On Total Edge Product Cordial Labeling

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Abstract—The total edge product cordial labeling is a variant of cordial labeling in general and total product cordial labeling in particular. Here we investigate total edge product cordial labeling of some cycle and path related graphs.

Index Terms—Cordial Labeling, Product Cordial Labeling, Total Edge Product Cordial Labeling.

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I. INTRODUCTION

The graph labeling is one of the concepts in graph theory which has attracted many researchers to work on it because of its diversified and rigorous application in fields such as design and analysis of communication networks, military surveillance, social sciences, optimization and linear algebra. Many types of graph labelings are available in the existing literature. A brief account of comprehensive bibliography of papers on the concept of graph labeling is available in a dynamic survey of graph labeling by Gallian [1].

We begin with finite, undirected and connected graph \( G = (V(G), E(G)) \) without loops and multiple edges. The members of \( V(G) \) and \( E(G) \) are commonly termed as graph elements. For the various graph theoretic notations and terminology, we follow Balakrishnan and Ranganathan [2].

Definition 1.1: A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of mapping is the set of vertices(edges) then the labeling is called a vertex(an edge) labeling.

In 1987, Cahit [3] has introduced the concept of cordial labeling. Some labeling schemes are also introduced with minor variations in cordial theme. Some of them are A - Cordial labeling, H - Cordial labeling, Prime cordial labeling, Product cordial labeling, Total product cordial labeling and Edge product cordial labeling.

Definition 1.2: For a graph \( G \), an edge labeling function \( f : E(G) \rightarrow \{0, 1\} \) induces a vertex labeling function \( f^* : V(G) \rightarrow \{0, 1\} \) defined as \( f^*(v) = \prod \{f(uv)\mod 2 \in E(G)\} \).

Let us denote \( v_f(i) \) is the number of vertices of \( G \) having label \( i \) under \( f^* \) and \( e_f(i) \) is the number of edges of \( G \) having label \( i \) under \( f \) for \( i = 0, 1 \).

The function \( f \) is called a total edge product cordial labeling of \( G \) if \( |v_f(0) + e_f(0) - v_f(1) - e_f(1)| \leq 1 \). A graph is called total edge product cordial if it admits total edge product cordial labeling.

Vaidya and Barasara [4] have introduced the concept of total edge product cordial labeling and in the same paper they have investigated total edge product cordial labeling for some common graph families.

For the definitions of various graphs considered in the succeeding section the readers are advised to refer Vaidya and Barasara [5].

Here we investigate total edge product cordial labeling for some cycle and path related graphs.

II. TOTAL EDGE PRODUCT CORDIAL LABELING OF SOME CYCLE RELATED GRAPHS

Theorem 2.1: \( C_n^2 \) admits total edge product cordial labeling.

Proof: For the graph \( C_n^2 \) we have \( |V(C_n^2)| = n \) and \( |E(C_n^2)| = 2n \). To define \( f : E(C_n^2) \rightarrow \{0, 1\} \), we will consider following four cases.

Case 1: When \( n = 3 \).

The graph \( C_3^2 \) is same as \( C_3 \) and it is total edge product cordial graph as proved by Vaidya and Barasara [4].

Case 2: For \( n = 4k \) or \( n = 4k + 2 \), \( k = 1, 2, \ldots \)

\[
f(v_{2i-1}v_{2i}) = 0; \quad 1 \leq i \leq \frac{n}{2}
\]

\[
f(v_iv_j) = 1; \quad \text{for remaining } 3n \text{ edges}
\]

Case 3: For \( n = 4k + 3 \), \( k = 1, 2, \ldots \)

\[
f(v_iv_{i+1}) = 0; \quad 1 \leq i \leq \frac{3n - 3}{4}
\]

\[
f(v_i v_j) = 1; \quad \text{for remaining } \frac{5n + 3}{4} \text{ edges}
\]

Case 4: For \( n = 4k + 3 \), \( k = 1, 2, \ldots \)

\[
f(v_iv_{i+1}) = 0; \quad 1 \leq i \leq \frac{3n - 1}{4}
\]

\[
f(v_i v_j) = 1; \quad \text{for remaining } \frac{5n + 1}{4} \text{ edges}
\]

Illustration 2.2: Total edge product cordial labeling of \( C_6^2 \) is shown in Figure 1.
**Theorem 2.3**: \( D_2(C_n) \) admits total edge product cordial labeling.

**Proof**: For graph \( D_2(C_n) \) let \( v'_1, v'_2, \ldots, v'_n \) be the vertices of first copy of cycle \( C_n \) and \( v''_1, v''_2, \ldots, v''_n \) be the vertices of second copy of cycle \( C_n \). Then \( |V(D_2(C_n))| = 2n \) and \( |E(D_2(C_n))| = 4n \). To define \( f : E(D_2(C_n)) \to \{0, 1\} \), we will consider following two cases.

**Case 1**: For \( n = 2k \), \( k = 1, 2, \ldots \)

\[
\begin{align*}
    f(v'_i v'_{i+1}) &= 0; & 1 \leq i \leq n - 1 \\
    f(v'_i v''_{i+1}) &= 0; & 1 \leq i \leq \frac{n}{2} \\
    f(v'_i v''_{i}) &= 1; \\
    f(v''_i v''_{i+1}) &= 1; & \text{for all 2n edges} \\
    f(v''_i v''_{i}) &= 1; \\
    f(v''_{i+1} v''_{i}) &= 1; & 1 \leq i \leq n - 1 \\
    f(v''_{i+1} v''_{i}) &= 1; \\
\end{align*}
\]

In both the cases the number of graph elements with label 0 is \( 3n \), the number of graph elements with label 1 is \( 3n \) and their difference is 0.

Hence the result.

**Illustration 2.4**: Total edge product cordial labeling of \( D_2(C_n) \) is shown in Figure 2.

![Figure 2](image_url)

**Case 2**: For \( n = 2k + 1 \), \( k = 1, 2, \ldots \)

\[
\begin{align*}
    f(v'_i v'_{i+1}) &= 0; & 1 \leq i \leq n - 1 \\
    f(v'_i v''_{i+1}) &= 0; & 1 \leq i \leq \frac{n+1}{2} \\
    f(v'_i v''_{i}) &= 1; \\
    f(v''_i v''_{i+1}) &= 1; & \text{for remaining } \frac{3n-1}{2} \text{ edges} \\
    f(v''_i v''_{i}) &= 1; \\
    f(v''_{i+1} v''_{i}) &= 1; & 1 \leq i \leq n - 1 \\
    f(v''_{i+1} v''_{i}) &= 1; \\
\end{align*}
\]

**Case 3**: For \( n = 4k + 2 \), \( k = 1, 2, \ldots \)

\[
\begin{align*}
    f(e_i e_{i+1}) &= 0; & 1 \leq i \leq n - 1 \\
    f(e_i v_i) &= 0; & 1 \leq i \leq \frac{n+2}{4} \\
    f(v''_i v''_{i+1}) &= 1; & \text{for remaining } \frac{7n-2}{4} \text{ edges} \\
\end{align*}
\]

Therefore the number of graph elements with label 0 is \( \frac{5n-1}{2} \), the number of graph elements with label 1 is \( \frac{5n+1}{2} \) and their difference is 1.

**Case 4**: For \( n = 4k + 3 \), \( k = 0, 1, \ldots \)

\[
\begin{align*}
    f(e_i e_{i+1}) &= 0; & 1 \leq i \leq n - 1 \\
    f(e_i v_i) &= 0; & 1 \leq i \leq \frac{n+1}{4} \\
    f(v''_i v''_{i+1}) &= 1; & \text{for remaining } \frac{7n-1}{4} \text{ edges} \\
\end{align*}
\]

Therefore the number of graph elements with label 0 is \( \frac{5n-1}{2} \), the number of graph elements with label 1 is \( \frac{5n+1}{2} \) and their difference is 1.

Thus in all the cases described above the function \( f \) satisfies the condition for total edge product cordial labeling. Hence the result.

**Illustration 2.6**: Total edge product cordial labeling of \( M(C_n) \) is shown in Figure 3.

![Figure 3](image_url)
Theorem 2.7: \( T(C_n) \) admits total edge product cordial labeling.

**Proof:** For the graph \( T(C_n) \) we have \( |V(T(C_n))| = 2n \) and \( |E(T(C_n))| = 4n \). To define \( f : E(T(C_n)) \rightarrow \{0, 1\} \), we will consider the following two cases.

**Case 1:** For \( n = 2k, k = 1, 2, \ldots \)

\[
\begin{align*}
&f(v_iv_{i+1}) = 0; \quad 1 \leq i \leq n - 1 \\
&f(e_ie_{i+1}) = 0; \quad 1 \leq i \leq \frac{n}{2} \\
&f(e_ie_j) = 1; \quad \frac{n + 2}{2} \leq i \leq n - 1 \\
&f(e_1e_n) = 1; \\
&f(v_iv_n) = 1; \\
&f(e_iv_j) = 1; \quad \text{for all } 2n \text{ edges}
\end{align*}
\]

In both the cases we have the number of graph elements with label 0 is \( 3n \), the number of graph elements with label 1 is \( 3n \) and their difference is 0.

**Hence the result.**

Illustration 2.8: Total edge product cordial labeling of \( T(C_6) \) is shown in Figure 4.

![Figure 4](image)

Theorem 2.9: \( S'(C_n) \) admits total edge product cordial labeling.

**Proof:** Let \( C_n \) be the graph with vertices \( v_1, v_2, \ldots, v_n \). For graph \( S'(C_n) \) added vertices corresponding to \( v_1, v_2, \ldots, v_n \) are \( v'_1, v'_2, \ldots, v'_n \). Then \( |V(S'(C_n))| = 2n \) and \( |E(S'(C_n))| = 3n \). To define \( f : E(S'(C_n)) \rightarrow \{0, 1\} \), we will consider the following four cases.

**Case 1:** For \( n = 4k, k = 1, 2, \ldots \)

\[
\begin{align*}
&f(v_iv_{i+1}) = 0; \quad 1 \leq i \leq n - 3 \\
&f(v_{i+1}v_{i+4}) = 0; \quad 1 \leq i \leq n + 4 \\
&f(v_{n-1}v_n) = 0; \\
&f(v_{n-2}v_{n-1}) = 1; \\
&f(v_nv_1) = 1; \\
&f(v_iv'_j) = 1; \quad \text{for remaining } \frac{7n - 4}{4} \text{ edges}
\end{align*}
\]

Therefore the number of graph elements with label 0 is \( \frac{5n}{2} \), the number of graph elements with label 1 is \( \frac{5n}{2} \) and their difference is 0.

**Case 2:** For \( n = 4k + 1, k = 1, 2, \ldots \)

\[
\begin{align*}
&f(v_iv_{i+1}) = 0; \quad 1 \leq i \leq n - 1 \\
&f(v_{i+1}v_{i+4}) = 0; \quad 1 \leq i \leq n + 3 \\
&f(v_nv_1) = 1; \\
&f(v_iv'_j) = 1; \quad \text{for remaining } \frac{7n - 3}{4} \text{ edges}
\end{align*}
\]

Therefore the number of graph elements with label 0 is \( \frac{5n + 1}{2} \), the number of graph elements with label 1 is \( \frac{5n - 1}{2} \) and their difference is 1.

**Case 3:** For \( n = 4k + 2, k = 1, 2, \ldots \)

\[
\begin{align*}
&f(v_iv_{i+1}) = 0; \quad 1 \leq i \leq n - 1 \\
&f(v_{i+1}v_{i+4}) = 0; \quad 1 \leq i \leq n + 2 \\
&f(v_nv_1) = 1; \\
&f(v_iv'_j) = 1; \quad \text{for remaining } \frac{7n - 2}{4} \text{ edges}
\end{align*}
\]

Therefore the number of graph elements with label 0 is \( \frac{5n}{2} \), the number of graph elements with label 1 is \( \frac{5n}{2} \) and their difference is 0.

**Case 4:** For \( n = 4k + 3, k = 0, 1, \ldots \)

\[
\begin{align*}
&f(v_iv_{i+1}) = 0; \quad 1 \leq i \leq n - 1 \\
&f(v_{i+1}v_{i+4}) = 0; \quad 1 \leq i \leq n + 1 \\
&f(v_nv_1) = 1; \\
&f(v_iv'_j) = 1; \quad \text{for remaining } \frac{7n - 1}{4} \text{ edges}
\end{align*}
\]

Therefore the number of graph elements with label 0 is \( \frac{5n - 1}{2} \), the number of graph elements with label 1 is \( \frac{5n + 1}{2} \) and their difference is 1.

Thus in all the cases described above the function \( f \) satisfies the condition for total edge product cordial labeling. Hence the result.

Illustration 2.10: Total edge product cordial labeling of \( S'(C_6) \) is shown in Figure 5.

![Figure 5](image)

III. TOTAL EDGE PRODUCT CORDLABELING OF SOME PATH RELATED GRAPHS

Theorem 3.1: \( P^2_n \) admits total edge product cordial labeling except for \( n = 2 \).
Proof: For the graph $P_n^2$, we have $|V(P_n^2)| = n$ and $|E(P_n^2)| = 2n - 3$. To define $f : E(P_n^2) \rightarrow \{0, 1\}$, we will consider following four cases.

Case 1: For $n = 2$

The graph $P_2^2$ is same as $P_2$ and it is not total edge product cordial graph as proved by Vaidya and Barasara [4].

Case 2: For $n = 4k + 1, k = 1, 2, \ldots$

\[
\begin{align*}
    f(v_i v_{i+1}) &= 0; \quad 1 \leq i \leq \frac{3n - 11}{4} \\
    f(v_{n-1} v_n) &= 0; \\
    f(v_i v_j) &= 1; \quad \text{for remaining } \frac{5n - 5}{4} \text{ edges}
\end{align*}
\]

Therefore the number of graph elements with label 0 is $\frac{3n - 11}{2}$, the number of graph elements with label 1 is $\frac{3n - 3}{2}$ and their difference is 0.

Case 3: For $n = 4k + 3, k = 0, 1, \ldots$

\[
\begin{align*}
    f(v_i v_{i+1}) &= 0; \quad 1 \leq i \leq \frac{3n - 5}{4} \\
    f(v_i v_j) &= 1; \quad \text{for remaining } \frac{5n - 7}{4} \text{ edges}
\end{align*}
\]

Therefore the number of graph elements with label 0 is $\frac{3n - 5}{2}$, the number of graph elements with label 1 is $\frac{3n - 3}{2}$ and their difference is 0.

Case 4: For $n = 4k$ or $n = 4k + 2, k = 1, 2, \ldots$

In Vaidya and Barasara [5] it has been proved that $P_2^2$ is edge product cordial graph for even $n$ and $P_2^2$ is of even order. Every edge product cordial graph of either even size or even order is total edge product cordial as proved by Vaidya and Barasara [4].

Hence from case 1 to case 4 the result follows.

Illustration 3.2: Total edge product cordial labeling of $P_6^2$ is shown in Figure 6.

Theorem 3.3: $D_2(P_n)$ admits total edge product cordial labeling except for $n = 2$.

Proof: For graph $D_2(P_n)$ let $v'_1, v'_2, \ldots, v'_n$ be the vertices of first copy of path $P_n$ and $v''_1, v''_2, \ldots, v''_n$ be the vertices of second copy of path $P_n$. Then $|V(D_2(P_n))| = 2n$ and $|E(D_2(P_n))| = 4n - 4$. To define $f : E(D_2(P_n)) \rightarrow \{0, 1\}$, we will consider following three cases.

Case 1: When $n = 2$

The graph $D_2(P_2)$ is same as $C_4$ and it is not total edge product cordial graph as proved by Vaidya and Barasara [4].

Case 2: For $n = 2k, k = 2, 3, \ldots$

\[
\begin{align*}
    f(v'_i v'_{i+1}) &= 0; \quad 1 \leq i \leq n - 1 \\
    f(v''_i v''_{i+1}) &= 0; \quad 1 \leq i \leq n - 2 \\
    f(v'_i v''_j) &= 1; \quad \text{for all } 2n - 2 \text{ edges} \\
    f(v'_i v'_j) &= 1; \quad \frac{n}{2} \leq i \leq n - 1
\end{align*}
\]

Case 3: For $n = 2k + 1, k = 1, 2, \ldots$

\[
\begin{align*}
    f(v'_i v'_{i+1}) &= 0; \quad 1 \leq i \leq n - 1 \\
    f(v''_i v''_{i+1}) &= 0; \quad 1 \leq i \leq \frac{n - 1}{2} \\
    f(v'_i v''_j) &= 1; \quad \text{for remaining } \frac{3n - 3}{2} \text{ edges} \\
    f(v'_i v'_j) &= 1; \quad 1 \leq i \leq n - 1
\end{align*}
\]

In case 2 and 3, we have the number of graph elements with label 0 is $3n - 2$, the number of graph elements with label 1 is $3n - 2$ and their difference is 0.

Hence the result.

Illustration 3.4: Total edge product cordial labeling of $D_2(P_6)$ is shown in Figure 7.

Theorem 3.5: $M(P_n)$ admits total edge product cordial graph.

Proof: For the graph $M(P_n)$ we have $|V(M(P_n))| = 2n - 1$ and $|E(M(P_n))| = 3n - 4$. To define $f : E(M(P_n)) \rightarrow \{0, 1\}$, we will consider following three cases.

Case 1: For $n = 4k + 1, k = 1, 2, \ldots$

\[
\begin{align*}
    f(e_i e_{i+1}) &= 0; \quad 1 \leq i \leq n - 2 \\
    f(e_i v_{i+1}) &= 0; \quad 1 \leq i \leq \frac{n - 1}{4} \\
    f(e_1 v_2) &= 0; \\
    f(e_i v_j) &= 1; \quad \text{for remaining } \frac{7n - 11}{4} \text{ edges}
\end{align*}
\]

Therefore the number of graph elements with label 0 is $\frac{5n - 5}{2}$, the number of graph elements with label 1 is $\frac{5n - 5}{2}$ and their difference is 0.

Case 2: For $n = 4k + 3, k = 0, 1, \ldots$

\[
\begin{align*}
    f(e_i e_{i+1}) &= 0; \quad 1 \leq i \leq n - 2 \\
    f(e_i v_{i+1}) &= 0; \quad 1 \leq i \leq \frac{n + 1}{4} \\
    f(e_i v_j) &= 1; \quad \text{for remaining } \frac{7n - 9}{4} \text{ edges}
\end{align*}
\]

Therefore the number of graph elements with label 0 is $\frac{5n - 5}{2}$, the number of graph elements with label 1 is $\frac{5n - 5}{2}$ and their difference is 0.

Case 3: For $n = 4k, k = 1, 2, \ldots$ or $n = 4k + 2, k = 0, 1, \ldots$

In Vaidya and Barasara [5] it has been proved that $M(P_n)$ is edge product cordial graph for even $n$ and $M(P_n)$ is of even size. Every edge product cordial graph of either even size or even order is total edge product cordial as proved by Vaidya and Barasara [4].

Hence from case 1 to case 3 the result follows.

Illustration 3.6: Total edge product cordial labeling of $M(P_6)$ is shown in Figure 8.
Theorem 3.7: $T(P_n)$ admits total edge product cordial graph.
Proof: For the graph $T(P_n)$ we have $|V(T(P_n))| = 2n - 1$ and $|E(T(P_n))| = 4n - 5$. To define $f : E(T(P_n)) \rightarrow \{0, 1\}$, we will consider following two cases.

Case 1: For $n = 2k + 1$, $k = 1, 2, \ldots$

In Vaidya and Barasara [5] it has been proved that $T(P_n)$ is edge product cordial graph for even $n$ and $T(P_n)$ is of even order. Every edge product cordial graph of either even size or even order is total edge product cordial as proved by Vaidya and Barasara [4].

Case 2: For $n = 2k$, $k = 1, 2, \ldots$

$$f(v_i v_{i+1}) = 0; \quad 1 \leq i \leq n - 1$$
$$f(v_i e_i) = 0; \quad 1 \leq i \leq \frac{n - 2}{2}$$
$$f(e_i e_j) = 1; \quad 1 \leq i \leq n - 2$$
$$f(e_i v_j) = 1; \quad \text{for remaining } \frac{3n - 2}{2} \text{ edges}$$

Thus we have the number of graph elements with label 0 is $3n - 3$, the number of graph elements with label 1 is $3n - 3$ and their difference is 0.

Hence the result.

Illustration 3.8: Total edge product cordial labeling of $T(P_6)$ is shown in Figure 9.

Figure 9

Theorem 3.9: $S'(P_n)$ admits total edge product cordial graph.
Proof: Let $P_n$ be the graph with vertices $v_1, v_2, \ldots, v_n$. For graph $S'(P_n)$ added vertices corresponding to $v_1, v_2, \ldots, v_n$ are $v'_1, v'_2, \ldots, v'_n$. Then $|V(S'(P_n))| = 2n$ and $|E(S'(P_n))| = 3n - 3$. To define $f : E(S'(P_n)) \rightarrow \{0, 1\}$, we will consider following four cases.

Case 1: For $n = 3$

Total edge product cordial labeling of $S'(P_3)$ is shown in Figure 10.

Figure 10

Case 2: For $n = 4k$, $k = 1, 2, \ldots$ or $n = 4k + 2$, $k = 0, 1, \ldots$

In Vaidya and Barasara [5] it has been proved that $S'(P_n)$ is edge product cordial graph for even $n$ and $S'(P_n)$ is of even order. Every edge product cordial graph of either even size or even order is total edge product cordial as proved by Vaidya and Barasara [4].

Case 3: For $n = 4k + 1$, $k = 1, 2, \ldots$

$$f(v_i v_{i+1}) = 0; \quad 1 \leq i \leq n - 1$$
$$f(v_i v'_{i+1}) = 0; \quad 1 \leq i \leq \frac{n - 1}{4}$$
$$f(v_i v'_j) = 1; \quad \text{for remaining } \frac{7n - 7}{4} \text{ edges}$$

Therefore the number of graph elements with label 0 is $5n - 3$, the number of graph elements with label 1 is $\frac{5n - 3}{2}$ and their difference is 0.

Case 4: For $n = 4k + 3$, $k = 1, 2, \ldots$

$$f(v_i v_{i+1}) = 0; \quad 1 \leq i \leq n - 1$$
$$f(v_i v'_{i+1}) = 0; \quad 1 \leq i \leq \frac{n - 3}{4}$$
$$f(v_i v'_j) = 1; \quad \text{for remaining } \frac{7n - 9}{4} \text{ edges}$$

Therefore the number of graph elements with label 0 is $5n - 3$, the number of graph elements with label 1 is $\frac{5n - 3}{2}$ and their difference is 0.

Hence from case 1 to case 4 the result follows.

Illustration 3.10: Total edge product cordial labeling of $S'(P_6)$ is shown in Figure 11.

Figure 11

IV. Concluding Remarks

The vertex analogue of the present work is termed as total product cordial labeling which was introduced by Sundaram et al. [6] while we discuss here the concept of total edge product cordial labeling. These two concepts are totally independent as mentioned below.

1) $T(C_n)$ is total edge product cordial as well as total product cordial.
2) $P_2$ is not total edge product cordial but it is total product cordial.
3) $K_{2,3}$ is total edge product cordial but not total product cordial.
4) $C_4$ is neither total edge product cordial nor total product cordial.

REFERENCES