Quasi-Metric Space and Fixed Point Theorems

G. Rano and T. Bag

***Abstract*—We introduce here the notions of Cauchy-ness and convergence of sequences, open-ness and closed-ness of sets in quasi-metric space and established some basic theorems like Cantor's and Baire's in complete quasi-metric spaces(QMS). The definitions of different kinds of contracting mappings are given and finally we able to prove some most important fixed point theorems of functional analysis such as Banach, Kannan and Caristi in this setting. Uniqueness of these theorems are studied.**

***Index Terms*—Quasi-metric space, Cantor's intersection theorem, Baire's category theorem, Fixed points.**

***MSC 2010 Codes* – 46S40, 47H10, 54H25.**

I. Introduction

It is well known that metric and norm structures play pivotal role in functional analysis. So in order to develop functional analysis one has to take care of the suitable generalization of these structures. Historically, the problem of generalization of the metric structure came first.

Different authors introduced ideas of quasi-metric space [1], generalized metric space([2], [3]), generalized quasi-metric space [4], dislocated metric space [5], fuzzy metric space([6], [7], [8]), statistical metric space [9], two metric space [10] etc.

In this paper, we introduce few well known sequential concepts in QMS and established some basic theorems like Cantor's intersection theorem and a category theorem of Baire's in complete QMS. The definitions of different kinds of contraction mapping are given here and established some fixed point theorem with uniqueness. Straightforward proofs are omitted throughout this manuscript.

We organize this manuscript in the following manner: Section II comprises of some preliminary definitions and properties. We established some basic theorems like Cantor's and Baire's in section III. Section IV consist of a few fixed point theorems including a discussion of uniqueness.

II. Some preliminary results.

This section of the paper consist of a collection of preliminary definitions and ideas related to quasi-metric spaces.

Definition 2.1 Consider a nonempty set X and a mapping $\rho_q : X \to [0, \infty)$. Then ρ_q is called a quasi-metric on X if it satisfies the following three conditions:

(QM1) $\rho_q(x, y) = 0$ iff $x = y$;

(QM2) $\rho_q(x, y) = \rho_q(y, x) \ \forall x, y \in X$;

(QM3) $\exists \ K \geq 1$ such that

$\rho_q(x, y) \leq K\{\rho_q(x, z) + \rho_q(z, y)\}$ for all $x, y, z \in X$.

The order pair (X, ρ_q) is said to be a quasi-metric space (QMS). The least value of the constant K satisfying (QM3) is known as the index of the quasi-metric ρ_q.

The space (X, ρ_q) is consider as a strong QMS if it satisfies the additional condition (QM4) given by:

(QM4) : $\exists \ K \geq 1$ such that

$$\rho_q(x_m, x_{m+p}) \leq K\{\sum_{i=0}^{p-1} \rho_q(x_{m+i}, x_{m+i+1})\} \ \forall x_{m+i} \in X, \forall p \in N.$$

Note 2.1 If $K = 1$ then the quasi-metric ρ_q is reduced to a metric and obviously (X, ρ_q) to a metric space.

Note 2.2 Every metric space is a QMS but not on the contrary, which is illustrated by an example given by 2.1.

Example 2.1 Let us consider $X = R^2$. Take $x = (x_1, x_2), y = (y_1, y_2) \in X$ and then we define

$$\rho_q(x, y) = (\sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|})^2 .$$

Then (X, ρ_q) is a QMS without having a metric space.

Proof. First we shall show ρ_q is a quasi-metric on X. Conditions (QM1) and (QM2) are directly followed from definition. For (QM3), it is not difficult to show that

$\rho_q(x, z) \leq 2\{ \rho_q(x, y) + \rho_q(y, z)\}$.

So (X, ρ_q) is a QMS.

Next we must now show that (X, ρ_q) is not a metric space. Take $x = (1, 0), y = (0, 0)$ and $z = (0, 1)$ then

$\rho_q(x, y) = 1, \rho_q(y, z) = 1$ and $\rho_q(x, z) = 4$.

So $\rho_q(x, z) > \rho_q(x, y) + \rho_q(y, z)$.

Hence ρ_q is not a metric i.e. the space (X, ρ_q) is not a metric space.

Note 2.3 Example 2.1 illustrate that (X, ρ_q) a QMS with quasi index $K = 2$.

Example 2.2 Suppose that $X = R^2$. For $x = (x_1, x_2), y = (y_1, y_2) \in X$ define

$$\rho_q(x, y) = (|x_1 - y_1| + |x_2 - y_2|)^2 .$$

Then (X, ρ_q) is a QMS having quasi-index $K = 3$ but not a metric space.

Example 2.3 Consider $X = R^n$. For $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in X$ define

$$\rho_q(x, y) = \sum_{i=1}^{n} |x_i - y_i|^2.$$

Then (X, ρ_q) is a QMS but not a metric space.

Proof. First we shall prove that (X, ρ_q) is a QMS. Conditions (QM1) and (QM2) are immediately followed. For (QM3), let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n), z = (z_1, z_2, ..., z_n) \in X$ in R^n then $2\{ \rho_q(x, y) + \rho_q(y, z)\}$;

G. Rano is a research scholar in the Department of Mathematics, Visva-Bharati University, Shantiniketan-731235, West Bengal, India (e-mail: gobardhanr@gmail.com)

T. Bag is a Reader in the Department of Mathematics, Visva-Bharati University, Shantiniketan-731235, West Bengal, India (e-mail: tarapadavb@gmail.com)

The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F. 510/4/DRS/2009(SAP-I)]

$= 2\{\sum_{i=1}^{n}|x_i - y_i|^2 + \sum_{i=1}^{n}|y_i - z_i|^2\};$

$= \sum_{i=1}^{n} \; 2\{|x_i - y_i|^2 + |y_i - z_i|^2\};$

$\geq \; \sum_{i=1}^{n} \; (x_i - y_i + y_i - z_i)^2 \; = \sum_{i=1}^{n}|x_i - z_i|^2 = \rho_q(x,\ z).$

Next we shall see $(X,\ \rho_q)$ is not a metric space. Take particular points $x = (1,\ 0,\ 0,\ ....,\ 0,\ 1)$, $y = (0,\ 0,\ ,0,\ ....,\ 0,\ 1)$ and $z = (-1,\ 0,\ 0,\ ....,\ 0,\ 0)$ in X then

$\rho_q(x,\ y) = 1, \quad \rho_q(y,\ z) = 2, \quad \rho_q(x,\ z) = 5.$

So $\rho_q(x,\ z) > \rho_q(x,\ y) + \rho_q(y,\ z)$.

Therefore the condition of the example is satisfied.

Definition 2.2 In a QMS $(X,\ \rho_q)$,

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said

(a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} \rho_q(x_n,\ x) = 0$;

(b) to be a Cauchy sequence if $\lim_{m,\ n\to\infty} \rho_q(x_n,\ x_m) = 0$;

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B;

(iii) A subset A of X is called be bounded if there exists a real number $M > 0$ such that $\rho_q(x,\ y) \leq M \quad \forall\ x,\ y \in A$;

(iv) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim_{n\to\infty} x_n = x$ implies $x \in A$;

(v) A set A in X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A.

Proposition 2.1 Suppose $(X,\ \rho_q)$ is a QMS.

(a) The limit of a sequence $\{x_n\}$ in X if exists is unique;

(b) The subsequences of a convergent sequence are also convergent and converges to the limit of the original sequence;

(c) Every sequence which is convergent is also a Cauchy sequence.

III. CANTOR'S AND BAIRE'S TYPE THEOREMS IN QMS

We advance here few basic properties of QMS and established some fundamental theorems of functional analysis like Cantor's and Baire's in complete QMS.

Definition 3.1 Suppose $(X,\ \rho_q)$ is a QMS, $r > 0$ and $x_0 \in X$. Let us define $S(x_0,\ r) = \{x \in X \ : \ \rho_q(x_0,\ x) < r\}$ and $\overline{S(x_0,\ r)} = \{x \in X \ : \ \rho_q(x_0,\ x) \leq r\}$. Then $S(x_0,\ r)$ and $\overline{S(x_0,\ r)}$ are respectively called an open ball(or open sphere) and a closed ball(or closed sphere) with center at x_0 and having radius r.

Definition 3.2 Consider $(X,\ \rho_q)$ is a QMS and $A \subset X$. The closure of A is denoted by $\overline{A}$ and is defined by

$\overline{A}$={ x : if $\exists$ a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = x\}$.

Proposition 3.1 Let us suppose $(X,\ \rho_q)$ is a QMS. Then

(a) a sequence $\{x_n\}$ in X converges to x iff every ball $S(x,\ r)$ with center at x contains all the points of the sequence except perhaps a finite number;

(b) for $A \subset X,\ x \in \overline{A}$ iff $S(x,\ r) \cap A \neq \phi$ for each ball $S(x,\ r)$ with center at x;

(c) the union of a finite number of closed sets in X is a closed set;

(d) the intersection of an arbitrary number of closed sets in X is a closed set.

Remark 3.1 If A is any set in $(X,\ \rho_q)$ and $x \notin \overline{A}$ then there exists a ball $S(x,\ r)$ which contains no points of A.

Proposition 3.2 For a set A in $(X,\ \rho_q)$, $\overline{A}$ is closed.

Proof. Consider a sequence $\{x_n\}$ in $\overline{A}$ and $\lim_{n\to\infty} x_n = x$. Then corresponding to any $\epsilon > 0$, there exists $n_1 \in N$ leads to

$\rho_q(x_{n_1},\ x) < \epsilon$. Since $x_{n_1} \in \overline{A}$, there exists $y_1 \in A$ such that

$\rho_q(x_{n_1},\ y_1) < \epsilon$.

Now $\rho_q(y_1,\ x) \leq K(\rho_q(y_1,\ x_{n_1}) + \rho_q(x_{n_1},\ x))$ for some $K \geq 1$;

$\Rightarrow \rho_q(y_1,\ x) \leq 2K\epsilon$. In the similar manner we can construct a sequence $\{y_n\}$ in A satisfying

$\rho_q(y_n,\ x) < \frac{2K\epsilon}{n} \quad \forall\ n \in N$.

$\Rightarrow \lim_{n\to\infty} \rho_q(y_n,\ x) = 0$.

So $x \in \overline{A}$ and $\overline{A}$ is closed.

Remark 3.2 $A = \overline{A}$ if A is closed.

Definition 3.3 Suppose that $(X,\ \rho_q)$ is a QMS and $A \subset X$. Then A is open if its complement $X - A$ is closed.

Theorem 3.1 *Consider $(X,\ \rho_q)$ is a QMS and A is a subset of X. Then A is open iff every x in A, there is a $S(x,\ r)$ having center at x so that $S(x,\ r) \subset A$.*

Proof. Let A be open. Then $X - A$ is closed i.e. $X - A = \overline{X - A}$ and $A = X - \overline{X - A}$. Let $x \in A$, then $x \notin \overline{X - A}$. By Remark 3.1, there exists a ball $S(x,\ r)$ which contains no points of $X - A$ i.e $S(x,\ r) \subset A$. Conversely, assume that for every point $x \in A$, there exists a ball $S(x,\ r)$ with center at x such that $S(x,\ r) \subset A$. By Remark 3.1, if $x \in A$ then $x \notin X - A$. Now we claim that $x \notin \overline{X - A}$. If not, suppose $x \in \overline{X - A}$ then there exists a sequence $\{x_n\}$ in $X - A$ leading to $\lim_{n\to\infty} \rho_q(x_n,\ x) = 0$. Therefore for $r > 0$ and there is a $M(r) \in N$ such that $\rho_q(x_n,\ x) < r \quad \forall\ n \geq M(r)$, which contradict the fact that $S(x,\ r) \subset A$. So $x \notin \overline{X - A}$ and $x \in X - \overline{X - A}$ and hence $A \subset X - \overline{X - A} \subset A$. So $A = X - \overline{X - A} \Rightarrow X - A = \overline{X - A}$. Which completes the proof of the theorem.

Proposition 3.3 Assume that $(X,\ \rho_q)$ is a QMS. Then

(a) the intersection of a finite number of open sets in $(X,\ \rho_q)$ is an open set;

(b) the union of an arbitrary number of open sets in $(X,\ \rho_q)$ is an open set.

Definition 3.4 Suppose that $(X,\ \rho_q)$ is a QMS and A a bounded subset of X. Then the diameter of A is denoted by $\delta\ (A)$ and is given by:

$\delta\ (A) = \bigvee\{\rho_q(x,\ y) \ : \ x,\ y \in A\}$.

Lemma(RBS) 3.1 *Let us consider two sequences $\{x_n\}$ and $\{y_n\}$ in a QMS $(X,\ \rho_q)$ having $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then*

(1) $\overline{\lim_{n\to\infty}} \rho_q(x_n,\ y_n) \leq K^2\rho_q(x,\ y)$;

(2) $\underline{\lim_{n\to\infty}} \rho_q(x_n,\ y_n) \geq \frac{\rho_q(x,\ y)}{K^2}$,

K being the index of the quasi-metric ρ_q.

Proof(1). Now $\rho_q(x_n,\ y_n) \leq K(\rho_q(x_n,\ x) + \rho_q(x,\ y_n))$

$\leq K\rho_q(x_n,\ x) + K^2\rho_q(x,\ y) + K^2\rho_q(y,\ y_n)$;

$\Rightarrow \overline{\lim_{n\to\infty}}\rho_q(x_n,\ y_n) \leq \overline{\lim_{n\to\infty}}(K\rho_q(x_n,\ x) + K^2\rho_q(x,\ y) + K^2\rho_q(y,\ y_n))$;
$\Rightarrow \overline{\lim_{n\to\infty}}\rho_q(x_n,\ y_n) \leq K^2\rho_q(x,\ y)$.

Proof(2). We know that
$\rho_q(x,\ y) \leq K(\rho_q(x,\ x_n) + \rho_q(x_n,\ y))$
$\leq K\rho_q(x,\ x_n) + K^2\rho_q(x_n,\ y_n) + K^2\rho_q(y_n,\ y)$;
$\Rightarrow \underline{\lim_{n\to\infty}}\rho_q(x,\ y) \leq \underline{\lim_{n\to\infty}}(K\rho_q(x,\ x_n) + K^2\rho_q(x_n,\ y_n) + K^2\rho_q(y_n,\ y)))$;
$\Rightarrow \underline{\lim_{n\to\infty}}\rho_q(x_n,\ y_n) \geq \frac{\rho_q(x,\ y)}{K^2}$.

Lemma(RBS) 3.2 *Let us suppose that $(X,\ \rho_q)$ is a QMS with quasi index K and A be a bounded subset of X. Then $\overline{A}$ is also bounded and $\delta(\overline{A}) \leq K^2\delta(A)$.*

Proof. Let $x,\ y \in \overline{A}$, then there exists sequences $\{x_n\},\ \{y_n\}$ in A such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. By Lemma 2.1(2),
$\rho_q(x,\ y) \leq K^2 \underline{\lim_{n\to\infty}}\rho_q(x_n,\ y_n)$
$\Rightarrow \rho_q(x,\ y) \leq \overline{K^2\delta}(A)$
$\Rightarrow \delta(\overline{A}) \leq K^2\delta(A)$.

Theorem 3.2(Cantor's intersection theorem) *A QMS $(X,\ \rho_q)$ is complete iff every nested sequence of non-empty closed subsets $A_1 \supset A_2 \supset A_3 .... \supset A_n ....$ of $(X,\ \rho_q)$ with $\delta(A_n) \to 0$ as $n \to \infty$, be such that $\cap_{n=1}^{\infty} A_n$ contains exactly one point.*

Proof. Consider $x_n \in A_n\ \forall\, n \in N$, then clearly $\{x_n\}$ is a sequence in X. Now
$\rho_q(x_n,\ x_{n+p}) \leq \delta(A_n) \to 0$ as $n \to \infty$. This confirms that $\{x_n\}$ is a Cauchy sequence. Again by the completeness property of the space, $\{x_n\}$ converges to a unique limit $x \in X$. Suppose that $T \in N$, then $\{x_{T+n}\}$ is a sequence in A_T which converges to x. As A_T is closed, $x \in A_T$. Therefore $x \in A_n \forall\, n \in N$ i.e. $x \in \cap_{n=1}^{\infty} A_n$ and so the intersection is nonempty.
If possible let, $y \in \cap_{n=1}^{\infty} A_n$. Then
$\rho_q(x,\ y) \leq \delta(A_n) \forall\, n \in N$
$\Rightarrow \rho_q(x,\ y) = 0$ and $x = y$.

Conversely, suppose that the condition of the theorem is satisfied. We must now show that $(X,\ \rho_q)$ is complete. For this, take any Cauchy sequence $\{x_n\}$ in X.
Consider $H_n = \{x_n,\ x_{n+1},\ x_{n+2}, ...\}$. If $\epsilon > 0$ is arbitrary, there is a positive integer N such that
$\rho_q(x_n,\ x_m) < \epsilon$ if $m,\ n \geq N$. Clearly $\delta(H_n) \to 0$ as $n \to \infty$ which implies $\delta(\overline{H_n}) \leq K^2\delta(H_n) \to 0$ as $n \to \infty$. Again $\overline{H}_1 \supset \overline{H}_2 \supset \overline{H}_3 .... \supset \overline{H}_n ....$ is a nested family of nonempty and closed sets. By hypothesis, there exists a unique x satisfying
$x \in \cap_{n=1}^{\infty}\overline{H}_n$. So $\rho_q(x_n,\ x) \leq \delta(\overline{H}_n) \to 0$ as $n \to \infty$ leads to $\{x_n\}$ converges to x. Hence the theorem.

Proposition 3.4 Suppose $(X,\ \rho_q)$ is a QMS and $Y \subset X$. Then $(Y,\ \rho_q)$ is complete iff it is closed.

Definition 3.5 Consider a QMS $(X,\ \rho_q)$ and a subset E of X. Then E is said to be non-dense (or nowhere dense) in X if for every open ball $S(x,\ \epsilon)$ in X there exists an open ball $S(x_1,\ \epsilon_1) \subset S(x,\ \epsilon)$ contains no points of E.

Theorem 3.3(Baire's Category) *Considering that $(X,\ \rho_q)$ is a complete QMS, where $X \neq \phi$. Then X can not be represented by*

$$X = \cup_{k=1}^{\infty} A_k$$

where A_k's are nondense in X.

Proof. If possible let

$$X = \cup_{k=1}^{\infty} A_k$$

where A_k's are nondense in X.
Since A_1 is nondense in X, there exists an open ball $S_1(x_1,\ \epsilon_1)$ which contains no points of A_1. Since A_2 is nondense in X, there exists an open ball $S_2(x_2,\ \epsilon_2) \subset S_1(x_1,\ \frac{\epsilon_1}{K^2})$ which contains no points of A_2, where K is the index of ρ_q. Similarly there exists an open ball $S_3(x_3,\ \epsilon_3) \subset S(x_2,\ \frac{\epsilon_2}{K^2})$ which contains no points of A_3. Proceeding similarly we observed
$\rho_q(x_n,\ x_{n+p}) < \frac{\epsilon_n}{K^2}$ (3.1).

Clearly the Cauchy-ness of $\{x_n\}$ is obvious. Further the completeness of $(X,\ \rho_q)$ asserts that there is a $x \in X$ satisfying
$\lim_{n\to\infty} \rho_q(x_n,\ x) = 0$.
Now from definition and (3.1), we have
$\rho_q(x_n,\ x) \leq K(\rho_q(x_n,\ x_{n+p}) + \rho_q(x_{n+p},\ x)) < \frac{\epsilon_n}{K} + K\rho_q(x_{n+p},\ x)$ (3.2).

Taking $p \to \infty$ in (3.2) we get $\rho_q(x_n,\ x) \leq \frac{\epsilon_n}{K} < \epsilon_n\ \forall\, n$. This $x \in S_n(x_n,\ \epsilon_n)\ \forall\, n \in N$ but does not belongs to any A_n. Again $x \in X = \cup_{k=1}^{\infty} A_k$, this leads to a contradiction.

Definition 3.6 Let us consider $(X,\ \rho_q)$ is a QMS. Then ρ_q is said to be continuous if for any sequences $\{x_n\}$ and $\{y_n\}$ in this space having $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ leads to $\lim_{n\to\infty} \rho_q(x_n, y_n) = \rho_q(x, y)$. In this case, the space $(X,\ \rho_q)$ is known as a continuous QMS.

Remark 3.3 Examples 2.1, 2.2 and 2.3 illustrate that $(X,\ \rho_q)$ is a continuous QMS.

Proposition 3.5 In a continuous QMS $(X,\ \rho_q)$, an open ball is an open set and a closed ball is a closed set.

Proof. Suppose $(X,\ \rho_q)$ is a continuous QMS and
$S(x_0,\ r) = \{x \in X : \rho_q(x_0,\ x) < r\}$ is an open ball in X. We have to prove $S(x_0,\ r)$ is an open set i.e. the complement $S^c(x_0,\ r)$ is a closed set. Let $\{x_n\}$ be a sequence in $S^c(x_0,\ r)$ with $\lim_{n\to\infty} x_n = x$. The theorem will be proved if we prove $x \in S^c(x_0,\ r)$. If not let $x \in S(x_0,\ r)$ then by the continuity of this space we have $\lim_{n\to\infty} \rho_q(x_n,\ x_0) = \rho_q(x,\ x_0) < r$. Therefore there is a $m \in N$ so that $\rho_q(x_n,\ x_0) < r$ whenever $n \geq m$, which contradicts the fact that $\{x_n\}$ belongs to $S^c(x_0,\ r)$.

For the second part we have to prove that $\overline{S(x_0,\ r)} = \{x \in X : \rho_q(x_0,\ x) \leq r\}$ is a closed set. Considering $\{x_n\}$ as a sequence $\overline{S(x_0,\ r)}$ having $\lim_{n\to\infty} x_n = x$. The theorem will be proved if we confirm that $x \in \overline{S(x_0,\ r)}$. If not, take $x \notin \overline{S(x_0,\ r)}$ which implies $\lim_{n\to\infty} \rho_q(x_n,\ x_0) = \rho_q(x,\ x_0) > r$, since the space is continuous. Consequently there is a $m \in N$ having $\rho_q(x_n,\ x_0) > r$ whenever $n \geq m$, which contradicts the fact that $\{x_n\}$ is a sequence in $\overline{S(x_0,\ r)}$.

IV. Fixed point theorems

Here we introduce the idea of contracting mappings and prove Banach, Kannan's and Caristi's fixed point theorems in this setting.

Definition 4.1 Let us take a QMS $(X,\ \rho_q)$ having quasi index K and an operator $T\ :\ X \to\ X$ satisfying

$\rho_q(Tx,\ Ty)\ \leq\ \delta\ \rho_q(x,\ y)\ \ \forall\ x,\ y \in\ X$, where $\delta \in (0,\ \frac{1}{K})$ (4.1).

Such type of operator will be known as a contracting operator.

Definition 4.2 Consider two QMS $(X,\ \rho_q)$ and $(Y,\ \lambda_q)$ and an operator $T : X \to\ Y$. Then T is called continuous at x in X if for any sequence $\{x_n\}$ of X with $x_n \to\ x$ i.e. $\lim_{n\to\infty} \rho_q(x_n,\ x) = 0$ implies $T(x_n) \to\ T(x)$ i.e. $\lim_{n\to\infty} \lambda_q(T(x_n),\ T(x)) = 0$. T is continuous if it is continuous at each point of the domain X.

Remark 4.1 Contracting mapping is continuous.

Lemma 4.1 *Assume that $(X,\ \rho_q)$ is a QMS and T : $X \to X$ is a contracting mapping. Then for any $x_0 \in\ X,\ \ x_n = \{T^n(x_0)\}$ is a Cauchy sequence.*

Proof. Let $x_0 \in\ X$ and $x_1 = T(x_0)$. Having defined x_1 we define $x_2 = T(x_1) = T^2(x_0)$ and in similar manner we get $x_n = T(x_{n-1}) = T^n(x_0)$ and so on.

Suppose $m < n$, let $n = m + p$, then

$\rho_q(x_m,\ x_{m+p})\ \leq\ K\rho_q(x_m,\ x_{m+1}) + K\rho_q(x_{m+1},\ x_{m+p})$;

$\leq K\ \delta^m\rho_q(x_0,\ x_1) + K^2\rho_q(x_{m+1},\ x_{m+2}) +\ K^2\rho_q(x_{m+2},\ x_{m+p})$;

$\leq K\ \delta^m\rho_q(x_0,\ x_1) + K^2\ \delta^{m+1}\rho_q(x_0,\ x_1) + K^3\ \delta^{m+2}\rho_q(x_0,\ x_1) + \dots\dots\dots + K^{p-1}\ \delta^{m+p-2}\rho_q(x_0,\ x_1) + K^{p-1}\ \delta^{m+p-1}\rho_q(x_0,\ x_1)$;

$\leq K\ \delta^m\rho_q(x_0,\ x_1)[1 + K\delta + (K\delta)^2 + (K\delta)^3 + \dots\dots\dots + (K\delta)^{p-1}]$, since $K \geq\ 1$;

$= K\ \delta^m\rho_q(x_0,\ x_1)\frac{1-(K\delta)^p}{1-K\delta}$;

$\leq\ K\ \delta^m\rho_q(x_0,\ x_1)\frac{1}{1-K\delta}$, since $K\delta < 1$;

$\Rightarrow \lim_{m\to\infty}\ \rho_q(x_m,\ x_{m+p})\ = 0$.

Hence the Cauchy-ness of $\{x_n\} = \{T^n(x_0)\}$ is satisfied.

Theorem 4.1(Banach) *Consider $(X,\ \rho_q)$ as a complete QMS and $T : X \to\ X$ is a contracting mapping. Then fixed point of T exists and unique.*

Proof. Consider $x_0 \in\ X$ and so far we imitate the method employed in Lemma 4.1 and get

$x_n = T(x_{n-1}) = T^n(x_0)\ \ \forall\ n \in\ N$.

By Lemma 4.1, the sequence $\{x_n\}$ is Cauchy. As $(X,\ \rho_q)$ is complete, there is a x in X so that $\lim_{n\to\infty}\ x_n = x$.

Since T is a contracting, it is continuous and $\lim_{n\to\infty}\ T(x_n) = Tx$.

Now $Tx = \lim_{n\to\infty}\ T(x_n) = \lim_{n\to\infty}\ x_{n+1} = x$.

Therefore x is a fixed point of the operator T.

Uniqueness: Suppose that $x,\ y\ \in\ X$ are any two fixed points of T. From (4.1) we obtain,

$\rho_q(x,\ y) = \rho_q(Tx,\ Ty)\ \leq\ \delta\ \rho_q(x,\ y)$;

$\Rightarrow\ (1-\delta)\rho_q(x,\ y) \leq\ 0$;

$\Rightarrow\ \rho_q(x,\ y) =\ 0$.

Therefore $x = y$.

Theorem 4.2(Kannan) *Let us consider $(X,\ \rho_q)$ as a complete QMS and $T : X \to\ X$ be a continuous mapping such that $\rho_q(Tx,\ Ty)\ \leq\ \delta\ [\rho_q(x,\ Tx) + \rho_q(y,\ Ty)]\ \ \ \forall\ x,\ y \in\ X$, where $0 < \delta < \frac{1}{1+K}$. Then the operator T has a unique fixed point.*

Proof. Let $x_0 \in\ X$ and $x_1 = T(x_0),\ x_2 = T(x_1) = T^2(x_0), \dots\dots\dots,\ x_n = T(x_{n-1}) = T^n(x_0)$ and so on. Then

$\rho_q(x_1,\ x_2) = \rho_q(T(x_0),\ T(x_1))\ \leq\ \delta\ [\rho_q(x_0,\ T(x_0)) + \rho_q(x_1,\ T(x_1))]$

$= \delta\ [\rho_q(x_0,\ T(x_0)) + \rho_q(x_1,\ x_2)]$

$\Rightarrow\ \rho_q(x_1,\ x_2) \leq \frac{\delta}{1-\delta}\rho_q(x_0,\ x_1)$.

Similarly we can show that

$\rho_q(x_2,\ x_3) \leq (\frac{\delta}{1-\delta})^2\rho_q(x_0,\ x_1)$,

$\rho_q(x_3,\ x_4) \leq (\frac{\delta}{1-\delta})^3\rho_q(x_0,\ x_1)$,

....

$\rho_q(x_n,\ x_{n+1}) \leq (\frac{\delta}{1-\delta})^n\rho_q(x_0,\ x_1)$.

By (QM3), there exists $K \geq\ 1$ such that

$\rho_q(x_n,\ x_{n+p})\ \leq\ K\{\rho_q(x_n,\ x_{n+1}) + \rho_q(x_{n+1},\ x_{n+p})\}$

$\leq\ K(\frac{\delta}{1-\delta})^n\rho_q(x_0,\ x_1) + K^2(\frac{\delta}{1-\delta})^{n+1}\rho_q(x_0,\ x_1) + K^3(\frac{\delta}{1-\delta})^{n+2}\rho_q(x_0,\ x_1) + \dots\dots\dots$

$\dots\dots + K^{p-1}(\frac{\delta}{1-\delta})^{n+p-2}\rho_q(x_0,\ x_1) + K^{p-1}(\frac{\delta}{1-\delta})^{n+p-1}\rho_q(x_0,\ x_1)$;

$\leq\ Kr^n\rho_q(x_0,\ x_1)[1+Kr+(Kr)^2+\dots\dots\dots+(Kr)^{p-1}]$, where $r = \frac{\delta}{1-\delta}$;

$= Kr^n\rho_q(x_0,\ x_1)\frac{1-(Kr)^p}{1-Kr}$;

$< Kr^n\rho_q(x_0,\ x_1)\frac{1}{1-Kr}$.

Now $0 < \delta < \frac{1}{1+K}$ implies $Kr \in (0,\ 1)$ and $\lim_{n\to\infty}\ \rho_q(x_n,\ x_{n+p}) = 0$.

So that the sequence $\{x_n\}$ is Cauchy sequence.

As $(X,\ \rho_q)$ is complete, therefore there exists $x \in\ X$ so that $\lim_{n\to\infty}\ x_n = x$.

By the continuity of T we have $\lim_{n\to\infty}\ T(x_n) = Tx$.

Now $Tx = \lim_{n\to\infty}\ T(x_n) = \lim_{n\to\infty}\ x_{n+1} = x$.

Therefore x is a fixed point of T.

Uniqueness: For uniqueness consider any two distinct fixed points $x,\ y\ \in\ X$ of T. Then

$\rho_q(x,\ y) = \rho_q(Tx,\ Ty) \leq\ \delta[\rho_q(x,\ Tx) + \rho_q(y,\ Ty)] = 0$

$\Rightarrow\ x = y$.

Theorem 4.3(Caristi) *Let us suppose that $(X,\ \rho_q)$ is a complete strong QMS and $T : X \to\ X$ be a continuous mapping. Assuming that there is a mapping $P : X \to\ (0,\ \infty)$ satisfying*

$$\rho_q(x,\ Tx)\ \leq\ P(x) - P(Tx)\ \ \forall\ x \in\ X,$$

then there exists fixed point of T in X.

Proof. Take any point $x_0 \in\ X$ and $x_1 = T(x_0),\ x_2 = T(x_1) = T^2(x_0), \dots\dots\dots,\ x_n = T(x_{n-1}) = T^n(x_0)$ and so on. For any positive integer v we have

$\rho_q(x_v,\ x_{v+1})\ = \rho_q(x_v,\ Tx_v) \leq\ P(x_v) - P(Tx_v)$

$\Rightarrow\ \rho_q(x_v,\ x_{v+1}) \leq\ P(x_v) - P(x_{v+1})$

$$\Rightarrow \sum_{v=0}^{n-1}\ \rho_q(x_v,\ x_{v+1}) \leq\ \sum_{v=0}^{n-1}[P(x_v) - P(x_{v+1})]$$

$$= P(x_0) - P(x_n)$$

$$\leq\ P(x_0).$$

So, the series $\sum_{v=0}^{\infty}\ \rho_q(x_v,\ x_{v+1})$ is convergent. If $n,\ m(= n + p) \in\ N$ then, by (QM4), there exists $K \geq\ 1$ such that

$$\rho_q(x_n,\ x_m) \leq\ K \sum_{v=n}^{m-1} \rho_q(x_v,\ x_{v+1}).$$

Since the series $\sum_{v=0}^{\infty} \rho_q(x_v,\ x_{v+1})$ is convergent, for any positive ϵ there is a natural number n_0 leads to
$\sum_{v=n}^{m-1} \rho_q(x_v,\ x_{v+1}) < \frac{\epsilon}{K}$ whenever $m >\ n \geq\ n_0$
$\Rightarrow\ \rho_q(x_n,\ x_m) < \epsilon$ whenever $m >\ n \geq\ n_0$.
Hence $\{x_n\}$ is a Cauchy sequence. Again the completeness of the space $(X,\ \rho_q)$ asserts that there is a $x \in\ X$ such that $\lim_{n\to\infty}\ x_n = x$.
As T is continuous, $\lim_{n\to\infty}\ T(x_n) = Tx$.
Now $Tx = \lim_{n\to\infty}\ T(x_n) = \lim_{n\to\infty}\ x_{n+1} = x$.
This completes the proof.

Conclusions

Metric and norm structure have a great role in the development of functional analysis. A thousands of research work have been done in metric and norm spaces. The concept of fuzzy metric space and fuzzy normed linear space generalize these structure in a some extent. These are also not a very recent subject. But, except a few work, there are no remarkable one in quasi-metric spaces. Knowing this, we think that there may be some possibility of successful research in this space. The concepts of Cauchy sequence, Convergent sequence, Open set, Closed set etc. are introduced here. We established some basic theorems like Cantor's intersection theorem and Baire's category theorem in complete QMS. Using the concept of contracting mapping we prove some fixed point theorems in complete QMS and uniqueness of this theorems are studied. Hope there is a scope to develop functional analysis with this structure in a large extent.

Acknowledgment

The authors are grateful to the referees and Editor-in-Chief of IJMSC for their valuable suggestions and kind cooperation in rewriting the paper in the present form.

References

[1] L. Kikina and K.Kikina, "A related fixed point theorem for m mapping on m complete quasi-metric spaces," *Mathematica Cluj,* (accepted).
[2] P. Das and L. K. Dey, "A fixed point theorem in generalized metric space," *Soochow Journal of Mathematics*, vol. 33, pp. 33–39, 2007.
[3] L. Kikina and K. Kikina, "Fixed point on two generalized metric spaces," *Int. Journal of Math. Analysis*, vol. 5, no.30, pp. 1459–1467, 2011.
[4] L. Kikina, K. Kikina and K. Gjino, "A new fixed point theorem on generalized quasi-metric spaces," (Communicated).
[5] C.T. Aage and J.N. Salunke, "Some results of fixed point theorem in dislocated quasi-metric spaces," *Bulletin of the Marathwada Mathematical Society*, vol. 9, no. 2, pp. 1–5, 2008.
[6] T. Bandyopadhyay, S.K. Samanta and P.Das, "Fuzzy metric spaces redefined and a fixed point theorem," *Bull.Cal.Math.Soc.,* vol. 81, pp. 247–252, 1989.
[7] O. Kaleva and S. Seikkala, "On fuzzy metric spaces," *Fuzzy Sets and Systems,* vol. 12, pp. 215–229, 1984.
[8] P.V. Subrahmanyam, "A common fixed point theorems in fuzzy metric spaces," *Information Seiences,* vol. 83, nos. 3-4, pp. 109–112, 1995.
[9] I. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces," *Kybernetica,* pp. 326–334, 1975.
[10] B. Fisher, "Fixed point in two metric spaces," *Glasnik Mat,* vol. 16, no. 36, pp. 333–337, 1981.