Deriving a Formula in Solving Fibonacci-like Sequence

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***Abstract*— Sequences have been fascinating topic for mathematicians for centuries. Inclusion of missing terms in arithmetic, harmonic and geometric sequence has been formulated for a long time. Other sequences like Fibonacci and Lucas sequences could be solved using the Binet's Formula. In this paper, derivation of formula in solving Fibonacci-like sequence, a derivative of Fibonacci sequence, will be shown by finding important patterns from basic formula integrating with Binet's Formula. Finding of missing terms in Fibonacci-like sequence will be answered easily using this formula,**

$$x_1 = \frac{b - \left[\dfrac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}\right] a}{\dfrac{\varphi^{n+1} - (1\varphi)^{n+1}}{\sqrt{5}}}, n \geq 1.$$

***Index Terms*— Binet's formula, Fibonacci sequence, Fibonacci-like Sequence, missing terms**

***MSC 2010 Codes* — 11B39, 11B99**

I. Introduction

Many sequences have been studied for many years now. Arithmetic, geometric, harmonic, Fibonacci, and Lucas sequences have been very well-defined in mathematical journals. On the other hand, Fibonacci-like sequence received little attention from mathematicians since Fibonacci sequence attracts them more.

Fibonacci sequence (1,1,2,3,5,8,13,21….) is a succession of numbers that are obtained through adding the two preceding numbers. A derivative of this sequence is called Fibonacci-like sequence. This sequence has properties like of Fibonacci except that it can start with any two numbers and follows the same pattern of adding them to the next term e.g. 12,32,44,76…. and so forth.

Lucas and Tribonacci sequences are examples of this sequence. Generally, Fibonacci-like sequence can be expressed as

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$$S_n = S_{n-1} + S_{n-2}; n \geq 2. \quad (1.1)$$

Shown by B. Singh, et.al. [1] in which a=2, it can also be written as

$$S_n = \sum_{k=0}^{[n/2]} a \binom{n-k}{k} = a\binom{n}{0} + a\binom{n-1}{1} + \ldots + a\binom{n/2}{n/2}$$

(1.2)

where [x] is defined as the greatest integer less than or equal to x. Various properties of Fibonacci-like sequence have been presented in papers of B. Singh, et.al [1] and other derivatives like that of composite number has been thoroughly explained in works of J. Nicol [2].

Inclusion of missing terms in arithmetic, harmonic, geometric sequence is well-solved in any high school and college mathematics textbooks. Also, finding the n[th] term of Fibonacci sequence can be solved using the Binet's formula given by $F_n = \dfrac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$

(1.3)

where $\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.61803399.....$ also known as the golden ratio. However, to the best of author's knowledge, there are no scholarly works or studies describing a formula on inclusion of missing terms in a Fibonacci-like sequence e.g. insert five consecutive terms between 2 and 50 to make it a Fibonacci-like sequence.

P. Howell [3] presented a proof for finding n[th] term of Fibonacci sequence using vectors and eigenvalues but did not account Fibonacci-like sequence. M. Agnes, *et al*. [4] provided a formula for inclusion of missing terms in Fibonacci-like sequence but only for three consecutive missing terms.

This paper will present how to derive a formula to solve a Fibonacci-like sequence from Binet's formula regardless of the number of consecutive missing terms.

II. Derivation

Before moving to the general formula, it is imperative to observe the specific formula from the basic problems. Basic formula will be studied and from this, a general formula will be deduced. In this derivation, *a* stands for the first term given and *b* stands for the last term given in any Fibonacci-like sequence.

2.1 One Missing Term

Consider the Fibonacci-like sequence $a, \_, b$ wherein there is one term missing denoted by x_1, we could solve the missing term because the next number in Fibonacci sequence is found by adding up the two numbers before it.

In mathematical statement, $a + x_1 = b$. Rearranging terms,

$$x_1 = b - a. \qquad (2.1.1)$$

This basic formula will be used in finding the other formulas.

2.2 Two Consecutive Missing Terms

Examining the Fibonacci-like sequence where a, x_1, x_2, b where x_1 is the first missing term and x_2 is the second missing term, we could solve the formula.

From (2.1.1),

$$x_1 = x_2 - a \qquad (2.2.1)$$

and

$$x_2 = b - x_1 \qquad (2.2.2)$$

Substituting (2.2.1) in (2.2.2) to find x_2,

$$x_2 = b - x_1 = b - (x_2 - a)$$

or

$$x_2 = \frac{b+a}{2} \qquad (2.2.3)$$

Substituting above in (2.2.1), we obtain

$$x_1 = \frac{b+a}{2} - a = \frac{b-a}{2} \qquad (2.2.4)$$

2.3 Three Consecutive Missing Terms

It is worthy to note that as the number of consecutive missing terms increases, finding the solution is getting complicated. Equation (2.1.1), (2.2.3), and (2.2.4) will be used to find the solution for this type of problem.

Assuming a Fibonacci-like sequence a, x_1, x_2, x_3, b from (2.1.1), (2.2.3), and (2.2.4),

$$x_1 = x_2 - a \qquad (2.3.1)$$

$$x_2 = \frac{x_3 + a}{2} \qquad (2.3.2)$$

$$x_3 = b - x_2. \qquad (2.3.3)$$

Substituting (2.3.2) in (2.3.3) to find x_3,

$$x_3 = b - x_2 = b - \left(\frac{x_3 + a}{2}\right)$$

or

$$x_3 = \frac{2b-a}{3} \qquad (2.3.4)$$

Rearranging terms in (2.3.3), we could solve x_2 as

$$x_2 = b - x_3 = b - \frac{2b-a}{3} = \frac{b+a}{3} \qquad (2.3.5)$$

Substituting above in (2.3.1) to find x_1, we obtain

$$x_1 = x_2 - a = \frac{b+a}{3} - a = \frac{b-2a}{3} \qquad (2.3.6)$$

2.4 Four Consecutive Missing Terms

We could easily find the formula for missing terms in Fibonacci-like sequence a, x_1, x_2, x_3, x_4, b using the same approach above.

Here are the solved formulas,

$$x_1 = \frac{b-3a}{5} \qquad (2.4.1)$$

$$x_2 = \frac{b+2a}{5} \qquad (2.4.2)$$

$$x_3 = \frac{2b-a}{5} \qquad (2.4.3)$$

$$x_4 = \frac{3b+a}{5}. \qquad (2.4.4)$$

III. THE GENERAL FORMULA

To find a general formula for all *n* missing terms, a pattern must be recognized. All the formula (2.1.1), (2.2.3), (2.2.4), (2.3.5), (2.3.6), (2.4.1), (2.4.2), (2.4.3), and (2.4.4) are tabulated in Table 1 to find a recognizable pattern easily.

Seemingly, all the missing terms in a Fibonacci-like sequence could be solved given the first missing term x_1 since it is a succession of numbers that are obtained through adding the two preceding numbers. If a general formula will be obtained in x_1, the other missing terms will be calculated easily.

A clear observation could be seen in Table 1 for x_1 that the numerical coefficient of *a* in numerator and the denominator of the formulas were following the Fibonacci Sequence as shown by Table 2. This can be illustrated as

$$x_1 = \frac{b - F_n a}{F_{n+1}} \qquad (3.1)$$

Note that the formula for finding the nth term of Fibonacci sequence (also known as Binet's formula) is

$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}. \quad (3.2)$$

Table 1. Formula for different missing terms in Fibonacci-like sequence.

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Number of missing term	Formula			
	x_1	x_2	x_3	x_4
1	$b-a$	--------	--------	--------
2	$\frac{b-a}{2}$	$\frac{b+a}{2}$	--------	--------
3	$\frac{b-2a}{3}$	$\frac{b+a}{3}$	$\frac{2b-a}{3}$	--------
4	$\frac{b-3a}{5}$	$\frac{b+2a}{5}$	$\frac{2b-a}{5}$	$\frac{3b+a}{5}$

Number of Missing Term	Coefficient of a in Numerator	Coefficient of Denominator
1	1	1
2	1	2
3	2	3
4	3	5
⋮	⋮	⋮
n	$\frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$	$\frac{\varphi^{n+1} - (1-\varphi)^{n+1}}{\sqrt{5}}$

Table 2. Relationship of Number of Missing Terms with Numerator and Denominator of Formulas

Using the observations above, the general formula for x_1 is

$$x_1 == \frac{b - \left[\frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}\right]a}{\frac{\varphi^{n+1} - (1-\varphi)^{n+1}}{\sqrt{5}}} \quad (3.3)$$

where x_1 is the first missing term in Fibonacci-like sequence, a is the first term given, b is the last term given, n is the number of missing terms and φ is known as the golden ration equal to $1.61803399\ldots\ldots$

To fully understand the formula, we answer the presented problem by inserting five terms between 2 and 50 to make it a Fibonacci-like sequence. In mathematical statement, the sequence is $2, x_1, x_2, x_3, x_4, x_5, 50, \ldots\ldots$

This problem will be solved by finding x_1 using (3.3).

Given: $a = 2,\ n = 5,\ b = 50$

$$x_1 = \frac{b - \left[\frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}\right]a}{\frac{\varphi^{n+1} - (1-\varphi)^{n+1}}{\sqrt{5}}}$$

$$x_1 = \frac{50 - \left[\frac{1.618034^5 - (1-1.618034)^5}{\sqrt{5}}\right]2}{\frac{1.618034^{5+1} - (1-1.68034)^{5+1}}{\sqrt{5}}} = 5$$

Now that we have solved for x_1, we could easily find x_2 which is $x_2 = x_1 + 2 = 7$. Similarly, we can find x_3, etc.

Therefore the Fibonacci-like sequence is 2, 5, 7, 12, 19, 31, 50…

IV. CONCLUSION

Fibonacci-like sequence is very similar to the other sequences. A formula was developed to solve Fibonacci like sequence given its first and last term. Recognizing patterns could be used to develop formula. Other sequences could be solved using this method.

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