Modified Adomian for Linear Partial Differential Equations

Ruchi Nigam

***Abstract*—In this paper we outline a strategy to use Adomian decomposition method properly for solving linear partial differential equation with homogeneous and inhomogeneous boundary. Our fundamental goal in this chapter is to present a further insight into partial solutions in the decomposition method, and the resolution of above cases by suitable transformation. The modifications are necessary in order to make the Adomian decomposition method efficient for boundary and initial value problems. The heat equation is chosen as an example and the cases where improper solution can be obtained are discussed.**

***Index Terms*—Adomian decomposition; Heat equation; Laplace Transform.**

***MSC 2010 Codes* – 44A10, 35K05**

I. INTRODUCTION

LET us look at the possible difficulties in the heat equation

$$\begin{aligned} &u_t = u_{xx}, \quad 0 < x < \ell, t > 0, \\ &u(0,t) = 0, \quad t > 0 \\ &u(\ell,t) = 0, \quad t > 0 \\ &u(x,0) = sinx, \quad 0 < x < \ell \end{aligned} \tag{1}$$

Thus, in the usual decomposition notation $L_t u = L_x u$ or

$$u = u(x,0) + L_t^{-1} L_x \sum_{n=0}^{\infty} u_n,$$

$$u_0 = sinx$$

$$u_1 = L_t^{-1} L_x \sum_{n=0}^{\infty} u_0 = -tsinx$$

$$u_2 = L_t^{-1} L_x \sum_{n=0}^{\infty} u_1 = \frac{t^2}{2!} sinx$$

$$u_3 = L_t^{-1} L_x \sum_{n=0}^{\infty} u_2 = -\frac{t^3}{3!} sinx$$

$$u_4 = L_t^{-1} L_x \sum_{n=0}^{\infty} u_3 = \frac{t^4}{4!} sinx$$

$$\vdots$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n = (1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - ....) \quad = e^{-t} sinx$$

while the x equation obtained by inverting the L_x operator gives zero [1].

In a problem suggested by Professor Y. Cherrault [1], we use the same boundary conditions but choose $u(x,0) = x(\ell - x)$. Again the x equation gives a zero result. However the t equation

$$u = u(x,0) + L_t^{-1} L_x \sum_{n=0}^{\infty} u_n \tag{2}$$

$$u_0 = x\ell - x^2$$
$$u_1 = -2t$$
$$u_{n \geq 2} = 0$$

yields $u = x\ell - x^2 + 2t$ which clearly does not satisfy the boundary conditions. The terms above, for $m > 0$, are

$$u_m = [L_t^{-1} L_x]^m u(x,0). \tag{3}$$

consequently,

$$u(x,t) = \sum_{m=0}^{\infty} [L_t^{-1} L_x]^m f(x) \tag{4}$$

where $u(x,0) = f(x)$. In the case $f(x) = x(\ell - x)$, we see that the operation $L_t^{-1} L_x f(x)$ annihilates the series in a finite number of terms and the result does not satisfy the boundary conditions. This annihilation of the series was not the case with $f(x) = sinx$. The resolution of such cases was obtained by Adomian and Rach [1] by appropriate expansion of the initial term without making a priori assumptions about the solution. However, this technique failed to obtain exact solution in the case of initial-boundary value problems with inhomogeneous boundary conditions.

A review of previous studies indicates that the difficulty arise while applying adomian decomposition method to partial differential equation with homogeneous and inhomogeneous boundary conditions. Hence we solve (1) using adomian with Laplace. For Adomian decomposition method one can refer to [1], [5], [6].

Applying Laplace both sides of equation (1)

$$p\overline{u}(x,p) - u(x,0) \quad = \quad \frac{d^2\overline{u}}{dx^2} \tag{5}$$

$$p\overline{u} - x(\ell - x) \quad = \quad \frac{d^2\overline{u}}{dx^2} \tag{6}$$

Solving by using adomian decomposition method

$$\begin{aligned} Lu \quad &= \quad \frac{d^2\overline{u}}{dx^2}, \; Ru = p\overline{u}, \; g = -x(\ell - x), \\ L^{-1}[.] \quad &= \quad \int_0^x \int_0^{x_1} [.] dx_2 dx_1 \end{aligned} \tag{7}$$

Ruchi Nigam was research scholar at Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667 and worked as assistant professor at IJCET, Tuticorin, India. Currently she is on child care leave. (E-mail: reply2rn@gmail.com)

$$\begin{aligned}
\overline{u}_0 &= u(0,p) + x\overline{F}(p) - \int_0^x \int_0^{x_1} x(\ell - x)dx_2dx_1 \\
&= \overline{F}(p)x - \left(\frac{x^3\ell}{3!} - \frac{x^4}{3.4}\right) \\
\overline{u}_1 &= \int_0^x \int_0^{x_1} p\overline{u}_0 dx_2dx_1 \\
&= p\frac{x^3}{3!}\overline{F}(p) - \left(\frac{x^5p\ell}{5!} - \frac{x^6p}{3.4.5.6}\right) \\
\overline{u}_2 &= p^2\frac{x^5}{5!}\overline{F}(p) - \left(\frac{x^7p^2\ell}{7!} - \frac{x^6p}{3.4.5.6.7.8}\right) \\
&\vdots
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{u}(x,p) &= \overline{F}(p)\left(x + \frac{px^3}{3!} + \frac{p^2x^5}{5!}...\right) \\
&- \ell\left(\frac{x^3}{3!} + \frac{px^5}{5!} + \frac{p^2x^7}{7!}.....\right) \\
&+ 2\left(\frac{x^4}{4!} + \frac{px^6}{6!} + \frac{p^2x^8}{8!} + ...\right) \qquad (8)
\end{aligned}$$

The value of $\overline{F}(p) = \overline{u}_x(0,p)$ are found by applying the condition at $x = \ell$. Substituting the value of $\overline{F}(p)$ in (8), we get a series

$$\begin{aligned}
\overline{u}(x,p) &= \left(\frac{x^4}{12} + \frac{px^6}{360} + \frac{p^2x^8}{20160}....\right) \\
&+ \left(-\frac{x^3}{6} - \frac{px^5}{120} - \frac{p^2x^7}{5040}....\right)\ell \\
&+ \left(\frac{x}{12} + \frac{px^3}{72} + \frac{p^2x^5}{1440}....\right)\ell^3 \\
&+ \left(\frac{-px}{120} - \frac{p^2x^3}{720} - \frac{p^3x^5}{14400}....\right)\ell^5 + ... \qquad (9)
\end{aligned}$$

in closed form above series can be written as

$$\overline{u}(x,p) = x(\ell - x)\frac{1}{p} - \frac{2}{p^2} + \frac{2}{p^2}\frac{\cosh\{\sqrt{p}(2x-\ell)/2\}}{\cosh(\sqrt{p}\ell/2)} \qquad (10)$$

Taking inverse Laplace Transform, we have

$$u(x,t) = x(\ell - x) - 2t + 2\left[\frac{1}{2}\{\frac{(2x-\ell)^2}{4} - \frac{\ell^2}{4}\} + t - \frac{16}{\pi^3}.\frac{\ell^2}{4}\times \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\, e^{\frac{-(2n-1)^2\pi^2 t}{4(\ell^2/4)}}\cosh\frac{(2n-1)\pi(2x-\ell)/2}{2.(\ell/2)}\right] \qquad (11)$$

Consider equation (1) with inhomogeneous boundary conditions $u(0,t) = T1, u(\ell,t) = T2$, and $u(x,0) = h(x)$. T_1, T_2 may be functions or numeric constants. In [1] steady state as well as transient solutions are obtained. However, for inhomogeneous boundary conditions the result obtained in [1] does not satisfy the boundary conditions. And there was no consideration about the wrong result. We can deal with this difficulty by using Laplace with adomian. For example, consider inhomogeneous conditions $u(0,t) = 0, u(1,t) = \sin 1e^{-t}$, and $u(x,0) = \sin x$.

Applying Laplace with adomian as stated above, we get

$$\begin{aligned}
\overline{u} &= \overline{u}(0,p) + x.\overline{u}_x(0,p) - \int_0^x \int_0^{x_1} \sin x dx_2dx_1 \\
&+ \int_0^x \int_0^{x_1} p\overline{u}dx_2dx_1 \qquad (12)
\end{aligned}$$

$$\begin{aligned}
\overline{u}_0 &= \overline{F}(p).x + \sin x \\
\overline{u}_1 &= \overline{F}(p)\frac{px^3}{3!} - p\sin x \\
\overline{u}_2 &= \overline{F}(p)\frac{p^2x^5}{5!} + p^2\sin x \\
\overline{u}_3 &= \overline{F}(p)\frac{p^3x^7}{7!} - p^3\sin x
\end{aligned} \qquad (13)$$

Thus,

$$\begin{aligned}
\overline{u}(x,p) &= \overline{F}(p)\left(x + \frac{px^3}{3!} + \frac{p^2x^5}{5!} + \frac{p^3x^7}{7!}...\right) \\
&+ \sin x\left(1 - p + p^2 - p^3 + ...\right) \qquad (14)
\end{aligned}$$

The series is clearly of

$$\overline{u}(x,p) = \overline{F}(p)\frac{\sinh\sqrt{p}x}{\sqrt{p}} + \frac{\sin x}{p+1}$$

Applying boundary conditions at $x = 1$, we have

$$\overline{F}(p)\frac{\sinh\sqrt{p}}{\sqrt{p}} + \frac{\sin 1}{p+1} = \frac{\sin 1}{p+1} \quad \Rightarrow \overline{F}(p) = 0 \qquad (15)$$

Solution in Laplace space is given by

$$\overline{u}(x,p) = \frac{\sin x}{p+1} \qquad (16)$$

Taking Laplace inverse, we get

$$u(x,t) = \sin xe^{-t} \qquad (17)$$

Although the above example can be solved by Adomian also because of the presence of the functions $sinx$ and e^{-t}. But the involvement of all the boundary conditions make laplace approach more reliable and efficient.

Next consider (1) with boundary conditions $u(x,0) = T_0, u_x(0,t) = 0, u(\ell,t) = T_1$. It is the case where it is difficult to obtain solution by decomposition method. Such a problem can be easily solve by using Laplace approach to decomposition method. Thus the solution is

$$\begin{aligned}
\frac{d^2\overline{u}}{dx^2} &= p\overline{u} - T_0 \\
\overline{u}_0 &= -\int_0^x \int_0^{x_1} T_0 dx_2dx_1 + \overline{F}(p) = -T_0\frac{x^2}{2!} + \overline{F}(p) \\
\overline{u}_1 &= \int_0^x \int_0^{x_1} p\overline{u}_0 dx_2dx_1 = p\left(\overline{F}(p)\frac{x^2}{2!} - pT_0\frac{x^4}{4!}\right) \\
\overline{u}_2 &= \int_0^x \int_0^{x_1} p\overline{u}_1 dx_2dx_1 = p^2\left(\overline{F}(p)\frac{x^4}{4!} - T_0\frac{x^6}{6!}\right) \\
\overline{u}_3 &= \int_0^x \int_0^{x_1} p\overline{u}_2 dx_2dx_1 = p^3\left(\overline{F}(p)\frac{x^6}{6!} - T_0\frac{x^8}{8!}\right)
\end{aligned} \qquad (18)$$

$$\begin{aligned}\overline{u}(x,t) &= \overline{F}(p)\left(1+p\frac{x^2}{2!}+p^2\frac{x^4}{4!}+p^2\frac{x^6}{6!}...\right)\\ &- T_0\left(\frac{x^2}{2!}+p.\frac{x^4}{4!}+p^2.\frac{x^6}{6!}+p^3.\frac{x^8}{8!}...\right) \quad (19)\end{aligned}$$

The series is clearly of

$$\overline{u}(x,p)=\overline{F}(p)\cosh\sqrt{p}x-\frac{T_0}{p}\cosh\sqrt{p}x+\frac{T_0}{p} \quad (20)$$

The function $\overline{F}(p)$ has to be determined by applying the condition at $x=\ell$. Substituting the value, we have

$$u(x,p)=(T_0-T_1)\frac{\cosh\sqrt{p}x}{p\cosh\sqrt{p}\ell}+\frac{T_0}{p} \quad (21)$$

Taking inverse Laplace

$$u(x,t)=T_1+\frac{4(T_1-T_0)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-\frac{(2n-1)^2\pi^2kt}{4t^2}}cos\frac{(2n-1)\pi x}{2\ell} \quad (22)$$

Another way of solving inhomogeneous boundary condition problem is by reducing the problem to homogeneous one by applying the transformation introduced in [7], [8]. Let

$$u(x,t)=\upsilon(x,t)+g(x), \quad f(x)=h(x)-g(x)$$
$$g(x)=\frac{(\ell-x)}{\ell}T_1+\frac{x}{\ell}T_2$$

substituting this in the problem yields

$$\begin{aligned}&\upsilon_t=u_{xx}, \quad 0<x<\ell, t>0,\\ &\upsilon(0,t)=0, \quad t>0\\ &\upsilon(\ell,t)=0, \quad t>0\\ &\upsilon(x,0)=f(x), \quad 0<x<\ell \quad (23)\end{aligned}$$

hence with the technique stated above we can obtain the solution.

II. Inhomogeneous heat equation

Consider the inhomogeneous case

$$u_t=u_{xx}+g(x,t) \quad (24)$$

$$u(x,0)=u(\ell,t)=u(0,t)=0 \quad (25)$$

write t equation

$$L_t=L_x+g(x,t) \quad (26)$$

According to adomian decomposition, we have

$$u=u_0+L_t^{-1}L_x\sum_{n=0}^{\infty}u_n, \quad (27)$$

with $u_0=L_t^{-1}g+u(x,0)=L_t^{-1}g$, since initial condition is zero.

$$u_m=(L_t^{-1}L_x)^m L_t^{-1}g \quad (28)$$

are components of u. If annihilation of series is not the case the adomian method is sufficient to obtain the solution. In the cases where L_x^m operator acting on g annihilates the series in a finite number of terms, we consider the Laplace Transform method. Thus, we have

$$p\overline{u}=\frac{d^2\overline{u}}{dx^2}+L\{g(x,t)\} \quad (29)$$

where $L\{g(x,t)\}$ denotes the Laplace transform of the function $g(x,t)$.
Proceeding as stated above we may obtain the solution.

In case of inhomogeneous heat equation with inhomogeneous boundary condition the equation can be reduced to homogeneous boundary with the help of the transformation stated earlier and hence can be solve.
For example, consider the equation (24) where $g(x,t)=2x$ and boundary conditions $u(0,t)=0, u(1,t)=0, u(x,0)=x-x^2$. In this case adomian method provides a solution $u(x,t)=2xt$ that does not satisfy the boundary conditions. While using the Laplace approach we obtain the series of the solution $u(x,t)=x(1-x)-\frac{4}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-e^{n^2\pi^2 t}}{n^3}\right)\sin n\pi x$.

III. Conclusion

The heat equations are solved Laplace approach to decomposition method. In cases where annihilation of the series is not the case, solution can be obtained by the usual decomposition method. In cases where L_x^m operator annihilates the series in a finite number of terms improper solution may be obtained when consider the initial boundary value problems involving inhomogeneous boundary conditions. We overcome the difficulties by using the Laplace approach and the results are successfully obtained. The transformation used in this paper is a simple technique for special cases. However, the powerful Adomian decomposition method can be applied to much more complicated physical problems [2], [3], [4], [5] in comparison with other methods. The above examples are used for clarity and comparison, since they are obviously solvable by separation of variables. In general, the decomposition is more widely applicable.

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