On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order

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Abstract— **In this paper, explicit formulas were derived for Fibonacci-like sequence of higher orders, specifically, order four, five, and six. Formulas were validated in any value of n using induction. This is an extension of work of Natividad and Policarpio on Fibonacci-like sequence of third order [2].**

Index Terms— **Fibonacci-like sequence of higher order, tetranacci, pentanacci, hexanacci**

MSC 2010 Codes **— 11B39, 11B99**

I. INTRODUCTION

Generalizations of Fibonacci sequence or Gibonacci sequences have been extensively studied in many ways. As mentioned by Waddill [3], Horadam stated that these sequences can be generalized in two ways: by giving arbitrary initial terms and by altering the recurrence relation. Fibonacci-like, tetranacci, pentanacci and so on are some names of these generalizations.

Tribonacci (order 3), tetranacci (order 4), pentanacci (order 5) sequences and among others are also called Fibonacci sequences of higher orders. When the initial terms are become arbitrary for the recurrence relation, some authors change their names by adding "like" to its original name e.g. Fibonacci-like, tribonacci-like, tetranacci-like… In this case, we called them as Fibonacci-like sequence of higher order.

There are many studies for Fibonacci sequence of order 4 like that of generalizations of Waddill[3]. He provided many interesting properties for this sequence mainly on its linear sum identities. Some initial values of Tetranacci sequence are 0, 0, 0, 1, 1, 2, 4, 8, 15, 29, 56…. (Sloane's A000078).

Definition 1.1 *The sequence* $Z_1, Z_2, Z_3, Z_4, \dots, Z_n$ *in which* $Z_n = Z_{n-4} + Z_{n-3} + Z_{n-2} + Z_{n-1}$ *is a generalized Fibonacci-like sequence of fourth order (tetranacci-like sequence). This sequence follows the pattern of Fibonacci sequence of fourth order (tetranacci sequence).*

However, we noted that there are only some studies concerning pentanacci and hexanacci sequences. Mendelsohn [1] derived some identities for pentanacci numbers and defined the sequences extensively. The first few terms for pentanacci sequence are 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 31… (Sloane's A001591).

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Definition 1.2 *The sequence* $Q_1, Q_2, Q_3, Q_4, Q_5, \dots, Q_n$ *in which* $Q_n = Q_{n-5} + Q_{n-4} + Q_{n-3} + Q_{n-2} + Q_{n-1}$ *is a generalized Fibonacci-like sequence of fifth order (pentanacci-like sequence). This sequence follows the pattern of Fibonacci sequence of fifth order (pentanacci sequence).*

First few terms for hexanacci sequence are 0, 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 32,…. (Sloane's A001592).

Definition 1.3 *The sequence* $I_1, I_2, I_3, I_4, I_5, I_6, \dots, I_n$ *in which* $I_n = I_{n-6} + I_{n-5} + I_{n-4} + I_{n-3} + I_{n-2} + I_{n-1}$ *is a generalized Fibonacci-like sequence of sixth order (hexanacci-like sequence). This sequence follows the pattern of Fibonacci sequence of sixth order (hexanacci sequence).*

In [2], Natividad and Policarpio provided and proved a formula for $n \geq 4$ concerning Fibonacci-like sequence of order 3. They found out that given the arbitrary terms S_1, S_2, and S_3, the formula to find the n^{th} term or S_n is

$$\boldsymbol{S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-1}S_3}$$

In this paper, we will attempt to extend the work of Natividad and Policarpio [2] on Fibonacci-like sequences of order 4, 5, and 6.

II. MAIN RESULT

From above discussions and preliminaries, the following theorems are proposed and proved.

Theorem 2.1 *For any real numbers* $Z_1, Z_2, Z_3,$ *and* Z_4, *the formula for finding the* n^{th} *term of generalized Fibonacci-like sequence of fourth order (tetranacci-like sequence) is*

$$\boldsymbol{Z_n = Y_{n-2}Z_1 + (Y_{n-2} + Y_{n-3})Z_2 + (Y_{n-2} + Y_{n-3} + Y_{n-4})Z_3 + Y_{n-1}Z_4}$$

where Z_n *is the* n^{th} *term of tetranacci-like sequence,* Z_1 *is the first term,* Z_2 *is the second term,* Z_3 *is the third term,* Z_4 *is the fourth term and* Y_{n-1}, Y_{n-2}, Y_{n-3}, Y_{n-4} *are the corresponding Fibonacci numbers of fourth order (tetranacci numbers).*

Proof. We begin by computing numerical coefficients for the first four terms of the sequence. Equations were derived and coefficients were listed on the chart for $5 \leq n \leq 10$:

n	Z_1	Z_2	Z_3	Z_4
5	1	1	1	1
6	1	2	2	2
7	2	3	4	4
8	4	6	7	8
9	8	12	14	15
10	15	23	27	29

Each coefficient corresponds to the tetranacci numbers. We note that the coefficient of Z_1 corresponds to Y_{n-2}, Z_2 corresponds to $Y_{n-2} + Y_{n-3}$. So, we can conclude that the n^{th} term (Z_n) is equal to $Y_{n-2}Z_1 + (Y_{n-2} + Y_{n-3})Z_2 + (Y_{n-2} + Y_{n-3} + Y_{n-4})Z_3 + Y_{n-1}Z_4$ completing the proof. □

By using mathematical induction, the formula can be validated in any values of n. The formula can be easily verified using n = 5, 6, 7 and so on. Let P(n) be taken as

$$Z_n = Y_{n-2}Z_1 + (Y_{n-2} + Y_{n-3})Z_2 + (Y_{n-2} + Y_{n-3} + Y_{n-4})Z_3 + Y_{n-1}Z_4$$

then P(k) is

$$Z_k = Y_{k-2}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-1}Z_4.$$

It also follows that P(k+1) is

$$Z_{k+1} = Y_{k-1}Z_1 + (Y_{k-1} + Y_{k-2})Z_2 + (Y_{k-1} + Y_{k-2} + Y_{k-3})Z_3 + Y_kZ_4.$$

Now to verify, we will provide the assumption of P(k) implies the truth of P(k+1). To do so, we will add Z_{k-1}, Z_{k-2} and Z_{k-3} to both sides of P(k). The equation will become

$$Z_{k-1} + Z_{k-2} + Z_{k-3} + Z_k = Y_{k-2}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-1}Z_4 + Z_{k-1} + Z_{k-2} + Z_{k-3}$$

But since $Z_{k-1} = Y_{k-3}Z_1 + (Y_{k-3} + Y_{k-4})Z_2 + (Y_{k-3} + Y_{k-4} + Y_{k-5})Z_3 + Y_{k-2}Z_4$,

$Z_{k-2} = Y_{k-4}Z_1 + (Y_{k-4} + Y_{k-5})Z_2 + (Y_{k-4} + Y_{k-5} + Y_{k-6})Z_3 + Y_{k-3}Z_4$,

$Z_{k-3} = Y_{k-5}Z_1 + (Y_{k-5} + Y_{k-6})Z_2 + (Y_{k-5} + Y_{k-6} + Y_{k-7})Z_3 + Y_{k-4}Z_4$,

$Z_{k+1} = Z_{k-1} + Z_{k-2} + Z_{k-3} + Z_k$

$$Z_{k-1} + Z_{k-2} + Z_{k-3} + Z_k = Y_{k-2}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-1}Z_4 + Z_{k-1} + Z_{k-2} + Z_{k-3}$$

$$= Y_{k-2}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-1}Z_4 + Y_{k-3}Z_1 + (Y_{k-3} + Y_{k-4})Z_2 + (Y_{k-3} + Y_{k-4} + Y_{k-5})Z_3 + Y_{k-2}Z_4 + Y_{k-4}Z_1 + (Y_{k-4} + Y_{k-5})Z_2 + (Y_{k-4} + Y_{k-5} + Y_{k-6})Z_3 + Y_{k-3}Z_4 + Y_{k-5}Z_1 + (Y_{k-5} + Y_{k-6})Z_2 + (Y_{k-5} + Y_{k-6} + Y_{k-7})Z_3 + Y_{k-4}Z_4$$

$$= Y_{k-2}Z_1 + Y_{k-3}Z_1 + Y_{k-4}Z_1 + Y_{k-5}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-3} + Y_{k-4})Z_2 + (Y_{k-4} + Y_{k-5})Z_2 + (Y_{k-5} + Y_{k-6})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + (Y_{k-3} + Y_{k-4} + Y_{k-5})Z_3 + (Y_{k-4} + Y_{k-5} + Y_{k-6})Z_3 + (Y_{k-5} + Y_{k-6} + Y_{k-7})Z_3 + Y_{k-4}Z_4 + Y_{k-3}Z_4 + Y_{k-2}Z_4 + Y_{k-1}Z_4$$

$$= (Y_{k-2} + Y_{k-3} + Y_{k-4} + Y_{k-5})Z_1 + [(Y_{k-2} + Y_{k-3} + Y_{k-4} + Y_{k-5}) + (Y_{k-3} + Y_{k-4} + Y_{k-5} + Y_{k-6})]Z_2 + [(Y_{k-2} + Y_{k-3} + Y_{k-4} + Y_{k-5}) + (Y_{k-3} + Y_{k-4} + Y_{k-5} + Y_{k-6}) + (Y_{k-4} + Y_{k-5} + Y_{k-6} + Y_{k-7})]Z_3 + (Y_{k-1} + Y_{k-2} + Y_{k-3} + Y_{k-4})Z_4$$

$$Z_{k+1} = Y_{k-1}Z_1 + (Y_{k-1} + Y_{k-2})Z_2 + (Y_{k-1} + Y_{k-2} + Y_{k-3})Z_3 + Y_kZ_4$$

The resulting equation is exactly our P(k+1), hence, the formula is valid for any value of n.

Theorem 2.2 *For any real numbers Q_1, Q_2, Q_3, Q_4 and Q_5, the formula for finding the n^{th} term of generalized Fibonacci-like sequence of fifth order (pentanacci-like sequence) is*

$$\boldsymbol{Q_n = P_{n-2}Q_1 + (P_{n-2} + P_{n-3})Q_2 + (P_{n-2} + P_{n-3} + P_{n-4})Q_3 + (P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5})Q_4 + P_{n-1}Q_5}$$

where Q_n is the n^{th} term of pentanacci-like sequence, Q_1 is the first term, Q_2 is the second term, Q_3 is the third term, Q_4 is the fourth term, Q_5 is the fifth term and P_{n-1}, P_{n-2}, P_{n-3}, P_{n-4}, P_{n-5} are the corresponding Fibonacci numbers of fifth order (pentanacci numbers).

Proof. For any $Q_1, Q_2, Q_3, Q_4, Q_5 \in \mathbb{R}$, the numerical coefficients in solving Q_n for $6 \le n \le 11$ are

n	Q_1	Q_2	Q_3	Q_4	Q_5
6	1	1	1	1	1
7	1	2	2	2	2
8	2	3	4	4	4
9	4	6	7	8	8
10	8	12	14	15	16
11	16	24	28	30	31

The coefficient corresponds to tetranacci number (P_n) resulting to the equation above. □

Just like we did in previous discussions, the formula will be validated in any values of n using mathematical induction. Again, the formula can be easily verified using n = 6, 7, 8 and so on. Let P(n) be taken as

$$Q_n = P_{n-2}Q_1 + (P_{n-2} + P_{n-3})Q_2 + (P_{n-2} + P_{n-3} + P_{n-4})Q_3 + (P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5})Q_4 + P_{n-1}Q_5$$

then P(k) is

$$Q_k = P_{k-2}Q_1 + (P_{k-2} + P_{k-3})Q_2 + (P_{k-2} + P_{k-3} + P_{k-4})Q_3 + (P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5})Q_4 + P_{k-1}Q_5$$

The equation for P(k+1) is

$$Q_{k+1} = P_{k-1}Q_1 + (P_{k-1} + P_{k-2})Q_2 + (P_{k-1} + P_{k-2} + P_{k-3})Q_3 + (P_{k-1} + P_{k-2} + P_{k-3} + P_{k-4})Q_4 + P_kQ_5$$

Adding terms to both sides of equations, substituting and rearranging the terms

$$Q_{k-4} + Q_{k-3} + Q_{k-2} + Q_{k-1} + Q_k = P_{k-2}Q_1 + (P_{k-2} + P_{k-3})Q_2 + (P_{k-2} + P_{k-3} + P_{k-4})Q_3 + (P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5})Q_4 + P_{k-1}Q_5 + Q_{k-4} + Q_{k-3} + Q_{k-2} + Q_{k-1} + Q_k$$

$$Q_{k+1} = P_{k-2}Q_1 + (P_{k-2} + P_{k-3})Q_2 + (P_{k-2} + P_{k-3} + P_{k-4})Q_3 + (P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5})Q_4 + P_{k-1}Q_5 + P_{k-6}Q_1 + (P_{k-6} + P_{k-7})Q_2 + (P_{k-6} + P_{k-7} + P_{k-8})Q_3 + (P_{k-6} + P_{k-7} + P_{k-8} + P_{k-9})Q_4 + P_{k-5}Q_5 + P_{k-5}Q_1 + (P_{k-5} + P_{k-6})Q_2 + (P_{k-5} + P_{k-6} + P_{k-7})Q_3 + (P_{k-5} + P_{k-6} + P_{k-7} + P_{k-8})Q_4 + P_{k-4}Q_5 + P_{k-4}Q_1 + (P_{k-4} + P_{k-5})Q_2 + (P_{k-4} + P_{k-5} + P_{k-6})Q_3 + (P_{k-4} + P_{k-5} + P_{k-6} + P_{k-7})Q_4 + P_{k-3}Q_5 + P_{k-3}Q_1 + (P_{k-3} + P_{k-4})Q_2 + (P_{k-3} + P_{k-4} + P_{k-5})Q_3 + (P_{k-3} + P_{k-4} + P_{k-5} + P_{k-6})Q_4 + P_{k-2}Q_5$$

$$= (P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5} + P_{k-6})Q_1 + [(P_{k-6} + P_{k-5} + P_{k-4} + P_{k-3} + P_{k-2}) + (P_{k-7} + P_{k-6} + P_{k-5} + P_{k-4} + P_{k-3})]Q_2 + [(P_{k-6} + P_{k-5} + P_{k-4} + P_{k-3} + P_{k-2}) + (P_{k-7} + P_{k-6} + P_{k-5} + P_{k-4} + P_{k-3}) + (P_{k-8} + P_{k-7} + P_{k-6} + P_{k-5} + P_{k-4})]Q_3 + [(P_{k-6} + P_{k-5} + P_{k-4} + P_{k-3} + P_{k-2}) + (P_{k-7} + P_{k-6} + P_{k-5} + P_{k-4} + P_{k-3}) + (P_{k-8} + P_{k-7} + P_{k-6} + P_{k-5} + P_{k-4}) + (P_{k-9} + P_{k-8} + P_{k-7} + P_{k-6} + P_{k-5})]Q_4 + (P_{k-5} + P_{k-4} + P_{k-3} + P_{k-2} + P_{k-1})Q_5$$

$$Q_{k+1} = P_{k-1}Q_1 + (P_{k-1} + P_{k-2})Q_2 + (P_{k-1} + P_{k-2} + P_{k-3})Q_3 + (P_{k-1} + P_{k-2} + P_{k-3} + P_{k-4})Q_4 + P_kQ_5$$

Based on strong mathematical induction, conclusion follows.

Theorem 2.3 *For any real numbers I_1, I_2, I_3, I_4, I_5 and I_6, the formula for finding the n^{th} term of generalized Fibonacci-like sequence of sixth order (hexanacci-like sequence) is*

$$\boldsymbol{I_n = H_{n-2}I_1 + (H_{n-2} + H_{n-3})I_2 + (H_{n-2} + H_{n-3} + H_{n-4})I_3 + (H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5})I_4 + (H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6})I_5 + H_{n-1}I_6}$$

where I_n is the n^{th} term of hexanacci-like sequence, I_1 is the first term, I_2 is the second term, I_3 is the third term, I_4 is the fourth term, I_5 is the fifth term, I_6 is the sixth term and H_{n-1}, H_{n-2}, H_{n-3}, H_{n-4}, H_{n-5}, H_{n-6} are the corresponding Fibonacci numbers of sixth order (hexanacci numbers).

The proof for this formula is just like the proof for Theorem 1 and 2. The formula has been validated for any values of n using induction just like we did in the last discussions and the details of the proof will be left to the readers.

III. CONCLUSION

We derived an explicit formula for the Fibonacci-like sequence of the fourth, fifth and sixth order. The way this formula has been derived may be extended to the other recursive sequences.

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