On Some Groups of Circulant Matrices

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Abstract—In this paper, we present some groups of circulant matrices. Furthermore, we show that these groups are subgroups of the general linear group.

Index Terms—circulant matrix, group, general linear group, subgroup

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I. INTRODUCTION

Given any finite sequence $c_0, c_1, ..., c_{n-1}$, we can form a circulant matrix. Circulant matrices have four types: the right circulant, the left circulant, the skew-right circulant and the skew-left circulant. The right and skew-right circulant matrices are Toeplitz matrices while the left and skew-left circulant matrices are Hankel matrices. A Toeplitz matrix $T$ and Hankel matrix $H$ take the following forms, respectively:

$$ T = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-(n-2)} & c_{-(n-3)} & \cdots & c_0 & c_1 \\ c_{-(n-1)} & c_{-(n-2)} & \cdots & c_{-1} & c_0 \end{pmatrix} $$

(1)

$$ H = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \\ c_2 & c_3 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & \cdots & c_{2n-4} & c_{2n-3} \\ c_{n-1} & c_n & \cdots & c_{2n-3} & c_{2n-2} \end{pmatrix} $$

(2)

These two matrices are closely related in a sense that Hankel matrices are upside-down Toeplitz matrices.

The right circulant matrix, left circulant matrix, skew-right circulant matrix and skew-left circulant matrix take the following forms, respectively:

$$ C_R(\vec{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix} $$

(3)

$$ S_R(\vec{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ -c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ -c_{n-2} & -c_{n-1} & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_2 & -c_3 & \cdots & c_0 & c_1 \\ -c_1 & -c_2 & \cdots & -c_{n-1} & c_0 \end{pmatrix} $$

(4)

$$ S_L(\vec{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & -c_0 \\ c_2 & c_3 & \cdots & -c_0 & -c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & \cdots & -c_{n-4} & -c_{n-3} \\ c_{n-1} & -c_0 & \cdots & -c_{n-3} & -c_{n-2} \end{pmatrix} $$

(5)

$$ C_L(\vec{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & \cdots & c_0 & c_1 \\ c_{n-1} & c_0 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix} $$

(6)

In each circulant matrix, the first row

$$ \vec{c} = (c_0, c_1, ..., c_{n-1}) \in \mathbb{C}^n $$

is called the circulant vector. It determines the circulant matrix.

II. PRELIMINARIES

In this section, we shall use $diag(c_0, c_1, \ldots, c_{n-1})$ to denote the diagonal matrix

$$ \begin{pmatrix} c_0 & 0 & \cdots & 0 \\ 0 & c_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{n-1} \end{pmatrix} $$

and $adiag(c_0, c_1, \ldots, c_{n-1})$ to denote the anti diagonal matrix

$$ \begin{pmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 0 & 0 & \cdots & c_1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{n-2} & \cdots & 0 & 0 \\ c_{n-1} & 0 & \cdots & 0 & 0 \end{pmatrix} $$

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Note that these matrices are not necessarily circulant. Now, we define the Fourier matrix to establish the relationship of the circulant matrices.

**Definition 2.1:** The unitary matrix $F_n$ given by

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \omega & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \ldots & \omega^{(n-1)(n-1)} \end{pmatrix} \quad (7)$$

where $\omega = e^{2\pi i/n}$ is called the Fourier matrix.

The circulant matrices are related by the following equations:

$$C_R(\vec{c}) = F_n D F_n^{-1}$$

where $\vec{c} = (c_0, c_1, ..., c_{n-1})$ and $D = diag(d_0, d_1, ..., d_{n-1})$.

$$C_L(\vec{c}) = \Pi C_R(\vec{c}) = \Pi F_n D F_n^{-1}$$

where $\vec{c} = (c_0, c_1, ..., c_{n-1})$, $D = diag(d_0, d_1, ..., d_{n-1})$,

$$\Pi = \begin{pmatrix} 1 & \Omega_1 \\ \Omega_1^T & \tilde{I}_{n-1} \end{pmatrix} \quad (an \text{ } nxn \text{ } \text{matrix}),$$

$$\tilde{I}_{n-1} = adiag(1,1,...,1,1) \quad (an \text{ } (n-1)x(n-1) \text{ } \text{matrix}), \quad \Omega_1 = (0,0,...,0,0) \quad (an \text{ } (n-1)x1 \text{ } \text{matrix}).$$

$$S_R(\vec{c}) = \Delta F_n D(\Delta F_n)^* = \Delta F_n D F_n^{-1} \Delta^*$$

where $\vec{c} = (c_0, c_1, ..., c_{n-1})$, $\Delta = diag(d_0, d_1, ..., d_{n-1})$, and $\theta = e^{i\pi/n}$.

$$S_L(\vec{c}) = \Sigma S_L C_R C_L(\vec{c}) = \Sigma \Delta F_n D(\Delta F_n)^*$$

where $\vec{c} = (c_0, c_1, ..., c_{n-1})$, $\Delta = diag(1, \theta, \ldots, \theta^{n-1})$,

$$\Sigma = \begin{pmatrix} 1 & \Omega_1 \\ \Omega_1^T & -\tilde{I}_{n-1} \end{pmatrix} \quad (an \text{ } nxn \text{ } \text{matrix}), \quad \tilde{I}_{n-1} = adiag(1,1,...,1,1) \quad (an \text{ } (n-1)x(n-1) \text{ } \text{matrix}), \quad \Omega_1 = (0,0,...,0,0).$$

**Remark:** The matrices $\Pi$ and $\Sigma$ are orthonormal.

**Definition 2.2:** A group $G$ is set $G$ closed under a binary operation $*$ that satisfy the following conditions:

1) **Associativity:** For every $a, b, c \in G$, $(a*b)*c = (a*c)*b$
2) **Existence of Identity Element:** There is $e \in G$ such that for every $a \in G$, $ea = ae = a$
3) **Existence of Inverses:** For every $a \in G$, $a^{-1} \in G$

**Definition 2.3:** A subset $H$ of $G$ is said to be a subgroup of $G$ if it is a group under the operation $*$ of $G$.

**Definition 2.4:** If $G$ and $G'$ are groups closed under the binary operations $*$ and $*$, respectively then $G$ is said to be isomorphic to another group $G'$ if there is a function $\phi : G \rightarrow G'$ that satisfy the following conditions:

1) **Injection:** For every $a, b \in G$, $\phi(a) = \phi(b) \rightarrow a = b.$
2) **Surjection:** $im \ G = G'$
3) **Homomorphism:** For every $a, b \in G$, $\phi(a*b) = \phi(a)\phi(b)$

For the rest of the paper, we will use the following notations:

- $GL_n(\mathbb{C}) :=$ the general linear group (the set of invertible n$x$n complex matrices)
- $RC_n(\mathbb{C}) :=$ the set of invertible n$x$n complex right circulant matrices
- $LC_n(\mathbb{C}) :=$ the set of invertible n$x$n complex left circulant matrices
- $RS_n(\mathbb{C}) :=$ the set of invertible n$x$n complex skew-right circulant matrices
- $LS_n(\mathbb{C}) :=$ the set of invertible n$x$n complex skew-left circulant matrices
- $C_n := RC_n(\mathbb{C}) \cup LC_n(\mathbb{C})$
- $S_n := RS_n(\mathbb{C}) \cup LS_n(\mathbb{C})$

**III. Preliminary Results**

The following lemmas will be used to prove the main results.

**Lemma 3.1:** $C_R(\tilde{a}) \Pi = C_L(\tilde{\gamma}) \in LC_n(\mathbb{C})$ where $\tilde{\gamma} = (c_0, c_{n-1}, c_{n-2}, \ldots, c_1)$

**Proof:** Note that $\Pi$ is a permutation matrix and right multiplication with a permutation matrix switches the columns. In $\Pi$ the column $C_1$ is fixed while $C_2$ and $C_{n-1}$, $C_3$ and $C_{n-2}$ etc. are switched. This means that after right multiplication with $\Pi$, we have

$$S_L(\tilde{\gamma}) = \begin{pmatrix} c_0 & c_{n-1} & c_{n-2} & \ldots & c_2 & c_1 \\ c_{n-1} & c_{n-2} & c_{n-3} & \ldots & c_0 & c_1 \\ c_{n-2} & c_{n-3} & c_{n-4} & \ldots & c_0 & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_1 & c_0 & \ldots & c_4 & c_3 \\ c_1 & c_0 & c_{n-1} & \ldots & c_3 & c_2 \end{pmatrix}$$

which is left circulant.

**Lemma 3.2:** $S_R(\tilde{\rho}) \Sigma = S_L(\tilde{\rho}) \in LS_n(\mathbb{C})$ where $\tilde{\rho} = (c_0, -c_{n-1}, -c_{n-2}, \ldots, -c_1)$

**Proof:** $\Sigma$ has the same effect as $\Pi$ but once the columns are switched, they will become negative of the original. Hence, after right multiplication we have

$$S_L(\tilde{\rho}) = \begin{pmatrix} c_0 & -c_{n-1} & -c_{n-2} & \ldots & -c_2 & -c_1 \\ -c_{n-1} & -c_{n-2} & -c_{n-3} & \ldots & -c_0 & -c_1 \\ -c_{n-2} & -c_{n-3} & -c_{n-4} & \ldots & -c_0 & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_2 & -c_1 & -c_0 & \ldots & c_4 & c_3 \\ -c_1 & -c_0 & c_{n-1} & \ldots & c_3 & c_2 \end{pmatrix}$$

which is skew-left circulant.

**Lemma 3.3:** $C_R(\tilde{a}) C_R(\tilde{b}) \in CR_n(\mathbb{C})$.

**Proof:**

$$C_R(\tilde{a}) C_R(\tilde{b}) = (FD_1 F^{-1})(FD_2 F^{-1}) = FD_1 D_2 F^{-1} \in CR_n(\mathbb{C})$$

**Lemma 3.4:** $C_R(\tilde{a}) C_L(\tilde{b})$ and $C_L(\tilde{b}) C_R(\tilde{a}) \in LC_n(\mathbb{C})$

**Proof:**

$$C_R(\tilde{a}) C_L(\tilde{b}) = C_R(\tilde{a}) \Pi C_R(\tilde{b})$$

$$= C_L(\tilde{a}) C_R(\tilde{b})$$

where $\tilde{a} = (a, a_{n-1}, \ldots, a_1)$

$$= \Pi C_R(\tilde{a}) C_R(\tilde{b}) = \Pi C_L(\tilde{a}) C_R(\tilde{b}) \in LC_n(\mathbb{C})$$
Lemma 3.5: \( S_R(\vec{a})S_R(\vec{b}) \in RS_n(\mathbb{C}) \)

Proof:
\[
S_R(\vec{a})S_R(\vec{b}) = (\Delta FD_1(\Delta F)^*)(\Delta FD_2(\Delta F)^*) = \Delta FD_1D_2(\Delta F)^* \in RS_n(\mathbb{C})
\]

Lemma 3.6: \( S_L(\vec{a})S_L(\vec{b}) \) and \( S_R(\vec{b})S_L(\vec{a}) \in LS_n(\mathbb{C}) \)

Proof:
\[
S_L(\vec{a})S_R(\vec{b}) = \Sigma S_R(\vec{a})S_R(\vec{b}) \in LS_n(\mathbb{C})
\]

Lemma 3.7: \( I_n \) is both in \( RC_n(\mathbb{C}) \) and \( RS_n(\mathbb{C}) \).

Proof:
\[
I_n = C_R(\vec{i}) = S_R(\vec{i}) \in RC_n(\mathbb{C}) \text{ and } RS_n(\mathbb{C}) \text{ where } \vec{i} = (1, 0, ... 0)
\]

Lemma 3.8: If \( A \) and \( B \) are both invertible, then \( AB \) is also invertible.

IV. MAIN RESULTS

Theorem 4.1: \( RC_n(\mathbb{C}) \) is a subgroup of \( C_n(\mathbb{C}) \).

Proof: Using Lemmas 3.7 and 3.3 completes the proof.

Theorem 4.2: \( RS_n(\mathbb{C}) \) is a subgroup of \( S_n(\mathbb{C}) \).

Proof: Using Lemmas 3.7 and 3.5 completes the proof.

Theorem 4.3: \( RC_n(\mathbb{C}) \) is a subgroup of \( GL_n(\mathbb{C}) \).

Proof: Using Lemmas 3.8, 3.7 and 3.3 completes the proof.

Theorem 4.4: \( RS_n(\mathbb{C}) \) is a subgroup of \( GL_n(\mathbb{C}) \).

Proof: Using Lemmas 3.8, 3.7, 3.3 and 3.4 completes the proof.

Theorem 4.5: \( C_n(\mathbb{C}) \) is a subgroup of \( GL_n(\mathbb{C}) \).

Proof: Using Lemmas 3.8, 3.7 and 3.5 completes the proof.

Theorem 4.6: \( S_n(\mathbb{C}) \) is a subgroup of \( GL_n(\mathbb{C}) \).

Proof: Using Lemmas 3.8, 3.7, 3.5 and 3.6 completes the proof.

Note: In all the proofs of the theorem above, the associativity and existence of inverse parts were omitted because matrix multiplication is already associative and we are dealing with invertible matrices.

Theorem 4.7: \( RC_n(\mathbb{C}) \) is isomorphic to \( RS_n(\mathbb{C}) \).

Proof: Consider the mapping from \( \Gamma : RC_n(\mathbb{C}) \to RS_n(\mathbb{C}) \) defined by
\[
\Gamma[C_R(\vec{c})] = \Gamma[FD_F^{-1}] = \Delta FD_F^{-1}\Delta^* = \Delta C_R(\vec{c})\Delta^*
\]