On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order

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Abstract—In this paper, explicit formulas were derived for Fibonacci-like sequence of higher orders, specifically, order four, five, and six. Formulas were validated in any value of n using induction. This is an extension of work of Natividad and Policarpio on Fibonacci-like sequence of third order [2].

Index Terms—Fibonacci-like sequence of higher order, tetranacci, pentanacci, hexanacci

MSC 2010 Codes—11B39, 11B99

I. INTRODUCTION

Generalizations of Fibonacci sequence or Gibonacci sequences have been extensively studied in many ways. As mentioned by Waddill [3], Horadam stated that these sequences can be generalized in two ways: by giving arbitrary initial terms and by altering the recurrence relation. Fibonacci-like, tetranacci, pentanacci and so on are some names of these generalizations.

Trinonacci (order 3), tetranacci (order 4), pentanacci (order 5) sequences and among others are also called Fibonacci sequences of higher orders. When the initial terms are become arbitrary for the recurrence relation, some authors change their names by adding “like” to its original name e.g. Fibonacci-like, trinonacalike, tetranacci-like… In this case, we called them as Fibonacci-like sequence of higher order.

There are many studies for Fibonacci sequence of order 4 like that of generalizations of Waddill[3]. He provided many interesting properties for this sequence mainly on its linear sum identities. Some initial values of Tetrannacii sequence are 0, 0, 0, 1, 1, 2, 4, 8, 15, 29, 56…. (Sloane’s A000078).

Definition 1.1 The sequence \( Z_1, Z_2, Z_3, Z_4, ..., Z_n \) in which \( Z_n = Z_{n-4} + Z_{n-3} + Z_{n-2} + Z_{n-1} \) is a generalized Fibonacci-like sequence of fourth order (tetranacci-like sequence). This sequence follows the pattern of Fibonacci sequence of fourth order (tetranacci sequence).

However, we noted that there are only some studies concerning pentanacci and hexanacci sequences. Mendelsohn [1] derived some identities for pentanacci numbers and defined the sequences extensively. The first few terms for pentanacci sequence are 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 31… (Sloane’s A001591).

Definition 1.2 The sequence \( Q_1, Q_2, Q_3, Q_4, ..., Q_n \) in which \( Q_n = Q_{n-5} + Q_{n-4} + Q_{n-3} + Q_{n-2} + Q_{n-1} \) is a generalized Fibonacci-like sequence of fifth order (pentanacci-like sequence). This sequence follows the pattern of Fibonacci sequence of fifth order (pentanacci sequence).

First few terms for hexanacci sequence are 0, 0, 0, 0, 0, 1, 1, 2, 4, 8, 16, 32,… (Sloane’s A000078).

Definition 1.3 The sequence \( I_1, I_2, I_3, I_4, I_5, I_6, ..., I_n \) in which \( I_n = I_{n-6} + I_{n-5} + I_{n-4} + I_{n-3} + I_{n-2} + I_{n-1} \) is a generalized Fibonacci-like sequence of sixth order (hexanacci-like sequence). This sequence follows the pattern of Fibonacci sequence of sixth order (hexanacci sequence).

In [2], Natividad and Policarpio provided and proved a formula for \( n \geq 4 \) concerning Fibonacci-like sequence of order 3. They found out that given the arbitrary terms \( S_1, S_2, \) and \( S_3, \) the formula to find the \( n^{th} \) term or \( S_n \) is

\[
S_n = T_{n-2}S_1 + (T_{n-2} + T_{n-3})S_2 + T_{n-4}S_3
\]

In this paper, we will attempt to extend the work of Natividad and Policarpio [2] on Fibonacci-like sequences of order 4, 5, and 6.

II. MAIN RESULT

From above discussions and preliminaries, the following theorems are proposed and proved.

Theorem 2.1 For any real numbers \( Z_1, Z_2, Z_3, \) and \( Z_4, \) the formula for finding the \( n^{th} \) term of generalized Fibonacci-like sequence of fourth order (tetranacci-like sequence) is

\[
Z_n = Y_{n-2}Z_1 + (Y_{n-2} + Y_{n-3})Z_2 + (Y_{n-2} + Y_{n-3} + Y_{n-4})Z_3 + Y_{n-1}Z_4
\]

where \( Z_n \) is the \( n^{th} \) term of tetranacci-like sequence, \( Z_i \) is the first term, \( Z_2 \) is the second term, \( Z_3 \) is the third term, \( Z_4 \) is the fourth term and \( Y_{n,1}, Y_{n,2}, Y_{n,3}, Y_{n,4} \) are the corresponding Fibonacci numbers of fourth order (tetranacci numbers).

Proof. We begin by computing numerical coefficients for the first four terms of the sequence. Equations were derived and coefficients were listed on the chart for \( 5 \leq n \leq 10.\)
Each coefficient corresponds to the tetranacci numbers. We note that the coefficient of $Z_1$ corresponds to $Y_{n-3}$, $Z_2$ corresponds to $Y_{n-2} + Y_{n-3}$. So, we can conclude that the $n^{th}$ term $(Z_n)$ is equal to $Y_{n-2}Z_1 + (Y_{n-2} + Y_{n-3})Z_2 + (Y_{n-2} + Y_{n-3} + Y_{n-4})Z_3 + Y_{n-1}Z_4$ completing the proof.

By using mathematical induction, the formula can be validated in any values of $n$. The formula can be easily verified using $n = 5, 6, 7$ and so on. Let $P(n)$ be taken as

\[ Z_n = Y_{n-2}Z_1 + (Y_{n-2} + Y_{n-3})Z_2 + (Y_{n-2} + Y_{n-3} + Y_{n-4})Z_3 + Y_{n-1}Z_4 \]

then $P(k)$ is

\[ Z_k = Y_{k-2}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-1}Z_4. \]

It also follows that $P(k+1)$ is

\[ Z_{k+1} = Y_{k-1}Z_1 + (Y_{k-1} + Y_{k-2})Z_2 + (Y_{k-1} + Y_{k-2} + Y_{k-3})Z_3 + Y_kZ_4. \]

Now to verify, we will provide the assumption of $P(k)$ implies the truth of $P(k+1)$. To do so, we will add $Z_{k-1}, Z_{k-2}$ and $Z_{k-3}$ to both sides of $P(k)$. The equation will become

\[ Z_{k-1} + Z_{k-2} + Z_{k-3} + Z_k = Y_{k-2}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-1}Z_4 + Z_{k-1} + Z_{k-2} + Z_{k-3} + Z_k \]

But since $Z_{k-1} = Y_{k-3}Z_1 + (Y_{k-3} + Y_{k-4})Z_2 + (Y_{k-3} + Y_{k-4} + Y_{k-5})Z_3 + Y_{k-4}Z_4$, then

\[ Z_{k-2} = Y_{k-4}Z_1 + (Y_{k-4} + Y_{k-5})Z_2 + (Y_{k-4} + Y_{k-5} + Y_{k-6})Z_3 + Y_{k-5}Z_4, \]

\[ Z_{k-3} = Y_{k-5}Z_1 + (Y_{k-5} + Y_{k-6})Z_2 + (Y_{k-5} + Y_{k-6} + Y_{k-7})Z_3 + Y_{k-6}Z_4, \]

\[ Z_{k+1} = Z_{k-1} + Z_{k-2} + Z_{k-3} + Z_k \]

\[ Z_{k-1} + Z_{k-2} + Z_{k-3} + Z_k = Y_{k-2}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-1}Z_4 + Z_{k-1} + Z_{k-2} + Z_{k-3} + Z_k \]

\[ Y_{k-2}Z_1 + (Y_{k-2} + Y_{k-3})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-1}Z_4 \]

\[ = (Y_{k-2} + Y_{k-3} + Y_{k-4})Z_3 + Y_{k-2}Z_4 + Y_{k-3}Z_4 + Y_{k-2}Z_4 + Y_{k-1}Z_4, \]

\[ = (Y_{k-2} + Y_{k-3} + Y_{k-4} + Y_{k-5})Z_1 + (Y_{k-2} + Y_{k-3} + Y_{k-4} + Y_{k-5} + Y_{k-6})Z_2 + (Y_{k-2} + Y_{k-3} + Y_{k-4} + Y_{k-5} + Y_{k-6} + Y_{k-7})Z_3 + (Y_{k-2} + Y_{k-3} + Y_{k-4} + Y_{k-5} + Y_{k-6} + Y_{k-7} + Y_{k-8})Z_4 \]

\[ = (Y_{k-2} + Y_{k-3} + Y_{k-4} + Y_{k-5} + Y_{k-6} + Y_{k-7} + Y_{k-8})Z_4 \]

The resulting equation is exactly our $P(k+1)$, hence, the formula is valid for any value of $n$.

**Theorem 2.2** For any real numbers $Q_1, Q_2, Q_3, Q_4$ and $Q_5$, the formula for finding the $n^{th}$ term of generalized Fibonacci-like sequence of fifth order (pentanacci-like sequence) is

\[ Q_n = P_{n-2}Q_1 + (P_{n-2} + P_{n-3})Q_2 + (P_{n-2} + P_{n-3} + P_{n-4})Q_3 + (P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5})Q_4 + P_{n-1}Q_5 \]

where $Q_n$ is the $n^{th}$ term of pentanacci-like sequence, $Q_1$ is the first term, $Q_2$ is the second term, $Q_3$ is the third term, $Q_4$ is the fourth term, $Q_5$ is the fifth term and $P_{n-1}, P_{n-2}, P_{n-3}, P_{n-4}, P_{n-5}$ are the corresponding Fibonacci numbers of fifth order (pentanacci numbers).

**Proof.** For any $Q_1, Q_2, Q_3, Q_4, Q_5 \in \mathbb{R}$, the numerical coefficients in solving $Q_n$ for $6 \leq n \leq 11$ are

<table>
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<tr>
<th>$n$</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$Q_3$</th>
<th>$Q_4$</th>
<th>$Q_5$</th>
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<tbody>
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<td>6</td>
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</tbody>
</table>

The coefficient corresponds to tetranacci number ($P_n$) resulting to the equation above.

Just like we did in previous discussions, the formula will be validated in any values of $n$ using mathematical induction. Again, the formula can be easily verified using $n = 6, 7, 8$ and so on. Let $P(n)$ be taken as

\[ Q_n = P_{n-2}Q_1 + (P_{n-2} + P_{n-3})Q_2 + (P_{n-2} + P_{n-3} + P_{n-4})Q_3 + (P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5})Q_4 + P_{n-1}Q_5 \]

then $P(k)$ is

\[ Q_k = P_{k-2}Q_1 + (P_{k-2} + P_{k-3})Q_2 + (P_{k-2} + P_{k-3} + P_{k-4})Q_3 + (P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5})Q_4 + P_{k-1}Q_5 \]

The equation for $P(k+1)$ is

\[ Q_{k+1} = P_{k-1}Q_1 + (P_{k-1} + P_{k-2})Q_2 + (P_{k-1} + P_{k-2} + P_{k-3})Q_3 + (P_{k-1} + P_{k-2} + P_{k-3} + P_{k-4})Q_4 + P_{k-1}Q_5 \]
Adding terms to both sides of equations, substituting and rearranging the terms

\[ Q_{k-4} + Q_{k-3} + Q_{k-2} + Q_{k-1} + Q_k = P_{k-2}Q_1 + (P_{k-2} + P_{k-3})Q_2 + (P_{k-2} + P_{k-3} + P_{k-4})Q_3 + (P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5})Q_4 + P_{k-4} + P_{k-5}Q_5 + Q_{k-3} + Q_{k-2} + Q_{k-1} + Q_k \]

\[ Q_{k+1} = P_{k-2}Q_1 + (P_{k-2} + P_{k-3})Q_2 + (P_{k-2} + P_{k-3} + P_{k-4})Q_3 + P_{k-4}Q_4 + P_{k-5}Q_5 + (P_{k-2} + P_{k-3})Q_2 + (P_{k-2} + P_{k-3} + P_{k-4} + P_{k-5})Q_4 + P_{k-3}Q_5 + P_{k-4}Q_4 + P_{k-5}Q_3 \]

Adding terms to both sides of equations, substituting and rearranging the terms

\[ Q_{k+1} = P_{k-2}Q_1 + (P_{k-2} + P_{k-3})Q_2 + (P_{k-2} + P_{k-3} + P_{k-4})Q_3 + P_{k-4}Q_4 + P_{k-5}Q_5 + P_{k-3}Q_6 \]

We derived an explicit formula for the Fibonacci-like sequence of the fourth, fifth and sixth order. The way this formula has been derived may be extended to the other recursive sequences.

**REFERENCES**


III. CONCLUSION

Theorem 2.3 For any real numbers \( I_1, I_2, I_3, I_4, I_5 \) and \( I_6 \), the formula for finding the \( n^{th} \) term of generalized Fibonacci-like sequence of sixth order (hexanacci-like sequence) is

\[ I_n = (H_{n-2}I_1 + (H_{n-2} + H_{n-3})I_2 + (H_{n-2} + H_{n-3} + H_{n-4})I_3 + (H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5})I_4 + (H_{n-2} + H_{n-3} + H_{n-4} + H_{n-5} + H_{n-6})I_5 + H_{n-1}I_6 \]

where \( I_n \) is the \( n^{th} \) term of hexanacci-like sequence, \( I_1 \) is the first term, \( I_2 \) is the second term, \( I_3 \) is the third term, \( I_4 \) is the fourth term, \( I_5 \) is the fifth term, \( I_6 \) is the sixth term and \( H_{n-1}, H_{n-2}, H_{n-3}, H_{n-4}, H_{n-5}, H_{n-6} \) are the corresponding Fibonacci numbers of sixth order (hexanacci numbers).

The proof for this formula is just like the proof for Theorem 1 and 2. The formula has been validated for any values of \( n \) using induction just like we did in the last discussions and the details of the proof will be left to the readers.