A Note on (k,h)-Jacobsthal Sequence

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***Abstract*—In this paper, we define the (k,h)-Jacobsthal sequence. Furthermore, we derive the formula for the n^{th} term and the sum of the first n terms of this sequence.**

***Index Terms*—Jabobsthal sequence, (k,h)-Jacobsthal sequence, n^{th} term of a sequence, sum of the first n terms of a sequence**

***MSC 2010 Codes* – 11B39, 11B50**

I. INTRODUCTION

THE numbers in the Jacobsthal sequence $\{J_n\}_{n=0}^{\infty}$ satisfy the recurrence relation given by

$$J_n = J_{n-1} + 2J_{n-2} \quad (1)$$

with initial values $J_0 = 0$ and $J_1 = 1$.

In [1], some generalizations on $\{J_n\}_{n=0}^{+\infty}$ were presented together with some identities. Now we give another generalization of the Jacobsthal sequence and it will be called the (k,h)-Jacobsthal sequence denoted by $\{T_n\}_{n=0}^{+\infty}$. The numbers in this sequence satisfy the recurrence relation

$$T_n = kT_{n-1} + 2hT_{n-2} \quad (2)$$

with initial values $T_0 = 0$ and $T_1 = k$ where $k, h \in \mathbb{Z}$, k and h are not both zero and $k^2 + 8h > 0$.

II. MAIN RESULTS

Theorem 2.1:

$$T_n = \frac{k\left(\alpha^n - \beta^n\right)}{p} \quad (3)$$

where

$$\alpha = \frac{k + \sqrt{k^2 + 8h}}{2},$$
$$\beta = \frac{k - \sqrt{k^2 + 8h}}{2},$$
$$k = \alpha + \beta,$$

and

$$p = \alpha - \beta.$$

Proof:

The recurrence relation of the sequence has the characteristic equation

$$x^2 - kx - 2h = 0$$

whose roots are

$$\alpha = \frac{k + \sqrt{k^2 + 8h}}{2}$$

and

$$\beta = \frac{k - \sqrt{k^2 + 8h}}{2}$$

. This means that

$$T_n = c_1\alpha^n + c_2\beta^n$$

Using the initial values $T_0 = 0$ and $T_1 = k$ leads to the linear system

$$c_1 + c_2 = 0$$
$$c_1\alpha + c_2\beta = k$$

whose solutions are $c_1 = \frac{k}{\alpha-\beta} = \frac{k}{p}$ and $c_2 = -\frac{k}{\alpha-\beta} = -\frac{k}{p}$. Hence

$$T_n = \frac{k\left(\alpha^n - \beta^n\right)}{p}$$

Theorem 2.2:

$$\sum_{m=0}^{n-1} T_m = \frac{T_n + 2kT_{n-1} - k}{2h + k - 1} \quad (4)$$

Proof:

$$\begin{aligned}
\sum_{m=0}^{n-1} T_m &= \frac{k}{p}\sum_{m=0}^{n-1}\left[\alpha^m - \beta^m\right] \\
&= \frac{k}{p}\left[\frac{1-\alpha^n}{1-\alpha} - \frac{1-\beta^n}{1-\beta}\right] \\
&= \frac{k}{p}\left[\frac{(1-\alpha^n)(1-\beta) - (1-\beta^n)(1-\alpha)}{(1-\alpha)(1-\beta)}\right] \\
&= \frac{k}{p(2k+k-1)}\Gamma \\
&= \frac{T_n + 2hT_{n-1} - k}{2h + k - 1}
\end{aligned}$$

where $\Gamma = \left[(\alpha^n - \beta^n) - (\alpha - \beta) - 2h(\alpha^{n-1} - \beta^{n-1})\right]$

III. CONCLUSION

In summary, we have obtained the equations for the n^{th} term and the sum of the first n terms of the (k,h)-Jacobsthal sequence. A possible extension of this work is by retaining the recurrence relation while setting the initial values to $T_0 = a$ and $T = b$ where $a, b \in \mathbb{Z}$ and a and b are not both zero.

REFERENCES

[1] K. Atanassov, "Short Remarks on Jacobsthal Numbers", *Notes on Number Theory and Discrete Mathematics*, vol. 18, no. 2, pp. 63–64, 2012.

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