On Some Groups of Circulant Matrices

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***Abstract*—In this paper, we present some groups of circulant matrices. Furthermore, we show that these groups are subgroups of the general linear group.**

***Index Terms*—circulant matrix, group, general linear group, subgroup**

***MSC 2010 Codes* – 05C50, 05B20, 05C25**

I. Introduction

GIVEN any finite sequence $c_0, c_1, ..., c_{n-1}$, we can form a circulant matrix. Circulant matrices have four types: the right circulant, the left circulant, the skew-right circulant and the skew-left circulant.The right and skew-right circulant matrices are Toeplitz matrices while the left and skew-left circulant matrices are Hankel matrices. A Toeplitz matrix T and Hankel matrix H take the following forms, respectively:

$$T = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ c_{-2} & c_{-1} & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-(n-2)} & c_{-(n-3} & \cdots & c_0 & c_1 \\ c_{-(n-1)} & c_{-(n-2)} & \cdots & c_{-1} & c_0 \end{pmatrix} \quad (1)$$

$$H = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \\ c_2 & c_3 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & \cdots & c_{2n-4} & c_{2n-3} \\ c_{n-1} & c_n & \cdots & c_{2n-3} & c_{2n-2} \end{pmatrix} \quad (2)$$

These two matrices are closely related in a sense that Hankel matrices are upside-down Toeplitz matrices.

The right circulant matrix, left circulant matrix, skew-right circulant matrix and skew-left circulant matrix take the following forms, respectively:

$$C_R(\vec{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix} \quad (3)$$

$$C_L(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \ldots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & \ldots & c_{n-1} & c_0 \\ c_2 & c_3 & c_4 & \ldots & c_0 & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & c_0 & \ldots & c_{n-4} & c_{n-3} \\ c_{n-1} & c_0 & c_1 & \ldots & c_{n-3} & c_{n-2} \end{pmatrix} \quad (4)$$

$$S_R(\vec{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ -c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ -c_{n-2} & -c_{n-1} & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_2 & -c_3 & \cdots & c_0 & c_1 \\ -c_1 & -c_2 & \cdots & -c_{n-1} & c_0 \end{pmatrix} \quad (5)$$

$$S_L(\vec{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & -c_0 \\ c_2 & c_3 & \cdots & -c_0 & -c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & \cdots & -c_{n-4} & -c_{n-3} \\ c_{n-1} & -c_0 & \cdots & -c_{n-3} & -c_{n-2} \end{pmatrix} \quad (6)$$

In each circulant matrix, the first row

$$\vec{c} = (c_0, c_1, ..., c_{n-1}) \in \mathbb{C}^n$$

is called the circulant vector. It determines the circulant matrix.

II. Preliminaries

In this section, we shall use $diag(c_0, c_1, \ldots, c_{n-1})$ to denote the diagonal matrix

$$\begin{pmatrix} c_0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & c_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & c_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & c_{n-2} & 0 \\ 0 & 0 & 0 & \ldots & 0 & c_{n-1} \end{pmatrix}$$

and adiag $adiag(c_0, c_1, \ldots, c_{n-1})$ to denote the anti diagonal matrix

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & c_0 \\ 0 & 0 & 0 & \ldots & c_1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{n-2} & 0 & \ldots & 0 & 0 \\ c_{n-1} & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

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Note that these matrices are not necessarily circulant.

Now, we define the Fourier matrix to establish the relationship of the circulant matrices.

Definition 2.1: The unitary matrix F_n given by

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix} \quad (7)$$

where $\omega = e^{2\pi i/n}$ is called the Fourier matrix.

The circulant matrices are related by the following equations:

$$C_R(\vec{c}) = F_n D F_n^{-1} \quad (8)$$

where $\vec{c} = (c_0, c_1, ..., c_{n-1})$ and $D = diag(d_0, d_1, ..., d_{n-1})$.

$$C_L(\vec{c}) = \Pi C_R(\vec{c}) = \Pi F_n D F_n^{-1} \quad (9)$$

where $\vec{c} = (c_0, c_1, ..., c_{n-1})$, $D = diag(d_0, d_1, ..., d_{n-1})$, $\Pi = \begin{pmatrix} 1 & \mathbb{O}_1 \\ \mathbb{O}_1^T & \tilde{I}_{n-1} \end{pmatrix}$ (an nxn matrix), $\tilde{I}_{n-1} = adiag\,(1, 1, ..., 1, 1)$ (an (n-1)x(n-1) matrix), $\mathbb{O}_1 = (0, 0, ..., 0, 0)$ (an (n-1)x1 matrix). .

$$S_R(\vec{c}) = \Delta F_n D (\Delta F_n)^* = \Delta F_n D F_n^{-1} \Delta^* \quad (10)$$

where $\vec{c} = (c_0, c_1, ..., c_{n-1})$, $D = diag(d_0, d_1, ..., d_{n-1})$, $\Delta = diag\left(1, \theta, \ldots, \theta^{n-1}\right)$, and $\theta = e^{i\pi/n}$.

$$S_L(\vec{c}) = \Sigma SRCIRC_n(\vec{c}) = \Sigma \Delta F_n D (\Delta F_n)^* \quad (11)$$

where $\vec{c} = (c_0, c_1, ..., c_{n-1})$, ..., $d_{n-1})$, $\Delta = diag\left(1, \theta, \ldots, \theta^{n-1}\right)$, $\Sigma = \begin{pmatrix} 1 & \mathbb{O}_1 \\ \mathbb{O}_1^T & -\tilde{I}_{n-1} \end{pmatrix}$, (an nxn matrix) $\tilde{I}_{n-1} = adiag\,(1, 1, ..., 1, 1)$ (an (n-1)x(n-1) matrix), $\mathbb{O}_1 = (0, 0, ..., 0, 0)$.

Remark: The matrices Π and Σ are orthonormal.

Definition 2.2: A group G is set G closed under a binary operation * that satisfy the following conditions:

1) **Associativity:** For every $a, b, c \in G$, $a*(b*c) = (a*b)*c$
2) **Existence of Identity Element:** There is $e \in G$ such that for every $a \in G$, $ea = ae = a$
3) **Existence of Inverses:** For every $a \in G$, $a^{-1} \in G$

Definition 2.3: A subset H of G is said to be a subgroup of G if it is a group under the operation * of G.

Definition 2.4: If G and G' are groups closed under tthe binary operations * and $\hat{*}$, respectively then G is said to be isomorphic to another group G' if there is a function $\phi : G \to G'$ that satisfy the following conditions:

1) **Injection:** For every $a, b \in G$, $\phi(a) = \phi(b) \to a = b$.
2) **Surjection:** $im\ G = G'$
3) **Homomorphism:** For every $a, b \in G$, $\phi(a * b) = \phi(a) \hat{*} \phi(b)$

For the rest of the paper, we will use the following notations:

- $GL_n(\mathbb{C})$:= the general linear group (the set of invertible nxn complex matrices)
- $RC_n(\mathbb{C})$:= the set of invertible nxn complex right circulant matrices
- $LC_n(\mathbb{C})$:= the set of invertible nxn complex left circulant matrices
- $RS_n(\mathbb{C})$:= the set of invertible nxn complex skew-right circulant matrices
- $LS_n(\mathbb{C})$:= the set of invertible nxn complex skew-left circulant matrices
- $C_n(\mathbb{C}) := RC_n(\mathbb{C}) \cup LC_n(\mathbb{C})$
- $S_n(\mathbb{C}) := RS_n(\mathbb{C}) \cup LS_n(\mathbb{C})$

III. Preliminary Results

The following lemmas will be used to prove the main results.

Lemma 3.1: $C_R(\vec{a})\Pi = C_L(\vec{\gamma}) \in LC_n(\mathbb{C})$ where $\vec{\gamma} = (c_0, c_{n-1}, c_{n-2}, ..., c_1)$

Proof: Note that Π is a permutation matrix and right multiplication with a permutation matrix switches the colums. In Π the column C_1 is fixed while C_2 and C_{n-1}, C_3 and C_{n-2} etc are switched. This means that after right multiplication with Π, we have

$$`S_L(\vec{\gamma}) = \begin{pmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_2 & c_1 \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_0 & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_1 & c_0 & \cdots & c_4 & c_3 \\ c_1 & c_0 & c_{n-1} & \cdots & c_3 & c_2 \end{pmatrix}$$

which is left circulant. □

Lemma 3.2: $S_R(\vec{a})\Sigma = S_L(\vec{\rho}) \in LS_n(\mathbb{C})$ where $\vec{\rho} = (c_0, -c_{n-1}, -c_{n-2}, ..., -c_1)$

Proof: Σ has the same effect as Π but once the columns are switched, they will become negative of the original. Hence, after right multiplication we have

$$S_L(\vec{\rho}) = \begin{pmatrix} c_0 & -c_{n-1} & -c_{n-2} & \cdots & -c_2 & -c_1 \\ -c_{n-1} & -c_{n-2} & -c_{n-3} & \cdots & -c_1 & -c_0 \\ -c_{n-2} & -c_{n-3} & -c_{n-4} & \cdots & -c_0 & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_2 & -c_1 & -c_0 & \cdots & c_4 & c_3 \\ -c_1 & -c_0 & c_{n-1} & \cdots & c_3 & c_2 \end{pmatrix}$$

which is skew-left circulant. □

Lemma 3.3: $C_R(\vec{a})C_R(\vec{b}) \in CR_n(\mathbb{C})$.

Proof:

$$\begin{aligned} C_R(\vec{a})C_R(\vec{b}) &= (FD_1F^{-1})(FD_2F^{-1}) \\ &= FD_1D_2F^{-1} \in CR_n(\mathbb{C}) \end{aligned}$$

□

Lemma 3.4: $C_R(\vec{a})C_L(\vec{b})$ and $C_L(\vec{b})C_R(\vec{a}) \in LC_n(\mathbb{C})$

Proof:

$$\begin{aligned} C_R(\vec{a})C_L(\vec{b}) &= C_R(\vec{a})\Pi C_R(\vec{b}) \\ &= C_L(\vec{\alpha})C_R(\vec{b}) \ where\ \vec{\alpha} = (a, a_{n-1}, ..., a_1) \\ &= \Pi C_R(\vec{\alpha})C_R(\vec{b}) \in LC_n(\mathbb{C}) \end{aligned}$$

$$C_L(\vec{b})C_R(\vec{a}) = \Pi C_R(\vec{b})C_R(\vec{a}) \in LC_n(\mathbb{C})$$

□

Lemma 3.5: $S_R(\vec{a})S_R(\vec{b}) \in RS_n(\mathbb{C})$

Proof:

$$\begin{aligned} S_R(\vec{a})S_R(\vec{b}) &= (\Delta F D_1(\Delta F)^*)(\Delta F D_2(\Delta F)^*) \\ &= \Delta F D_1 D_2(\Delta F)^* \in RS_n(\mathbb{C}) \end{aligned}$$

□

Lemma 3.6: $S_L(\vec{a})S_R(\vec{b})$ and $S_R(\vec{b})S_L(\vec{a}) \in LS_n(\mathbb{C})$

Proof:

$$S_L(\vec{a})S_R(\vec{b}) = \Sigma S_R(\vec{a})S_R(\vec{b}) \in LS_n(\mathbb{C})$$

$$\begin{aligned} S_R(\vec{b})S_L(\vec{a}) &= S_R(\vec{b})\Sigma S_R(\vec{a}) \\ &= S_L(\vec{\beta})S_R(\vec{a}) \; where \; \vec{\beta} = (b_0, -b_{n-1}, ..., -b_1) \\ &= \Sigma S_R(\vec{\beta})S_R(\vec{a}) in LS_n(\mathbb{C}) \end{aligned}$$

□

Lemma 3.7: I_n is both in $RC_n(\mathbb{C})$ and $RS_n(\mathbb{C})$.

Proof:

$$I_n = C_R(\vec{i}) = S_R(\vec{i}) \in RC_n(\mathbb{C}) \; and \; RS_n(\mathbb{C}) \; where \; \vec{i} = (1, 0, ...0)$$

□

Lemma 3.8: If A and B are both invertible, then AB is also invertible.

IV. Main Results

Theorem 4.1: $RC_n(\mathbb{C})$ is a subgroup of $C_n(\mathbb{C})$.

Proof: Using Lemmas 3.7 and 3.3 completes the proof. □

Theorem 4.2: $RS_n(\mathbb{C})$ is a subgroup of $S_n(\mathbb{C})$.

Proof: Using Lemmas 3.7 and 3.5 completes the proof. □

Theorem 4.3: $RC_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$.

Proof: Using Lemmas 3.8, 3.7 and 3.3 completes the proof. □

Theorem 4.4: $RS_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$.

Proof: Using Lemmas 3.8, 3.7, 3.3 and 3.4 completes the proof. □

Theorem 4.5: $C_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$.

Proof: Using Lemmas 3.8, 3.7 and 3.5 completes the proof. □

Theorem 4.6: $S_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$.

Proof: Using Lemmas 3.8, 3.7, 3.5 and 3.6 completes the proof. □

Note: In all the proofs of the theorem above, the associativity and existence of inverse parts were omitted because matrix multiplication is already associative and we are dealing with invertible matrices.

Theorem 4.7: $RC_n(\mathbb{C})$ is isomorphic to $RS_n(\mathbb{C})$.

Proof:

Consider the mapping from $\Gamma : RC_n(\mathbb{C}) \to RS_n(\mathbb{C})$ defined by

$$\Gamma[C_R(\vec{c})] = \Gamma[FDF^{-1}] = \Delta FDF^{-1}\Delta^* = \Delta C_R(\vec{c})\Delta^*$$

- Injection

$$\begin{aligned} \Gamma[C_R(\vec{a})] &= \Gamma[C_R(\vec{b})] \\ \Delta F D_1 F^{-1}\Delta^* &= \Delta F D_2 F^{-1}\Delta^* \\ F D_1 F^{-1} &= F D_2 F^{-1} \\ C_R(\vec{a}) &= C_R(\vec{b}) \end{aligned}$$

- Surjection

$$\begin{aligned} & im\,\Gamma \\ &= \{C_R(\vec{a}) \in RC_n(\mathbb{C}) : \Gamma[C_R(\vec{a})] \in RS_n(\mathbb{C})\} \\ &= \{C_R(\vec{a}) \in RC_n(\mathbb{C}) : \Delta C_R(\vec{a})\Delta^* \in RS_n(\mathbb{C})\} \\ &= \left\{C_R(\vec{a}) \in RC_n(\mathbb{C}) : \Delta F D_1 F^{-1}\Delta^* \in RS_n(\mathbb{C})\right\} \\ &= RS_n(\mathbb{C}) \end{aligned}$$

- Homomorphism

$$\begin{aligned} & \Gamma[C_R(\vec{a})C_R(\vec{b})] \\ &= \Delta C_R(\vec{a})C_R(\vec{b})\Delta^* \\ &= (\Delta C_R(\vec{a})\Delta^*)(\Delta C_R(\vec{b})\Delta^*) \\ &= \Gamma[C_R(\vec{a})]\Gamma[C_R(\vec{b})] \end{aligned}$$

□

V. Conclusion

In summary, we have shown that $RC_n(\mathbb{C})$ and $RS_n(\mathbb{C})$ are subgroups of $C_n(\mathbb{C})$ and $S_n(\mathbb{C})$, respectively. Furthermore, we have also shown that $RC_n(\mathbb{C}), RS_n(\mathbb{C})$,
$C_n(\mathbb{C})$ and $S_n(\mathbb{C})$ are all subgroups of $GL_n(\mathbb{C})$. Lastly, we have established an isomorphism between $RC_n(\mathbb{C})$ and $RS_n(\mathbb{C})$.

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