Weak convergence of an Iteration Method for Solution of General Equilibrium Problem

Binayak S.Choudhury and Subhajit Kundu

***Abstract*—In this paper, we introduce a new two step iteration to find a solution of a general equilibrium problem. The iteration is a viscosity type iteration with a family of nonexpansive operators which form a semigroup. Viscosity iterations are a class of recently introduced iterations which have several applications . Here we prove that our iteration has a weak limit which is a common point of the set of solutions of the equilibrium problem and the set of common fixed points of nonexpansive semigroups. We have also demonstrated the consistency of the control conditions with an illustration.**

***Index Terms*—Equilibrium problem, Nonexpansive semigroup, Viscosity approximation, Fixed point, Weak convergence, Hilbert space.**

***MSC 2010 Codes* – 46C05, 47H10, 91B50.**

I. Introduction

This paper deals with an equilibrium problem which is a generalization of several problems in physics, optimization and economics [1], [2].

Several iterative methods are employed for obtaining solutions of this equilibrium problem in Hilbert space and also in the more general setting of Banach spaces [3], [4], [5]. Viscosity approximation method is an iterative method which involves a contraction mapping. In particular viscosity approximation methods were applied to the equilibrium problem in a number of works. The iterations used for solving equilibrium problems generally contain nonlinear operators. The sequences generated by these iterations converge to some common elements in the set of solutions of the equilibrium problem and the set of fixed points of the nonlinear operators involved in the iteration. Semigroup of nonexpansive operators have been considered in the problems of constructing fixed point iterations in Banach and Hilbert spaces. This paper aims to obtain a solution of the equilibrium problem by applying a two step viscosity iteration which involves a nonexpansive semigroup. In the next section, we discuss certain definitions and concepts which we use in this paper.

II. Some Preliminary Results

We use the notation $x_n \to x$ to mean strong convergence and $x_n \rightharpoonup x$ to indicate weak convergence.

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Let H be a Hilbert space and C be a nonempty closed convex subset of H.

Definition 2.1 A mapping M from C to C is said to be a nonexpansive mapping if for all $p, q \in C$

$$\|Mp - Mq\| \leq \|p - q\|. \quad (1)\square$$

Definition 2.2 A mapping l is said to be a θ contraction if for each $p, q \in C$,

$$\|lp - lq\| \leq \theta\|p - q\| \quad (2)$$

when $0 < \theta < 1$. $\square$

Definition 2.3 An operator A from C to C is called strongly positive if there is a constant $c > 0$ such that

$$\langle Ap, p\rangle \geq c\|p\|^2 \text{ for all } p \in H. \quad (3)\ \square$$

Definition 2.4 Let H be a real Hilbert space and and C is a nonempty closed convex subset of a real Hilbert space H. For any $p \in H$, we define the metric projection P_C from H into C as P_Cp= $\{z \in C : \|z - p\| = \inf_{y\in C} \|y - p\|\}$. (4)

It is well known that P_C is a firmly nonexpansive mapping from H onto C, that is,
$\|P_Cp - P_Cq\|^2 \leq \langle P_Cp - P_Cq, p - q\rangle$ for all $p, q \in H$.} P_C is also nonexpansive mapping from H onto C. Obviously, $\|p - P_Cp\| \leq \|p - q\|$ for all $q \in C$. It is clear that P_C is a firmly nonexpansive mapping from H onto C, that is, $\|P_Cp - P_Cq\|^2 \leq \langle P_Cp - P_Cq, p - q\rangle$ for all $p, q \in H$. P_C is also nonexpansive mapping from H onto C.

The fixed point set of an operator M from H to H is denoted by $Fix(M)$, that is, $Fix(M)$=$\{x \in H : Mx = x\}$ $\square$

Definition 2.5 A family $B = (M(b))_{b\geq 0}$ is said to be a nonexpansive semigroup on H if :
(A1) $M(0)p = p$ for all $p \in H$,
(A2) $M(b + d) = M(b)M(d)$ for all $b, d \geq 0$,
(A3) $\|M(b)p - M(b)q\| \leq \|p - q\|$ for all $p, q \in H$ and $b \geq 0$,
(A4) $b \to M(b)p$ is continuous for all $p \in H$. (5)

Definition 2.6 A sequence $\{x_n\}$ of elements of a Banach space X is said to converge weakly to an element $x \in X$ if $f(x_k) \to f(x)$ as $k \to \infty$ for all $f \in X'$ where f is a linear functional from X to X and X' is the dual of X. (6)

Definition 2.7 A sequence $\{x_n\}$ is said to have a weak limit point l if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to l. (7)

We denote $w_\omega(x_n)$as the set of all weak limit point of $\{x_n\}$. We denote the set of fixed point of $M(b)$ by $Fix(M(b))$. The set of all common fixed points of B is denoted by $Fix(B)$. So $Fix(B)$= $\cap_{s\geq 0}Fix(M(b))$.

Nonexpansive semigroups appeared in the work of Ballion and Brezis where it was shown that the continuous scheme $x_t = \frac{1}{t}\int_0^t M(b)x_t db$, $t \in (0, 1)$, $s \in R^+$, converges obeying some control conditions where R^+ is the set of positive real

numbers, converges to common fixed point of the nonexpansive semigroup $B = \{M(b) : 0 \leq b < \infty\}$.
Later Plubtieng and Punpaeng used the single step viscosity iteration method defined by

$$x_{n+1} = a_n l(x_n) + g_n x_n + (1 - a_n - g_n)\frac{1}{b_n}\int_0^{b_n} M(b)x_n db, \quad n \geq 0$$

to approximate common fixed point of the nonexpansive semigroup $B = \{M(b) : 0 \leq b < \infty\}$.
The description of the general equilibrium problem is the following:
Let $I : C\times C\rightarrow R$ be a bifunction where R is the set of real numbers. To find $p \in C$ such that

$$I(p, q) \geq 0, \text{ for all } q \in C. \tag{8}$$

The solution set of (2.8) is denoted by $E(I)$, that is, $E(I) = \{p \in C : I(p, q) \geq 0 \text{ for all } q \in C\}$.
The problem reduces to a variational inequality problem if we take $I(p, q) = \langle Ap, q - p\rangle$ for all $p, q \in C$ where $A : C \to H$ is a mapping.

For the bifunction I from $C\times C$ present in the equilibrium problem $\rightarrow R$, it is supposed that I satisfies:
(C1) $I(p, p) = 0$ for all $p \in C$,
(C2) I is monotone, that is, $I(p, q) + I(q, p) \leq 0$,
(C3) for each $p, q, r \in C$,

$$\lim_{d\to 0^+} I(dr + (1 - d)p, q) \leq I(p, q),$$

(C4) for each $p \in C$, $q \to I(p, q)$ is convex and lower semicontinuous.

Lemma 2.1 Let C be a nonempty closed convex subset of H. Let I be a bifunction from $C\times C$ into R. I satisfies conditions (C1)- (C4). Then for any $s > 0$ and $p \in H$ there exists $r \in C$ such that

$$I(r, q) + \frac{1}{s}\langle q - r, r - p\rangle \geq 0 \text{ for all } q \in C.$$

Further, if $M_s x = \{r \in C : I(r, q) + \frac{1}{s}\langle q - r, r - p\rangle \geq 0$ for all $q \in C\}$ then the following statements are true:
(1) M_s is single valued,
(2) M_s is firmly nonexpansive that is for any $p, q \in H$

$$\|M_s p - M_s q\|^2 \leq \langle M_s p - M_s q, p - q\rangle,$$

(3) $Fix(M_s) = E(I)$,
(4) $E(I)$ is closed and convex.

Lemma 2.2 Let $C \subset H$ and $C \neq \phi$. C is bounded closed and convex. Let $(M(b))_{b\geq 0}$ be a nonexpansive semigroup on C. Then for every $k \geq 0$,

$$\lim_{d\to\infty} \sup_{p\in C}\|\frac{1}{d}\int_0^d M(b)pdb - M(k)\frac{1}{d}\int_0^d M(b)pdb\| = 0.$$

Lemma 2.3 $C \subset X$ and $C \neq \phi$ where X be a uniformly convex Banach space. C is closed convex subset of X and $M : C \to X$ be a nonexpansive mapping. Then, the mapping $(I - M)$ is demiclosed on C, that is, if $\{x_n\}$ is weakly convergent to p and $\{(I - M)x_n\}$ is strongly convergent to q, then $(I - T)p = q$.

Lemma 2.4 Let us suppose (C1)-(C4) hold. Let $p, q \in H$, $s_1, s_2 > 0$. Then

$$\|M_{s_2} q - M_{s_1} p\| \leq \|p - q\| + |\frac{s_2 - s_1}{s_2}|\|M_{s_2} q - q\|.$$

Lemma 2.5 Let $\{b_n\}$ be a sequence. b_n is real and $b_n \geq 0$ for all n satisfying
$b_{n+1} \leq (1 - \pi_n)b_n + \lambda_n + \tau_n$, for all $n \geq 0$, where $\{\pi_n\}$ is a sequence in (0,1) and $\{\lambda_n\}$, $\{\tau_n\}$ are sequences of real numbers such that

(a) $\lim_{n\to\infty} \pi_n = 0$ and $\sum_{n=0}^{\infty} \pi_n = \infty$,
(b) $\limsup_{n\to\infty} \frac{\lambda_n}{\pi_n} \leq 0$,
(c) $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then $\{b_n\}$ converges to zero.

Lemma 2.6 Let X be a Banach space. If X is reflexive then every bounded sequence in X has a weakly convergent subsequence.

Lemma 2.7 Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficients $c > 0$ and $0 < \xi \leq \|A\|^{-1}$. Then $\|I - \xi A\| \leq 1 - \xi c$.

Lemma 2.8 Let X be a Banach space. Suppose that X is uniformly convex. $B_s(0)$ be a closed ball of X. Then a continuous strictly increasing convex function $e : [0, \infty) \to [0, \infty)$ exists with $e(0) = 0$ such that

$$\|\sigma p + \nu q + \pi r\|^2 \leq \sigma\|p\|^2 + \nu\|q\|^2 + \pi\|r\|^2 - \sigma\nu e(\|p - q\|)$$

for all $p, q, r \in B_s(0)$ and $\sigma, \nu, \pi \in [0, 1]$ with $\sigma + \nu + \pi = 1$.

III. Main Result

Let C be a nonempty bounded closed convex subset of a real Hilbert space H and $M(b)_{b\geq 0}$ be a nonexpansive semigroup on C. Let $I : C\times C\rightarrow R$ be a bifunction. Assume that $Fix(B) \cap E(I) \neq \phi$. Let A be a strongly positive linear bounded operator on C with coefficient $c > 0$. Let $l : C \to C$ be a α contraction where $0 < \alpha < 1$. For any $0 < \gamma < \frac{c}{\alpha}$, the sequence $\{x_n\}$ is defined as

$$\begin{aligned} &x_0 \in C,\\ &I(v_n, y) + \frac{1}{s_n}\langle y - v_n, v_n - x_n\rangle \geq 0, \quad \text{for all } y \in H.\\ &x_{n+1} = a_n\gamma l(x_n) + g_n x_n + ((1 - g_n)I - a_n A)\frac{1}{t_n}\int_o^{t_n} M(b)y_n db,\\ &y_n = \delta_n\gamma l(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \delta_n A)v_n, \text{ for all } n \geq 0, \end{aligned} \tag{9}$$

where $\{a_n\}, \{g_n\}, \{\gamma_n\}, \{\delta_n\}, \{s_n\}$, and $\{t_n\}$ satisfy following conditions:

(i) $a_n \in [0, 1]$, $a_n + g_n < 1$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} |a_n - a_{n-1}| < \infty$
(ii) $g_n \in [0, 1]$, $\lim_{n\to\infty} g_n = 0, \sum_{n=0}^{\infty} | g_{n+1} - g_n| < \infty$,
(iii) $\gamma_n \in [0, 1]$, $\lim_{n\to\infty} \gamma_n = 0, \sum_{n=0}^{\infty} | \gamma_{n+1} - \gamma_n| < \infty$,
(iv) $\delta_n \in [0, 1]$, $\lim_{n\to\infty} \delta_n = 0, \sum_{n=0}^{\infty} | \delta_{n+1} - \delta_n| < \infty$,
(v) $\lim_{n\to\infty} t_n = \infty$ and
$\lim_{n\to\infty} \frac{U_n}{V_n} = 0$, where
$U_n = |t_n - t_{n-1}|$, $V_n = t_n(\delta_n(c - \gamma\alpha)$
$+ a_n(c - \delta_n c^2 + 2\gamma_n c + \gamma c\delta_n\alpha - \gamma\alpha) - g_n(2 - \delta_n c + 2\gamma_n +$

$\gamma\delta_n\alpha) - 2\gamma_n)$,

(vi) $\liminf_{n\to\infty} s_n > 0, \sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$.

Then a weak limit of $\{x_n\}$ is an equilibrium point of I and a common fixed point of $(M(b))_{b\geq 0}$.

Note: A possible choice of the sequences $\{a_n\}, \{g_n\}, \{\delta_n\}, \{\gamma_n\}$ and $\{t_n\}$ satisfying the conditions of the above theorem is $a_n = g_n = \delta_n = \gamma_n = \frac{1}{\sqrt{n+1}}$ and $t_n = n + 1$. This shows the consistency of the control conditions (i)-(vi).

Proof. We may assume that $a_n \leq (1 - g_n)\|A\|^{-1}$ for all $n \in N$ as $\alpha_n \to 0$ when $n \to \infty$. With the help of Lemma 2.7 we have $\|J - \xi A\| \leq 1 - \xi c$ since $0 \leq \xi \leq \|A\|^{-1}$. We can write $\|J - A\| \leq 1 - c$.
(J denotes identity mapping).

Note that A is a strongly positive bounded linear operator on H. Thus, it follows that
$\|A\|= \sup\{|\langle Ap, p\rangle| : p \in H, \|p\| = 1\}$.

$$\text{Now, } \langle((1 - g_n)J - a_nA)p, p\rangle = 1 - g_n - a_n\langle Ap, p\rangle \geq 1 - g_n - a_n\|A\| \geq 0.$$

which shows that $(1 - g_n)J - a_nA$ is positive. Then,

$$\begin{aligned}&\|(1 - g_n)J - a_nA\|\\ &= \sup\ \{|\langle((1 - g_n)J - a_nA)p, p\rangle| : p \in H, \|p\| = 1\}\\ &= \sup\ \{1 - g_n - a_n\langle Ap, p\rangle : p \in H, \|p\| = 1\}\\ &\leq 1 - g_n - a_nc\ . \quad (10)\end{aligned}$$

In the same way we can prove that $(1-\gamma_n)J-\delta_nA$ is positive and

$$\|(1 - \gamma_n)J - \delta_nA\| \leq 1 - \gamma_n - \delta_nc. \quad (11)$$

We know that $Fix(B)$ and $E(I)$ are nonempty closed convex sets. Now we establish that $\{x_n\}$ is bounded. Let $z \in Fix(B) \cap E(I)$.

Then,

$$\begin{aligned}&\|x_{n+1} - z\|\\ &= \|a_n\gamma l(x_n) + g_nx_n + ((1 - g_n)J - a_nA)\tfrac{1}{t_n}\int_o^{t_n} M(b)y_ndb - z\|\\ &\leq a_n\|\gamma l(x_n) - Az\| + g_n\|x_n - z\| + \|(1 - g_n)J - a_nA\|\frac{1}{t_n}\int_o^{t_n} \|M(b)y_n - z\|\,db\\ &\leq a_n\|\gamma l(x_n) - Az\| + g_n\|x_n - z\| + \|(1 - g_n)J - a_nA\|\frac{1}{t_n}\int_o^{t_n} \|M(b)y_n - z\|\,db\\ &\quad + (1 - g_n - a_nc)\|y_n - z\|\\ &\leq a_n\gamma\alpha\|x_n - z\| + a_n\|\gamma l(z) - Az\| + g_n\|x_n - z\| + (1 - g_n - a_nc)\|y_n - z\|(\text{using}(2))\\ &= (a_n\gamma\alpha + g_n)\|x_n - z\| + g_n\|\gamma l(z) - Az\| + (1-g_n-a_nc)\|y_n-z\| \quad (12)\end{aligned}$$

Now,

$$\begin{aligned}&\|y_n - z\|\\ &= \|\delta_n\gamma l(x_n) + \gamma_nx_n + ((1 - \gamma_n)I - \delta_nA)v_n - z\|\\ &\leq \delta_n\|\gamma l(x_n) - Az\| + \gamma_n\|x_n - z\| + \|(1 - \gamma_n)I - \delta_nA\|\|v_n - z\|\\ &\leq \delta_n\|\gamma l(x_n) - Az\| + \gamma_n\|x_n - z\| + (1 - \gamma_n - \delta_nc)\|v_n - z\| \ \ (\text{using}(11)). \quad (13)\end{aligned}$$

Now by the Lemma 2.1 , $v_n = M_{s_n}x_n$ and $z = M_{s_n}z$.

$$\text{Therefore, } \|v_n - z\| = \|M_{s_n}x_n - M_{s_n}z\| \leq \|x_n - z\|. \quad (14)$$

Then, from (13) and (14) we have,

$$\begin{aligned}\|y_n - z\| &\leq \delta_n\|\gamma l(x_n) - Az\| + \gamma_n\|x_n - z\| + (1 - \gamma_n - \delta_nc)\|x_n - z\|\\ &\leq \delta_n[\gamma\|l(x_n) - l(z)\| + \|\gamma l(z) - Az\|] + \gamma_n\|x_n - z\| + (1 - \gamma_n - \delta_nc)\|x_n - z\|\\ &\leq \delta_n\gamma\alpha\|x_n - z\| + \delta_n\|\gamma l(z) - Az\| + \gamma_n\|x_n - z\| + (1 - \gamma_n - \delta_nc)\|x_n - z\|\\ &= (\delta_n\gamma\alpha + \gamma_n + 1 - \gamma_n - \delta_nc)\|x_n - z\| + \delta_n\|\gamma l(z) - Az\|\\ &= \{(1 - \delta_n(c - \gamma\alpha)\}\|x_n - z\| + \delta_n\|\gamma l(z) - Az\|\ . \quad (15)\end{aligned}$$

Then, from (12) and (15), we have,

$$\begin{aligned}&\|x_{n+1} - z\|\\ &\leq a_n\gamma\alpha\|x_n - z\| + a_n\|\gamma l(z) - Az\| + g_n\|x_n - z\|\\ &\quad + (1 - g_n - a_nc)[1 - \delta_n(c - \gamma\alpha)\|x_n - z\| + \delta_n\|\gamma l(z) - Az\|]\\ &= [a_n\gamma\alpha+g_n+1-g_n-a_nc-(1-g_n-a_nc)\delta_n(c-\gamma\alpha)]\|x_n-z\| + [\alpha_n + (1 - \beta_n - a_nc)\delta_n]\|\gamma l(z) - Az\|\\ &= [1 - \{a_n + (1 - g_n - a_nc)\delta_n\}(c - \gamma\alpha)]\|x_n - z\| + [a_n + (1 - g_n - a_nc)\delta_n]\|\gamma l(z) - Az\|\\ &\leq \max\{\|x_n - z\|, \tfrac{1}{c-\gamma\alpha}\|\gamma l(z) - Az\|\}.\end{aligned}$$

By induction, $\|x_{n+1} - z\| \leq \max\{\|x_0 - z\|, \frac{1}{c-\gamma\alpha}\|\gamma l(z) - Az\|\}$ for all $n \geq 0$. Therefore, $\{x_n\}$ is bounded.

$$\begin{aligned}\text{Now, } \|y_n - x_n\| &= \|\delta_n\gamma l(x_n) + \gamma_nx_n + ((1 - \gamma_n)I - \delta_nA)v_n - x_n\|\\ &\leq \delta_n\|\gamma l(x_n) - Av_n\| + \|\gamma_nx_n + v_n - \gamma_nv_n - x_n\|\\ &\leq \delta_n\|\gamma l(x_n)-Av_n\|+(1-\gamma_n)\|x_n-v_n\|\ . \quad (16)\end{aligned}$$

From (9), we have
$x_{n+1} = a_n\gamma l(x_n) + g_nx_n + ((1 - g_n)I - a_nA)z_n$ where
$z_n = \frac{1}{t_n}\int_o^{t_n} M(b)y_ndb$. Now,

$$\begin{aligned}&\|x_{n+1} - x_n\|\\ &= \|(I - a_nA)(z_n - z_{n-1}) - (a_n - a_{n-1})Az_{n-1} + \gamma a_n(l(x_n) - l(x_{n-1})) + \gamma(a_n - a_{n-1})l(x_{n-1})\\ &\quad + g_n(x_n - x_{n-1}) + (g_n - g_{n-1})x_{n-1} + g_n(z_{n-1} - z_n) + (g_{n-1} - g_n)z_{n-1}\|\\ &\leq (1-a_nc)\|z_n-z_{n-1}\|+|a_n-a_{n-1}|[\|Az_{n-1}\|+\gamma\|l(x_{n-1})\|] + \gamma a_n\alpha\|x_n - x_{n-1}\| + g_n\|x_n - x_{n-1}\| + g_n\|z_n - z_{n-1}\|\\ &\quad +|g_{n-1}-g_n|[\|x_{n-1}\|+\|z_{n-1}\|]. \quad (17)\end{aligned}$$

Now,

$$\begin{aligned}&\|z_n - z_{n-1}\|\\ &= \|\tfrac{1}{t_n}\int_o^{t_n}(M(b)y_n - M(b)y_{n-1})db + (\tfrac{1}{t_n} - \tfrac{1}{t_{n-1}})\int_o^{t_{n-1}} M(b)y_{n-1}db + \frac{1}{t_n}\int_{t_{n-1}}^{t_n} M(b)y_{n-1}db\|.\\ &= \|\tfrac{1}{t_n}\int_o^{t_n}(M(b)y_n - M(b)y_{n-1})db + (\tfrac{1}{t_n} - \tfrac{1}{t_{n-1}})\int_o^{t_{n-1}}(M(b)y_{n-1} - M(b)z)db\\ &\quad + \frac{1}{t_n}\int_{t_{n-1}}^{t_n}(M(b)y_{n-1} - M(b)z)db\|.\end{aligned}$$

$$\text{Therefore, } \|z_n - z_{n-1}\| \leq \|y_n - y_{n-1}\| + \tfrac{2|t_n-t_{n-1}|}{t_n}\|y_{n-1}-z\|. \quad (18)$$

Again, from (9), $y_n = \delta_n\gamma l(x_n)+\gamma_nx_n+((1-\gamma_n)I-\delta_nA)v_n$, we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|(I - \delta_nA)(v_n - v_{n-1}) - (\delta_n - \delta_{n-1})Av_{n-1} + \gamma\delta_n(l(x_n) - l(x_{n-1})) + \gamma(\delta_n - \delta_{n-1})l(x_{n-1})\\ &\quad + \gamma_n(x_n - x_{n-1}) + (\gamma_n - \gamma_{n-1})x_{n-1} + \gamma_n(v_{n-1} - v_n) + (\gamma_{n-1} - \gamma_n)v_{n-1}\|\end{aligned}$$

$$\le (1-\delta_n c)\|v_n - v_{n-1}\| + |\delta_n - \delta_{n-1}|\|Av_{n-1}\| + \gamma\delta_n\alpha\|x_n - x_{n-1}\| + \gamma|\delta_n - \delta_{n-1}|\|l(x_{n-1})\| + \gamma_n\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1}\| + \gamma_n\|v_{n-1} - v_n\| + |\gamma_n - \gamma_{n-1}|\|v_{n-1}\|$$
$$= (1-\delta_n c + \gamma_n)\|v_n - v_{n-1}\| + (\gamma\delta_n\alpha + \gamma_n)\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|Av_{n-1}\| + |\gamma_n - \gamma_{n-1}|[\|x_{n-1}\| + \|v_{n-1}\|] + \gamma|\delta_n - \delta_{n-1}|\|l(x_{n-1})\|. \quad (19)$$

Now, using Lemma 2.4, we have $\|v_n - v_{n-1}\| \le \|x_n - x_{n-1}\| + \frac{|s_n - s_{n-1}|}{s_n}\|v_n - x_n\|$.
Since $\liminf_{n\to\infty} s_n > 0$, there exists $u > 0$ for large $n \in N$ such that
$$\|v_n - v_{n-1}\| \le \|x_n - x_{n-1}\| + \frac{|s_n - s_{n-1}|}{u}\|v_n - x_n\|. \quad (20)$$

Then, from (19) and (20), we have
$$\|y_n - y_{n-1}\| \le (1-\delta_n c + \gamma_n)\|x_n - x_{n-1}\| + (1-\delta_n c + \gamma_n)\frac{|s_n - s_{n-1}|}{u}\|v_n - x_n\| + (\gamma\delta_n\alpha + \gamma_n)\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|Av_{n-1}\| + |\gamma_n - \gamma_{n-1}|[\|x_{n-1}\| + \|v_{n-1}\|] + \gamma|\delta_n - \delta_{n-1}|\|l(x_{n-1})\|$$
$$= (1-\delta_n c + 2\gamma_n + \gamma\delta_n\alpha)\|x_n - x_{n-1}\| + (1-\delta_n c + \gamma_n)\frac{|s_n - s_{n-1}|}{u}\|v_n - x_n\| + |\delta_n - \delta_{n-1}|\|Av_{n-1}\| + |\gamma_n - \gamma_{n-1}|[\|x_{n-1}\| + \|v_{n-1}\|] + \gamma|\delta_n - \delta_{n-1}|\|l(x_{n-1})\|. \quad (21)$$

Then, from (18) and (21), we have
$$\|z_n - z_{n-1}\| \le (1-\delta_n c + 2\gamma_n + \gamma\delta_n\alpha)\|x_n - x_{n-1}\| + (1-\delta_n c + \gamma_n)\frac{|s_n - s_{n-1}|}{u}\|v_n - x_n\| + |\delta_n - \delta_{n-1}|\|Av_{n-1}\| + |\gamma_n - \gamma_{n-1}|[\|x_{n-1}\| + \|v_{n-1}\|] + \gamma|\delta_n - \delta_{n-1}|\|l(x_{n-1})\| + \frac{2|t_n - t_{n-1}|}{t_n}\|y_{n-1} - z\| . \quad (22)$$

Again, from (17) and (22), we have
$$\|x_{n+1} - x_n\| \le (1-a_n c + \beta_n)[(1-\delta_n c + 2\gamma_n + \gamma\delta_n\alpha)\|x_n - x_{n-1}\| + (1-\delta_n c + \gamma_n)\frac{|s_n - s_{n-1}|}{u}\|v_n - x_n\| + |\delta_n - \delta_{n-1}|\|Av_{n-1}\| + |\gamma_n - \gamma_{n-1}|(\|x_{n-1}\| + \|v_{n-1}\|) + \gamma|\delta_n - \delta_{n-1}|\|l(x_{n-1})\| + \frac{2|t_n - t_{n-1}|}{t_n}\|y_{n-1} - z\|] + |a_n - a_{n-1}|[\|Az_{n-1}\| + \gamma\|l(x_{n-1})\|] + \gamma a_n\alpha\|x_n - x_{n-1}\| + g_n\|x_n - x_{n-1}\| + |g_n - g_{n-1}|[\|x_{n-1}\| + \|z_{n-1}\|]$$
$$\le [(1-a_n c + g_n)(1-\delta_n c + 2\gamma_n + \gamma\delta_n\alpha) + \gamma a_n\alpha + g_n]\|x_n - x_{n-1}\| + (1-\delta_n c + \gamma_n)(1-a_n c + g_n)\frac{|s_n - s_{n-1}|}{u}\|v_n - x_n\| + (1-a_n c + g_n)|\delta_n - \delta_{n-1}|\|Av_{n-1}\| + (1-a_n c + g_n)|\gamma_n - \gamma_{n-1}|(\|x_{n-1}\| + \|v_{n-1}\|) + (1-v_n c + g_n)\gamma|\delta_n - \delta_{n-1}|\|l(x_{n-1})\| + (1-a_n c + g_n)\frac{2|t_n - t_{n-1}|}{t_n}\|y_{n-1} - z\|] + |a_n - a_{n-1}|[\|Az_{n-1}\| + \gamma\|l(x_{n-1})\|] + |g_n - g_{n-1}|[\|x_{n-1}\| + \|z_{n-1}\|] - g_n(2-\delta_n c + 2\gamma_n + \gamma\delta_n\alpha) - 2\gamma_n]\}\|x_n - x_{n-1}\| + K(1-\delta_n c + \gamma_n)\frac{|s_n - s_{n-1}|}{u} + K|\delta_n - \delta_{n-1}| + K|\gamma_n - \gamma_{n-1}|.$$

Therefore, $\|x_{n+1} - x_n\| \le (1-A_n)\|x_n - x_{n-1}\| + B_n + C_n$ where $K = \max\{\sup_{n\in N}(1-\delta_n c + \gamma_n)(1-a_n c + g_n)\|v_n - x_n\|, (1-a_n c + g_n)\|Av_{n-1}\|,$ $(1-a_n c + g_n)(\|x_{n-1}\| + \|v_{n-1}\|),$ $(1-a_n c + g_n)\gamma\|l(x_{n-1})\|, (1-a_n c + g_n)\|y_{n-1} - z\|,$ $\|Az_{n-1}\| + \gamma\|l(x_{n-1})\|, \|x_{n-1}\| + \|z_{n-1}\|\},$
$A_n = \delta_n(c - \gamma\alpha) + a_n(c - \delta_n c^2 + 2\gamma_n c + \gamma c\delta_n\alpha - \gamma\alpha) - g_n(2-\delta_n c + 2\gamma_n + \gamma\delta_n\alpha) - 2\gamma_n,$
$B_n = 2K\frac{|t_n - t_{n-1}|}{t_n}, C_n = K[\frac{|s_n - s_{n-1}|}{u} + 2|\delta_n - \delta_{n-1}| + |a_n - a_{n-1}| + |g_n - g_{n-1}| + |\gamma_n - \gamma_{n-1}|].$

By using Lemma 2.5 and conditions (i)-(vi), we have
$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \quad (23)$$

Again, $\|v_n - z\|^2 = \langle v_n - z, M_{s_n}x_n - M_{s_n}z\rangle \le \langle v_n - z, x_n - z\rangle = \frac{1}{2}(\|v_n - z\|^2 + \|x_n - z\|^2 - \|x_n - v_n\|^2)$

Therefore,
$$\|v_n - z\|^2 \le \|x_n - z\|^2 - \|x_n - v_n\|^2. \quad (24)$$

Now for $x^* \in Fix(S) \cap E(I)$,
$$\|y_n - x^*\|^2 \le \delta_n\|\gamma l(x_n) - Ax^*\|^2 + \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n - \delta_n c)\|v_n - x^*\|^2 \text{(using Lemma 8)}$$
$$\le \delta_n\|\gamma l(x_n) - Ax^*\|^2 + \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n - \delta_n c)\|x_n - x^*\|^2 - (1-\gamma_n - \delta_n c)\|x_n - v_n\|^2 \text{ [using (24)]}$$
$$= \delta_n\|\gamma l(x_n) - Ax^*\|^2 + (1-\delta_n c)\|x_n - x^*\|^2 - (1-\gamma_n - \delta_n c)\|x_n - v_n\|^2 \quad (25)$$

From (9) and (25), using Lemma 2.8, we have
$$\|x_{n+1} - x^*\|^2 = \|a_n(\gamma l(x_n) - Ax^*) + g_n(x_n - x^*) + ((1-g_n)J - a_n A)(\frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*)\|^2$$
$$= a_n^2\|\gamma l(x_n) - Ax^*\|^2 + \|g_n(x_n - x^*) + ((1-g_n)J - a_n A)(\frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*)\|^2 + 2a_n\langle\gamma l(x_n) - Ax^*, g_n(x_n - x^*) + ((1-g_n)J - a_n A)(\frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*)\rangle$$
$$= a_n^2\|\gamma l(x_n) - Ax^*\|^2 + g_n^2\|x_n - x^*\|^2 + (1-g_n - a_n c)^2\|\frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*\|^2 + 2g_n\langle x_n - x^*, ((1-g_n)J - a_n A)(\frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*)\rangle + 2a_n g_n\langle\gamma l(x_n) - Ax^*, x_n - x^*\rangle + 2a_n((1-g_n) - g_n c)\langle\gamma l(x_n) - Ax^*, \frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*\rangle$$
$$\le a_n^2\|\gamma l(x_n) - Ax^*\|^2 + g_n^2\|x_n - x^*\|^2 + (1-g_n - a_n c)^2\|y_n - x^*\|^2 + 2g_n\langle x_n - x^*, ((1-g_n)J - a_n A)(\frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*)\rangle + 2a_n g_n\langle\gamma l(x_n) - Ax^*, x_n - x^*\rangle + 2a_n((1-g_n) - a_n c)\langle\gamma l(x_n) - Ax^*, \frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*\rangle$$
$$\le a_n^2\|\gamma l(x_n) - Ax^*\|^2 + g_n^2\|x_n - x^*\|^2 + \delta_n(1-g_n - a_n c)^2\|\gamma l(x_n) - Ax^*\|^2 + (1-g_n - a_n c)^2\|x_n - x^*\|^2 - (1-g_n - a_n c)^2(1-\gamma_n - \delta_n c)\|x_n - v_n\|^2 + 2g_n\langle x_n - x^*, ((1-g_n)I - a_n A)(\frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*)\rangle + 2a_n g_n\langle\gamma l(x_n) - Ax^*, x_n - x^*\rangle + 2a_n(1-g_n - a_n c)\langle\gamma l(x_n) - Ax^*, \frac{1}{t_n}\int_o^{t_n} M(b)y_n db - x^*\rangle. \quad (26)$$

Now,
$$[(1-g_n - a_n c)^2 + g_n^2]\|x_n - x^*\|^2 \le [(1-g_n - a_n c) + g_n]\|x_n - x^*\|^2 \le (1-a_n c)\|x_n - x^*\|^2 \le \|x_n - x^*\|^2. \quad (27)$$

Then, from (26) and (27), we have

$\|x_{n+1} - x^*\|^2 \le [a_n^2 + \delta_n(1 - g_n - a_n c)^2]\|\gamma l(x_n) - Ax^*\|^2$
$+ \|x_n - x^*\|^2 - (1 - g_n - a_n c)^2(1 - \gamma_n - \delta_n c)\|x_n - v_n\|^2$
$+ 2g_n(1 - g_n - a_n c)\|x_n - x^*\|\|y_n - x^*\|$
$+ 2a_n g_n\|\gamma l(x_n) - Ax^*\|\|x_n - x^*\|$
$+ 2a_n(1 - g_n - a_n c)\|\gamma l(x_n) - Ax^*\|\|y_n - x^*\|.$ (28)

From (28), it follows that
$(1 - g_n - a_n c)^2(1 - \gamma_n - \delta_n c)\|x_n - v_n\|^2$
$\le \{\|x_{n+1} - x^*\| + \|x_n - x^*\|\}\|x_{n+1} - x_n\|$
$+ [a_n^2 + \delta_n(1 - g_n - a_n c)^2]\|\gamma l(x_n) - Ax^*\|^2$
$+ 2g_n(1 - g_n - a_n c)\|x_n - x^*\|\|y_n - x^*\|$
$+ 2a_n g_n\|\gamma l(x_n) - Ax^*\|\|x_n - x^*\|$
$+ 2a_n(1 - g_n - a_n c)\|\gamma l(x_n) - Ax^*\| + \|y_n - x^*\|.$

Using (24), $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} g_n = 0$, $\lim_{n\to\infty} \delta_n = 0$ and the bounded property of $\{l(x_n)\}$, $\{x_n\}$, $\{y_n\}$, we have,
$\|x_n - v_n\| \to 0$ as $n \to \infty$. (29)

Again, from (18), (19) and conditions (iii) and (vi), we have
$\|z_n - z_{n-1}\| \to 0$ as $n \to \infty$. (30)

Now , $\|x_n - z_n\|$
$= \|a_{n-1}\gamma l(x_{n-1}) + g_{n-1}x_{n-1}$
$+ ((1 - g_{n-1}I - a_{n-1}A)z_{n-1} - z_n\|$
$= \|z_{n-1} - z_n\| + a_{n-1}\gamma\|l(x_{n-1})\| + g_{n-1}\|x_{n-1} - z_{n-1}\|$
$+ a_{n-1}\|Az_{n-1}\|.$

Then, using (30) and conditions (i) and (ii), we have
$\|x_n - z_n\| \to 0 \; as \; n \to \infty$. (31)

Using (29) and (31), we have
$\|v_n - z_n\| \le \|v_n - x_n\| + \|x_n - z_n\| \to 0$ as $n \to \infty$. (32)

Also we have, $\|y_n - v_n\|$
$= \|\delta_n\gamma l(x_n) + \gamma_n x_n + v_n - \gamma_n v_n - a_n A v_n - v_n\|$
$= \delta_n\gamma\|l(x_n)\| + \gamma_n\|x_n - v_n\| + a_n\|Av_n\|.$

Using (29) and conditions (i), (iii), (iv) we have
$\|y_n - v_n\| \to 0$ as $n \to \infty$. (33)

By (29), (31), (32) and (33) we can say that one of the sequences $\{x_n\}, \{v_n\}, \{z_n\}, \{y_n\}$ converge if and only if the other three converge to the same limit.

By (29), (31), (32) and (33) we have
$\omega_w(x_n)$=$\omega_w(v_n)$= $\omega_w(z_n)$ =$\omega_w(y_n)$. (34)

Since $\{y_{n_i}\}$ is bounded and the Hilbert space H is reflexive, by Lemma 2.6, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to x^*. Then x^* is also a weak limit of $\{x_n\}$. Let v_0=$P_{Fix(S)\cap EP(F)}x_0$. Since $\{x_n\}$ is a bounded sequence, there exists K such that $B(v_0, K)$ contains $\{x_n\}$. Moreover, $B(v_0, K)$ is $M(b)$-invariant for every $b \ge 0$. Therefore we can assume that $(M(b))_{s\ge 0}$ is a nonexpansive semigroup on $B(v_0, K)$. By (34), $x^* \in \omega_w(z_n)$= $\omega_w(v_n)$. Then, from Lemma 2.2, we have , for every $k \ge 0, \lim_{n\to\infty} \|\frac{1}{t_n}\int_o^{t_n} M(b)y_n db - M(k)\frac{1}{t_n}\int_o^{t_n} M(b)y_n db\|$= $\lim_{n\to\infty} \|z_n - T(k)z_n\| = 0$. Therefore from Lemma 2.3 , we have $x^* \in Fix(B)$. Next we prove that $x^* \in E(I)$. Let $\{x_{n_{i_j}}\}$ be a subsequence of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup x^*$ as $j \to \infty$. From (29) we can say that $v_j \rightharpoonup x^*$. Moreover, by (C2), we obtain
$(1/s_j)\langle y - v_j, v_j - x_j\rangle \ge F(y, u_j),$ for all $y \in H$.

By condition (C4), for fixed $x \in H$, the function $I(x, .)$ is lower semicontinuous and convex and thus is weakly lower semicontinuous.

Since $v_j \rightharpoonup x$, by (29) and the fact that $\liminf_{n\to\infty} s_n = u > 0$, we get $(v_j - x_j)/s_j \to 0$. Letting $j \to \infty$, we have, $I(y, x^*) \le \liminf_{j\to\infty} I(y, v_j) \le 0,$ for all $y \in H$.

Substituting y by y_d where $y_d = dy + (1 - d)x^*$, $d \in [0, 1]$ and using (C1) and (C4), we get
0=$I(y_d, y_d) \le dI(y_d, y) + (1 - d)I(y_d, x^*) \le I(y_d, y)$.

Therefore, $I(dy + (1 - d)x^*, y) \ge 0,$ $d \in [0, 1]$, $y \in H$.

Letting $d \to 0^+$ and using (C3), we conclude that
$I(x^*, y) \ge 0,$ $y \in H$. Therefore, $x^* \in E(I)$. So $x^* \in Fix(B) \cap E(I)$.

This completes the proof.

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