases} \frac{17}{4}, & if 1 \le x < 2 \\ \frac{1}{2} + \frac{3}{4}x, & if\ 2 \le x < 5. \end{cases}$$

Here $f(X) = [2, \frac{11}{4}),\ g(X) = [2, \frac{7}{2}),\ S(X) = [2, \frac{7}{2}) \cup \{4\}$ and $T(X) = [2, \frac{17}{4})$, so that $g(X) \subset S(X)$ and $T(X)$ is closed. The pair (g, T) satisfies property (E.A) with the sequence $\{x_n\},\ x_n = 2 + \frac{1}{n},\ n = 1,\ 2,\ 3, \ldots$. Each of the pair (f, S) is (g, T) are *owc*.

Now we define $\psi,\ \varphi : R^+ \to R^+$ by

$$\psi(t) = \begin{cases} 1 + \frac{t}{2}, & if\ 0 \le t \le 3 \\ \frac{3}{t^2} + \frac{13}{6}, & if\ t > 3. \end{cases} \quad \text{and}$$

$$\varphi(t) = \begin{cases} \frac{t}{10}, & if\ 0 \le t < 2 \\ \frac{t}{40}, & if\ t \ge 2. \end{cases}$$

Then $\psi \in \Psi$ and $\varphi \in \Phi$. We show that f, g, S and T satisfy '$(\psi, \varphi)-$weak quasi contraction'.

Case(i): $1 \le x < 2,\ 1 \le y < 2$

In this case $d(fx, gy) = 1,\ M(x, y) = \frac{11}{4}$.

Hence

$$\begin{aligned} \psi(d(fx, gy)) = \frac{3}{2} \le \frac{369}{160} &= \frac{19}{8} - \frac{11}{160} \\ &= \psi(M(x, y)) - \varphi(M(x, y)). \end{aligned}$$

Hence inequality (1.7.1) holds.

Case(ii): $1 \le x < 2,\ 2 \le y < 5$

$d(fx, gy) = \frac{y}{2} - \frac{1}{2}$ and $\psi(d(fx, gy)) = \frac{y}{4} + \frac{3}{4}$.

$$M(x, y) = \begin{cases} \frac{5}{2}, & if\ 2 \le y < \frac{14}{3} \\ \frac{3}{4}y - 1, & if\ \frac{14}{3} \le y < 5 \end{cases}$$

$$\psi(M(x, y)) = \begin{cases} \frac{9}{4}, & if\ 2 \le y < \frac{14}{3} \\ \frac{3}{8}y + \frac{1}{2}, & if\ \frac{14}{3} \le y < 5. \end{cases}$$

$$\varphi(M(x, y)) = \begin{cases} \frac{1}{16}, & if\ 2 \le y < \frac{14}{3} \\ \frac{3}{160}y - \frac{1}{40}, & if\ \frac{14}{3} \le y < 5 \end{cases}$$

Now it is easy to verify that the inequality (1.7.1) holds.

Case(iii): $2 \le x < 5\ , 1 \le y < 2$

$$\text{In this case } d(fx, gy) = \begin{cases} 1 - \frac{x}{4}, & if\ 2 \le x < 4 \\ \frac{x}{4}y - 1, & if\ 4 \le x < 5. \end{cases}$$

so that

$$\psi(d(fx, gy)) = \begin{cases} \frac{3}{2} - \frac{x}{8}, & if\ 2 \le x < 4 \\ \frac{1}{2} + \frac{x}{8}, & if\ 4 \le x < 5. \end{cases}$$

$$M(x, y) = \begin{cases} \frac{11}{4} - \frac{x}{4}, & if\ 2 \le y < 4 \\ \frac{7}{4}, & if\ 4 \le y < 5. \end{cases} \quad \text{so that}$$

$$\psi(M(x, y)) = \begin{cases} \frac{19}{8} - \frac{x}{8}, & if\ 2 \le y < 4 \\ \frac{15}{8}, & if\ 4 \le y < 5. \end{cases}$$

$$\varphi(M(x,y)) = \begin{cases} \frac{11}{160} - \frac{x}{160}, & if\ 2 \le y < 3 \\ \frac{11}{40}y - \frac{x}{40}, & if\ 3 \le y < 4 \\ \frac{7}{40}, & if\ 4 \le x < 5 \end{cases}$$

Now it is easy to verify that the inequality (1.7.1) holds.

Case(iv): $2 \le x < 5,\ 2 \le y < 5$

In this case $d(fx, gy) = |\frac{x}{4} + \frac{1}{2} - \frac{y}{2}|$.

For $x > y$,

$M(x,y) = \max\{|\frac{1}{2} + \frac{x}{2} - \frac{3}{4}y|,\ \frac{x}{4} - \frac{1}{2},\ \frac{y}{4} - \frac{1}{2},\ |1 + \frac{x}{4} - \frac{3}{4}y|\}$

$$= \begin{cases} \max\{\frac{x}{2} - \frac{y}{2},\ \frac{x}{4} - \frac{1}{2}\}, \\ \quad if\ \frac{3}{4}y < 1 + \frac{x}{4} < \frac{1}{2} + \frac{x}{2} \\ \quad or\ 1 + \frac{x}{4} < \frac{3}{4}y < \frac{1}{2} + \frac{x}{2} \\ \frac{3}{4}y - 1 - \frac{x}{4}, \\ \quad if\ 1 + \frac{x}{4} < \frac{1}{2} + \frac{x}{2} \le \frac{3}{4}y \end{cases}$$

so that

$\psi(M(x,y))$

$$= \begin{cases} \max\{1 + \frac{x}{4} - \frac{y}{4},\ \frac{x}{8} + \frac{3}{4}\}, \\ \quad if\ \frac{3}{4}y < 1 + \frac{x}{4} < \frac{1}{2} + \frac{x}{2} \\ \quad or 1 + \frac{x}{4} < \frac{3}{4}y < \frac{1}{2} + \frac{x}{2} \\ \frac{3}{8}y - \frac{x}{4} + \frac{1}{2}, \\ \quad if\ 1 + \frac{x}{4} < \frac{1}{2} + \frac{x}{2} \le \frac{3}{4}y \end{cases}$$

$\varphi(M(x,y))$

$$= \begin{cases} \max\{\frac{x}{20} - \frac{y}{20},\ \frac{x}{40} - \frac{1}{20}\}, \\ \quad if\ \frac{3}{4}y < 1 + \frac{x}{4} < \frac{1}{2} + \frac{x}{2} \\ \quad or\ 1 + \frac{x}{4} < \frac{3}{4}y < \frac{1}{2} + \frac{x}{2} \\ \frac{3}{80}y - \frac{x}{40} + \frac{1}{20}, \\ \quad if\ 1 + \frac{x}{4} < \frac{1}{2} + \frac{x}{2} \le \frac{3}{4}y \end{cases}$$

For $x \le y,\ M(x,y) = \frac{3}{4}y - 1 - \frac{x}{4}$ so that

$\psi(M(x,y)) = \frac{3}{8}y - \frac{x}{4} + \frac{1}{2}$ and $\varphi(M(x,y)) = \frac{3}{80}y - \frac{x}{40} + \frac{1}{20}$.

In this case also, it is not difficult to verify the inequality (1.7.1).

Hence from all the above cases, we conclude that f, g, S and T satisfy '$(\psi, \varphi)-$weak quasi contraction' so that all the hypotheses of Theorem 3.1 hold and 2 is the unique common fixed point f, g, S and T.

We now give an example in support of Theorem 3.2.

Example 3.10: Let $X = [0, 1)$ with the usual metric. We define selfmaps f, g, S and T on X by

$$fx = \begin{cases} \frac{3}{4}, & if\ 0 \le x < \frac{1}{2} \\ x, & if\ \frac{1}{2} \le x < 1, \end{cases}$$

$$gx = \begin{cases} \frac{1+x}{2}, & if\ 0 \le x < \frac{1}{2} \\ \frac{1}{2}, & if\ \frac{1}{2} \le x < 1, \end{cases}$$

$$Sx = \begin{cases} \frac{1}{4}, & if\ 0 \le x < \frac{1}{2} \\ \frac{3}{4} - \frac{x}{2}, & if\ \frac{1}{2} \le x < 1, \end{cases}$$ and

$$Tx = \begin{cases} 0, & if\ 0 \le x < \frac{1}{2} \\ 1 - x, & if\ \frac{1}{2} \le x < 1. \end{cases}$$

Here $f(X) = [\frac{1}{2}, 1),\ g(X) = [\frac{1}{2}, \frac{3}{4}),\ S(X) = [\frac{1}{2}, \frac{1}{4}]$ and $T(X) = [0, \frac{1}{2}]$ so that $S(X)$ and $T(X)$ are closed. The pairs (f, S) and (g, T) satisfy common property (E.A) with the sequence $\{x_n\}$ defined by $x_n = \frac{1}{2} + \frac{1}{n+3},\ n = 1, 2, 3, ..\ ...$ Each of the pairs (f, S) and (g, T) are owc.

Now we define $\psi,\ \varphi : R^+ \to R^+$ by

$\psi(t) = \frac{1+t}{2},\ t \ge 0$ and $\varphi(t) = \begin{cases} \frac{t}{8}, & if\ 0 \le t < 2 \\ \frac{t}{6}, & if\ t \ge 2. \end{cases}$

Then $\psi \in \Psi$ and $\varphi \in \Phi$. We now show that f, g, S and T satisfy '$(\psi, \varphi)-$weak quasi contraction'.

Case(i): $0 \le x < \frac{1}{2},\ 0 \le y < \frac{1}{2}$

In this case $d(fx, gy) = \frac{1}{4} - \frac{y}{2},\ M(x,y) = \frac{3}{4}$.

Hence $\psi(d(fx, gy)) = \frac{3}{8} - \frac{y}{4},\ \psi(M(x,y)) = \frac{11}{16}$ and $\varphi(M(x,y)) = \frac{3}{24}$; with these values, clearly the inequality (1.7.1) holds.

Case(ii): $0 \le x < \frac{1}{2},\ \frac{1}{2} \le y < 1$

Here $d(fx, gy) = \frac{1}{4}$, $\psi(d(fx, gy)) = \frac{3}{8}$.

$$M(x,y) = \begin{cases} \frac{1}{2}, & if\ \frac{1}{2} \le y < \frac{3}{4} \\ y - \frac{1}{4}, & if\ \frac{3}{4} \le y < 1. \end{cases}$$ so that

$$\psi(M(x,y)) = \begin{cases} \frac{3}{4}, & if\ \frac{1}{2} \le y < \frac{3}{4} \\ \frac{y}{2} + \frac{3}{8}, & if\ \frac{3}{4} \le y < 1. \end{cases}$$

$$\varphi(M(x,y)) = \begin{cases} \frac{1}{16}, & if\ \frac{1}{2} \le y < \frac{3}{4} \\ \frac{y}{8} - \frac{1}{32}, & if\ \frac{3}{4} \le y < 1. \end{cases}$$

Now it is easy to verify that the inequality (1.7.1) holds.

Case(iii): $\frac{1}{2} \le x < 1,\ 0 \le y < \frac{1}{2}$

In this case $d(fx, gy) = x - \frac{1}{2},\ M(x,y) = x$.

Hence $\psi(d(fx, gy)) = \frac{x}{2} + \frac{1}{4},\ \psi(M(x,y)) = \frac{1+x}{2}$ and

$$\varphi(M(x,y)) = \begin{cases} \frac{x}{8}, & if\ \frac{1}{2} \le y < \frac{3}{4} \\ \frac{x}{6}, & if\ \frac{3}{4} \le y < 1. \end{cases}$$

Hence the inequality (1.7.1) holds.

Case(iv): $\frac{1}{2} \le x < 1,\ \frac{1}{2} \le y < 1$

In this case

$d(fx, gy) = x - \frac{1}{2},\ M(x,y) = \max\{\frac{3}{2}x - 34, x + y - 1\}$.

Hence $\psi(d(fx, gy)) = \frac{x}{2} + \frac{1}{4},\ \psi(M(x,y)) = \frac{1+x}{2}$ so that the inequality (1.7.1) holds for each choice of $M(x,y)$.

Thus from all the above cases we conclude that f, g, S and T satisfy all the hypotheses of Theorem 3.2 and $\frac{1}{2}$ is the unique common fixed point of f, g, S and T.

IV. Conclusion

In this paper we introduced '$(\psi, \varphi)-$weak quasi contraction' for four selfmaps and established the existence of common fixed points of occasionally weakly compatible maps, using property (E.A.)/common property(E.A.) (Theorem 3.1 and Theorem 3.2). Examples are provided in support of the results.

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