Adaptive Control and Synchronization of the Uncertain Sprott J System

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***Abstract*—This paper investigates the adaptive control and synchronization of the uncertain Sprott J system with unknown parameters. The Sprott J system is one of the important and simple paradigms of three-dimensional chaotic systems discovered by J.C. Sprott in 1994. In this paper, we first design an adaptive control law so as to stabilize the uncertain Sprott J system to its unstable equilibrium at the origin based on the adaptive control theory and Lyapunov stability theory. Then we derive adaptive control laws to achieve global chaos synchronization of identical uncertain Sprott J systems with unknown parameters. Numerical simulations are shown to demonstrate the effectiveness of the proposed adaptive control and synchronization schemes for the uncertain Sprott J system.**

***Index Terms*—adaptive control, stabilization, chaos, synchronization, Sprott J system.**

***MSC 2010 Codes* – 34H10, 34D06, 93C10, 93C15.**

I. INTRODUCTION

CHAOTIC systems are dynamical systems that are highly sensitive to initial conditions. This sensitivity is popularly known as the butterfly effect [1]. Since chaos phenomenon in weather models was first observed by Lorenz in 1963 [2], a large number of chaos phenomena and chaos behavior have been discovered in social, economical, biological and electrical systems.

The control of chaotic systems is to design state feedback control laws that stabilize the chaotic systems around the unstable equilibrium points. Active control technique is used when the system parameters are known and adaptive control is used when the system parameters are unknown ([3]-[5]).

The synchronization of chaotic systems is a phenomenon that may occur when two or more chaotic oscillators are coupled or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect, which causes the exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, synchronizing two chaotic systems is seemingly a very challenging problem in the chaos literature ([6]-[22]).

In 1990, Pecora and Carroll [6] introduced a method to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. From then onwards, chaos synchronization has been widely explored in a variety of fields including physical systems [7], chemical systems [8], ecological systems [9], secure communications ([10]-[12]), etc.

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In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Since the seminal work by Pecora and Carroll [6], a variety of impressive techniques have been proposed for synchronization of chaotic systems such as the sampled-data feedback synchronization method [13], OGY method [14], time-delay feedback method [15], backstepping method [16], active control method ([17]-[19]), adaptive control method ([20]-[21]), sliding mode control method [22], etc.

In this paper, we apply adaptive control theory to derive new results for the stabilization of the uncertain Sprott J system ([23], 1994) with unknown parameters to its unstable equilibrium at the origin. We also apply adaptive control theory to derive state feedback control laws to solve the global chaos synchronization problem for identical uncertain Sprott J systems with unknown parameters.

This paper is organized as follows. In Section II, we derive results for the adaptive stabilization of the Sprott J system with unknown parameters. In Section III, we derive results for the adaptive synchronization of identical Sprott J systems with unknown parameters. In Section IV, we summarize the main results obtained in this paper.

II. ADAPTIVE CONTROL OF UNCERTAIN SPROTT J SYSTEM

In this section, we discuss the adaptive control of the uncertain Sprott J system ([23], 1994), which is one the paradigms of three-dimensional chaotic systems.

A. System Description

The Sprott J system is described by

$$\begin{aligned} \dot{x}_1 &= ax_3 \\ \dot{x}_2 &= -bx_2 + x_3 \\ \dot{x}_3 &= -x_1 + x_2 + x_2^2 \end{aligned} \tag{1}$$

where x_1, x_2, x_3 are the state variables and a, b are parameters of the system.

The Sprott J system (1) is chaotic when

$$a = 2 \text{ and } b = 2. \tag{2}$$

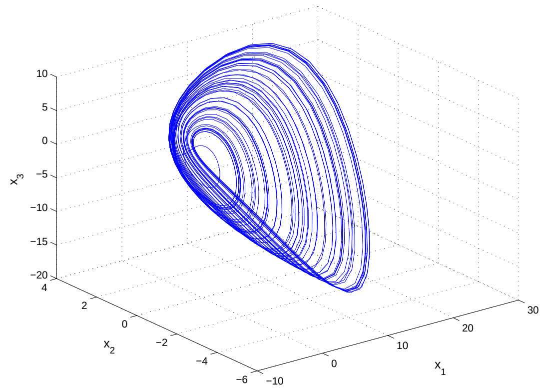

Fig. 1: State Orbits of the Sprott J System

The state orbits of the Sprott J system are illustrated in Figure 1.

When the parameter values are taken as in (2), the system (1) is chaotic and the system linearization matrix at the equilibrium point $E_0 = (0, 0, 0)$ is given by

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

which has the eigenvalues

$$\begin{aligned} \lambda_1 &= 0.1573 + 1.3025 \text{ i} \\ \lambda_2 &= 0.1573 - 1.3025 \text{ i} \\ \lambda_3 &= -2.3146 \end{aligned}$$

Since λ_1 and λ_2 are eigenvalues with positive real part, it is immediate from Lyapunov's stability theory [24] that the system (1) is unstable at the equilibrium point $E_0 = (0, 0, 0)$.

B. Theoretical Results

Here, we design an adaptive control law for globally stabilizing the uncertain Sprott J system (1) with unknown parameters.

Thus, we consider the controlled uncertain Sprott J system, which is described by

$$\begin{aligned} \dot{x}_1 &= ax_3 + u_1 \\ \dot{x}_2 &= -bx_2 + x_3 + u_2 \\ \dot{x}_3 &= -x_1 + x_2 + x_2^2 + u_3 \end{aligned} \tag{3}$$

where u_1, u_2 and u_3 are the feedback controllers to be designed using the states x_1, x_2, x_3 and estimates of the unknown parameters a, b of the uncertain Sprott J system.

In order to ensure that the controlled Sprott J system (3) globally converges to the origin asymptotically, we consider the following adaptive control functions

$$\begin{aligned} u_1 &= -\hat{a}\, x_3 - k_1 x_1 \\ u_2 &= \hat{b}\, x_2 - k_2 x_2 \\ u_3 &= x_1 - x_2 - x_2^2 - k_3 x_3 \end{aligned} \tag{4}$$

where $\hat{a}$ and $\hat{b}$ are estimates of the parameters a and b respectively and k_1, k_2, k_3 are positive constants.

Substituting the control law (4) into the controlled Sprott system (3), we obtain

$$\begin{aligned} \dot{x}_1 &= (a - \hat{a})\, x_3 - k_1 x_1 \\ \dot{x}_2 &= -(b - \hat{b})\, x_2 - k_2 x_2 \\ \dot{x}_3 &= -k_3 x_3 \end{aligned} \tag{5}$$

Let us now define the parameter estimation error as

$$\begin{aligned} e_a &= a - \hat{a} \\ e_b &= b - \hat{b} \end{aligned} \tag{6}$$

Using (6), the closed-loop dynamics (5) can be written compactly as

$$\begin{aligned} \dot{x}_1 &= e_a\, x_3 - k_1 x_1 \\ \dot{x}_2 &= -e_b\, x_2 - k_2 x_2 \\ \dot{x}_3 &= -k_3 x_3 \end{aligned} \tag{7}$$

For the derivation of the update law for adjusting the parameter estimates $\hat{a}$ and $\hat{b}$, the Lyapunov approach is used.

We consider the quadratic Lyapunov function

$$V = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2\right), \tag{8}$$

Clearly, V is a positive definite function on $\mathbb{R}^5$.

We note also that

$$\begin{aligned} \dot{e}_a &= -\dot{\hat{a}} \\ \dot{e}_b &= -\dot{\hat{b}} \end{aligned} \tag{9}$$

Differentiating V along the trajectories of (7) and using (9), we obtain

$$\begin{aligned} \dot{V} = & -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 + e_a\left[x_1 x_3 - \dot{\hat{a}}\right] \\ & + e_b\left[-x_2^2 - \dot{\hat{b}}\right] \end{aligned} \tag{10}$$

In view of Eq. (10), the estimated parameters are updated by the following law:

$$\begin{aligned} \dot{\hat{a}} &= x_1 x_3 + k_4 e_a \\ \dot{\hat{b}} &= -x_2^2 + k_5 e_b \end{aligned} \tag{11}$$

Substituting (11) into (10), we obtain

$$\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 e_a^2 - k_5 e_b^2, \tag{12}$$

which is a negative definite function on $\mathbb{R}^5$.

Thus, by Lyapunov stability theory [24], we obtain the following result.

Theorem 2.1: The Sprott J system (3) with unknown parameters is globally and exponentially stabilized for all initial conditions $x(0) \in \mathbb{R}^3$ by the adaptive control law (4), where the update law for the estimates of the parameters is given by (11) and $k_i, (i = 1, 2, \ldots, 5)$ are positive constants. ∎

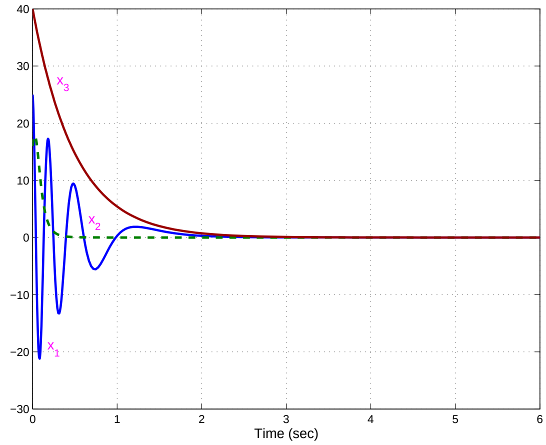

Fig. 2: Time Responses of the Controlled Sprott J System

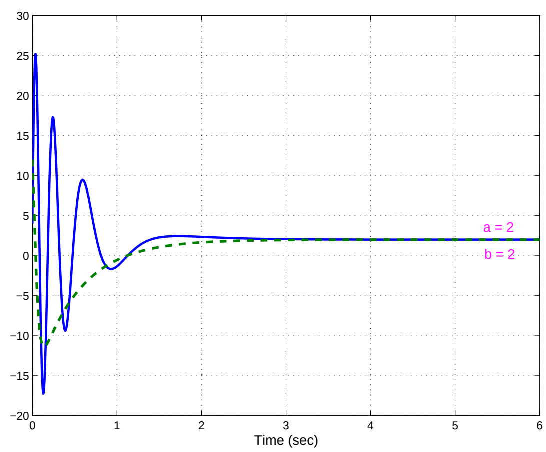

Fig. 3: Parameter Estimates $\hat{a}(t), \hat{b}(t)$

C. Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the controlled Sprott J system (3) with the adaptive control law (4) and the parameter update law (11).

The parameters of the Sprott J system (3) are selected as $a = 2$ and $b = 2$. We take the constants k_i as $k_i = 2$ for $i = 1, \ldots, 5$.

We take the initial values of the estimated parameters as

$$\hat{a}(0) = 4 \quad \text{and} \quad \hat{b}(0) = 12.$$

We take the initial values of the Sprott J system (3) as

$$x_1(0) = 25, \quad x_2(0) = 16, \quad x_3(0) = 40.$$

When the adaptive control law (4) and the parameter update law (11) are applied, the controlled Sprott J system (3) converges to the equilibrium $E_0 = (0, 0, 0)$ exponentially as shown in Figure 2.

Figure 3 shows that the parameter estimates $\hat{a}$ and $\hat{b}$ converge to the system parameters $a = 2$ and $b = 2$ exponentially.

III. Adaptive Synchronization of Uncertain Sprott J Systems

A. Theoretical Results

In this section, we discuss the adaptive synchronization of identical Sprott J systems ([23], 1994) with unknown parameters.

As the master system, we consider the Sprott J system described by

$$\begin{aligned} \dot{x}_1 &= a x_3 \\ \dot{x}_2 &= -b x_2 + x_3 \\ \dot{x}_3 &= -x_1 + x_2 + x_2^2 \end{aligned} \tag{13}$$

where x_1, x_2, x_3 are the states and a, b are unknown system parameters.

As the slave system, we consider the controlled Sprott J system described by

$$\begin{aligned} \dot{y}_1 &= a y_3 + u_1 \\ \dot{y}_2 &= -b y_2 + y_3 + u_2 \\ \dot{y}_3 &= -y_1 + y_2 + y_2^2 + u_3 \end{aligned} \tag{14}$$

where y_1, y_2, y_3 are the states and u_1, u_2, u_3 are the controllers to be designed so as to achieve global chaos synchronization between the uncertain systems (13) and (14).

The synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3) \tag{15}$$

Then the error dynamics is obtained as

$$\begin{aligned} \dot{e}_1 &= a e_3 + u_1 \\ \dot{e}_2 &= -b e_2 + e_3 + u_2 \\ \dot{e}_3 &= -e_1 + e_2 + y_2^2 - x_2^2 + u_3 \end{aligned} \tag{16}$$

Let us now define the adaptive control functions $u_1(t), u_2(t)$ and $u_3(t)$ as

$$\begin{aligned} u_1 &= -\hat{a}\, e_3 - k_1 e_1 \\ u_2 &= -\hat{b}\, e_2 - e_3 - k_2 e_2 \\ u_3 &= e_1 - e_2 - y_2^2 + x_2^2 - k_3 e_3 \end{aligned} \tag{17}$$

where $\hat{a}$ and $\hat{b}$ are estimates of the unknown parameters a and b, respectively.

Substituting (17) into (16), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= (a - \hat{a}) e_3 - k_1 e_1 \\ \dot{e}_2 &= -(b - \hat{b}) e_2 - k_2 e_2 \\ \dot{e}_3 &= -k_3 e_3 \end{aligned} \tag{18}$$

Let us now define the parameter estimation error as

$$\begin{aligned} e_a &= a - \hat{a} \\ e_b &= b - \hat{b} \end{aligned} \tag{19}$$

Substituting (19) into (18), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= e_a e_3 - k_1 e_1 \\ \dot{e}_2 &= -e_b e_2 - k_2 e_2 \\ \dot{e}_3 &= -k_3 e_3 \end{aligned} \tag{20}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2\right), \tag{21}$$

which is a positive definite function on $\mathbb{R}^5$.

We note also that

$$\begin{aligned} \dot{e}_a &= -\dot{\hat{a}} \\ \dot{e}_b &= -\dot{\hat{b}} \end{aligned} \tag{22}$$

Differentiating (21) along the trajectories of (20) and using (22), we obtain

$$\begin{aligned} \dot{V} &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a\left[e_1 e_3 - \dot{\hat{a}}\right] \\ &\quad + e_b\left[-e_2^2 - \dot{\hat{b}}\right] \end{aligned} \tag{23}$$

In view of Eq. (23), the estimated parameters are updated by the following law:

$$\begin{aligned} \dot{\hat{a}} &= e_1 e_3 + k_4 e_a \\ \dot{\hat{b}} &= -e_2^2 + k_5 e_b \end{aligned} \tag{24}$$

Substituting (24) into (23), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_a^2 - k_5 e_b^2, \tag{25}$$

which is a negative definite function on $\mathbb{R}^5$.

Thus, by Lyapunov stability theory [24], we obtain the following result.

Theorem 3.1: The identical uncertain Sprott J systems (13) and (14) are globally and exponentially synchronized for all initial conditions by the adaptive control law (17), where the update law for the estimates of parameters is given by (24) and k_i, $(i = 1, 2, \ldots, 5)$ are positive constants. ∎

B. Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the Sprott J systems (13) and (14) with the adaptive control law (17) and the parameter update law (24).

We consider the adaptive synchronization of the Sprott J systems for the chaotic case with the parameter values taken as $a = 2$ and $b = 2$. We take the constants k_i as $k_i = 2$ for $i = 1, \ldots, 5$.

We take the initial values of the parameter estimates as

$$\hat{a}(0) = 4 \quad \text{and} \quad \hat{b}(0) = 12.$$

We take the initial values of the master system (13) as

$$x_1(0) = 10, \quad x_2(0) = 4, \quad x_3(0) = 1.$$

We take the initial values of the master system (13) as

$$y_1(0) = 3, \quad y_2(0) = 5, \quad y_3(0) = 8.$$

Figure 4 shows the adaptive chaos synchronization of the identical uncertain Sprott J systems.

Figure 5 shows that the estimated values of the parameters $\hat{a}$ and $\hat{b}$ converge to the system parameters $a = 2$ and $b = 2$ exponentially.

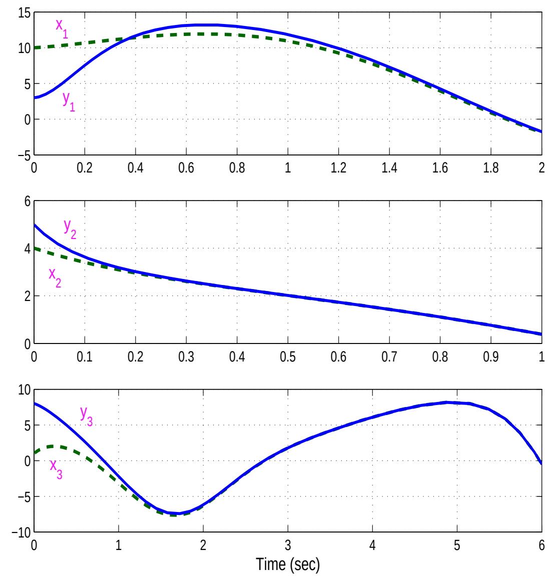

Fig. 4: Synchronization of the Identical Sprott J Systems

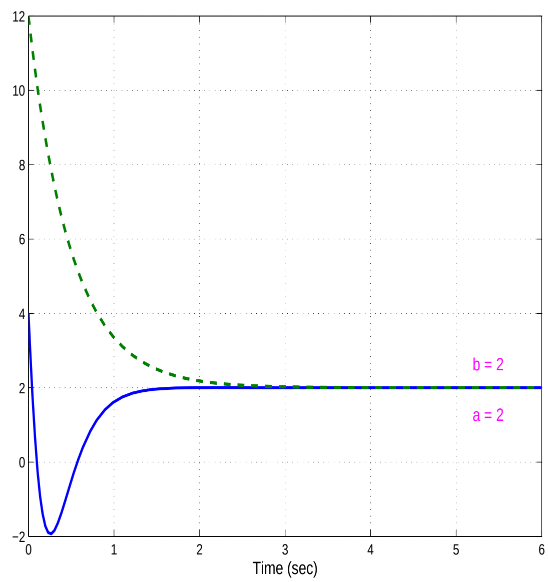

Fig. 5: Parameter Estimates $\hat{a}(t), \hat{b}(t)$

IV. Conclusion

In this paper, we applied adaptive control theory for the control and synchronization of the Sprott J chaotic system (Sprott, 1994) with unknown system parameters. First, we designed adaptive control laws to stabilize the Sprott J system to its unstable equilibrium at the origin based on the adaptive control theory and the Lyapunov stability theory. Then we derived adaptive synchronization scheme and update law for the estimation of system parameters for the identical Sprott J systems with unknown parameters. Our synchronization schemes were established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the proposed adaptive control method is very effective and convenient to achieve chaos control and synchronization of the Sprott J system.

REFERENCES

[1] K. T. Alligood, T. Sauer and J. A. Sauer, *Chaos: An Introduction to Dynamical Systems*, New York: Springer, 1997.

[2] E. N. Lorenz, "Deterministic nonperioidic flow," *J. Atmos. Sci.,* vol. 20, pp. 130–141, 1963.

[3] S. S. Ge, C. Wang and T. H. Lee, "Adaptive backstepping control of a class of chaotic systems," *Internat. J. Bifurcation and Chaos,* vol. 10, pp. 1149–1156, 2000.

[4] X. Wang, L. Tian and L. Yu, "Adaptive control and slow manifold analysis of a new chaotic system," *Internat. J. Nonlinear Science,* vol. 21, pp. 43–49, 2000.

[5] M. Sun, L. Tian, S. Jiang and J. Xun, "Feedback control and adaptive control of the energy resource chaotic system," *Chaos, Solitons and Fractals*, vol. 32, pp. 168–180, 2007.

[6] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.,* vol. 64, pp. 821–824, 1990.

[7] M. Lakshmanan and K. Murali, *Nonlinear Oscillators: Controlling and Synchronization,* Singapore: World Scientific, 1996.

[8] S. K. Han, C. Kerrer and Y. Kuramoto, "D-phasing and bursting in coupled neural oscillators," *Phys. Rev. Lett.,* vol. 75, pp. 3190–3193, 1995.

[9] B. Blasius, A. Huppert and L. Stone, "Complex dynamics and phase synchronization in spatially extended ecological system," *Nature,* vol. 399, pp. 354-359, 1999.

[10] M. Feki, "An adaptive chaos synchronization scheme applied to secure communication," *Chaos, Solitons and Fractals,* vol. 18, pp. 141–148, 2003.

[11] K. Murali and M. Lakshmanan, "Secure communication using a compound signal from generalized synchronizable systems," *Physical Review Letters A,* vol. 241, pp. 303–310, 1998.

[12] T. Yang, "A survey of chaotic secure communication systems," *Internat. J. Computational Cognition* vol. 2, no. 2, pp. 81–130, 2004.

[13] T. Yang and L. O. Chua, "Control of chaos using sampled-data feedback control" *Internat. J. Bifurcat. Chaos,* vol. 9, pp. 215–219, 1999.

[14] E. Ott, C. Grebogi and J. A. Yorke, "Controlling chaos," *Phys. Rev. Lett.* vol. 64, pp. 1196-1199, 1990.

[15] J. H. Park and O. M. Kwon, "A novel criterion for delayed feedback control of time-delay chaotic systems," *Chaos, Solitons and Fractals,* vol. 17, pp. 709–716, 2003.

[16] Y. G. Yu and S. C. Zhang, "Adaptive backstepping synchronization of uncertain chaotic systems," *Chaos, Solitons and Fractals,* vol. 27, pp. 1369–1375, 2006.

[17] J. H. Park, "Chaos synchronization of a chaotic system via nonlinear control," *Chaos, Solitons and Fractals,* vol. 25, no. 3, pp. 579–584, 2005.

[18] U. E. Vincent, "Chaos synchronization using active control and backstepping control," *Nonlinear Analysis: Modelling and Control,* vol. 13, pp. 253–261, 2008.

[19] V. Sundarapandian, "Global chaos synchronization of four-scroll and four-wing attractors by active nonlinear control," *International Journal on Computer Science and Engineering,* vol. 3, no. 5, pp. 2145–2155, 2011.

[20] S. Boccaletti, A. Farini and F. T. Arecchi, "Adaptive synchronization of chaos for secure communication,"*Phys. Rev. E,* vol. 55, pp. 4979–4981, 1997.

[21] T. L. Liao and S. H. Tsai, "Adaptive synchronization of chaotic system and its applications to secure communications," *Chaos, Solitons and Fractals,* vol. 11, pp. 1387–1396, 2000.

[22] K. Konishi, M. Hirai and H. Kokame, "Sliding mode control for a class of chaotic systems," *Physics Letters A,* vol. 245, pp. 511-517, 1998.

[23] J. C. Sprott, "Some simple chaotic flows," *Phys. Rev. E.,* vol. 50, pp. 647–650, 1994.

[24] W. Hahn, *The Stability of Motion,* Berlin: Springer, 1967.