Compact Quasi-Metric Spaces

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***Abstract*—The skeleton of this manuscript consists of a concept of compact quasi-metric space and some fundamental behaviors of this space. We define continuous function and prove generalized Weierstrass theorem in this work.**

***Index Terms*— Quasi-metric, Cauchy sequence, Compact set, Bounded set, ϵ net, Totally bounded set, Closed sets, generalized Weierstrass theorem.**

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I. INTRODUCTION

IT is well known that metric and norm structures play pivotal role in functional analysis. So in order to develop functional analysis one has to take care of the suitable generalization of these structures. Historically, the problem of generalization of metric structure came first. Different authors introduced ideas of quasi-metric space [8], generalized metric space([3], [6]), generalized quasi-metric space [7], dislocated metric space [1], fuzzy metric space([2], [5], [10]), statistical metric space [9], two metric space [4] etc.

In [11], G.Rano introduced the concepts of Cauchy sequence, Convergent sequence, Open set, Closed set etc. in a quasi-metric space and established some basic theorems like Cantor's intersection theorem, Baire's category theorem etc. in complete quasi-metric spaces. We define a Contraction mapping and establish some fixed point theorem with uniqueness.

In this paper, we define a compact quasi-metric space and prove a few basic properties of this space. We define continuous function and prove generalized Weierstrass theorem in this setting.

Section I consists of some preliminary results. Section II introduces the definition of compact quasi-metric space. Section III defines continuous functions on a quasi-metric space and studies some properties of this function. This new setting enables a generalization of Weierstrass theorem.

II. PRELIMINARY RESULTS FOR THIS PAPER

In this section, we define quasi-metric space and illustrate it with an example [11].

Definition 2.1[11] Let X be a nonempty set and $\rho_q : X \to [0, \infty)$. Then ρ_q is said to be a quasi-metric on X if it satisfies the following conditions:
(QM1) $\rho_q(x, y) = 0$ iff $x = y$;
(QM2) $\rho_q(x, y) = \rho_q(y, x) \ \forall x, y \in X$;
(QM3) $\exists \ K \geq 1$ such that
$\rho_q(x, y) \leq K\{\rho_q(x, z) + \rho_q(z, y)\}$ for all $x, y, z \in X$.
The order pair (X, ρ_q) is said to be a quasi-metric space.

The quasi-metric space (X, ρ_q) is said to be a strong quasi-metric space if it satisfies the following additional condition:
(QM4) There exists $K \geq 1$ such that
$\rho_q(x_m, x_{m+p}) \leq K\{\sum_{i=0}^{p-1} \rho_q(x_{m+i}, x_{m+i+1})\}$
$\forall x_{m+i} \in X, \forall p \in N$.

The least value of the constant K satisfying (QM3) is called the *index* of the quasi-metric ρ_q.

Example 2.1[11] Let $X = R^n$ be a linear space. Let for any $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in X$ define

$$\rho_q(x, y) = \sum_{i=1}^{n} |x_i - y_i|^2.$$

Then (X, ρ_q) is a quasi-metric space but not a metric space.

Definition 2.2[11] Let (X, ρ_q) be a quasi-metric space.
(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said
(a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} \rho_q(x_n, x) = 0$;
(b) to be a Cauchy sequence if $\lim_{m, n\to\infty} \rho_q(x_n, x_m) = 0$;
(ii) A set $B \subset X$ is said to be complete if every Cauchy sequence in B converges to some point in B;
(iii) A subset A of X is said to be bounded if there exists a real number $M > 0$ such that $\rho_q(x, y) \leq M \ \forall x, y \in A$;
(iv) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim_{n\to\infty} x_n = x$ implies $x \in A$;
(v) A subset A of X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A;
(vi) A subset A of X is said to be compact in X if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in X;
(vii) The closure of $A \subset X$ is denoted by $\overline{A}$ and is defined by $\overline{A}$={ x : if $\exists$ a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = x\}$.

Proposition 2.1[11] Let (X, ρ_q) be a quasi-metric space.
(a) The limit of a sequence $\{x_n\}$ in X if exists is unique;
(b) Every subsequence of a convergent sequence converges to the same limit;
(c) Every convergent sequence in X is a Cauchy sequence.

Definition 2.3[11] Let (X, ρ_q) be a quasi-metric space and A a bounded subset of X. Then the diameter of A is denoted by $\delta(A)$ and is defined by:
$\delta(A) = \bigvee\{\rho_q(x, y) : x, y \in A\}$.

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III. Compact set in quasi-metric space

Here we establish some basic results on compact quasi-metric space and study their interrelations which are most vital for functional analysis.

Theorem 3.1 A compact quasi-metric space $(X,\ \rho_q)$ is complete.

Proof. Let the quasi-metric space $(X,\ \rho_q)$ be compact and $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in this space i.e., $\lim_{m,n\to\infty} \rho_q(x_n,\ x_m) = 0$. Since $(X,\ \rho_q)$ is compact, $\exists$ a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to a point x(say) in X i.e., $\lim_{k\to\infty} \rho_q(x_{n_k},\ x) = 0$. Now
$\rho_q(x_n,\ x) \leq\ K(\rho_q(x_n,\ x_{n_k}) + \rho_q(x_{n_k},\ x))$;
$\Rightarrow \lim_{n,\ k\to\infty} \rho_q(x_n,\ x) \leq K(\lim_{n,\ k\to\infty} \rho_q(x_n,\ x_{n_k}) + \lim_{n,\ k\to\infty} \rho_q(x_{n_k},\ x)) = 0$;
$\Rightarrow \lim_{n\to\infty} \rho_q(x_n,\ x) = 0$.
Hence the theorem.

Theorem 3.2 Any compact subset M of a quasi-metric space $(X,\ \rho_q)$ is closed and bounded.

Proof. We consider a sequence $\{x_n\}$ of M such that $\lim_{n\to\infty} x_n = x$. Since M is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point in M. Now $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} x_{n_k} = x$ [Proposition 2.1(b)] and hence $x \in\ M$. So M is closed.
If possible let M be unbounded. Let $x_0 \in\ M$ be any fixed element. Then, because M is not bounded, there exists $x_1 \in\ M$ such that $\rho_q(x_1,\ x_0) > K$. Next, applying the same argument we can say that, there exists a point $x_2 \in\ M$ such that $\rho_q(x_2,\ x_0) > K(1 + \rho_q(x_1,\ x_0))$.

By proceeding like this, we obtain a sequence of points $x_1,\ x_2,\ x_3,$......... of M such that
$\rho_q(x_n,\ x_0) >\ K\{\rho_q(x_1,\ x_0) + \rho_q(x_2,\ x_0) + \rho_q(x_3,\ x_0) + ... + \rho_q(x_{n-1},\ x_0) + 1\}$.

So, for $n > m$, $\quad \rho_q(x_n,\ x_0) > K(1 + \rho_q(x_m,\ x_0)) \quad$ (3.1).

Now, $\ \rho_q(x_n,\ x_0) \leq\ K(\rho_q(x_n,\ x_m) + \rho_q(x_m,\ x_0)) \quad$ (3.2).

and using (3.1) and (3.2) we get $\rho_q(x_n,\ x_m) > 1$ whenever $n > m$. Therefore neither the sequence $\{x_n\}$ nor any subsequence of it can converge. This is a contradiction because M is compact. This completes the proof.

Note 3.1 The converse of the above theorem is not always true because it does not hold in metric space.

Proposition 3.1 Every closed subset M of a compact quasi-metric space $(X,\ \rho_q)$ is compact.

Proof. Suppose that M is a closed subset of the quasi-metric space $(X,\ \rho_q)$ which is compact. We have to prove that M is compact. Let us consider a sequence $\{x_n\}$ of $M \subset\ X$. Clearly $\{x_n\}$ is a sequence in the compact space $(X,\ \rho_q)$. From the definition of compactness, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converging to a point $x \in\ X$. Now $\{x_{n_k}\}$ is a sequence in a closed set M converging to x. This implies $x \in\ M$. Thus every sequence in M has a subsequence converging to some point in M. Therefore M is compact.

Theorem 3.3 Let $(X,\ \rho_q)$ be a quasi-metric space. Then $A \subset\ X$ is compact in X iff $\overline{A}$ is compact.

Proof. Let $\overline{A}$ be compact and $\{x_n\}$ a sequence in A. Since $A \subset\ \overline{A}$, $\{x_n\}$ is also a sequence in the set $\overline{A}$. Again $\overline{A}$ is a compact set, which implies $\{x_n\}$ has a convergent subsequence converging to an element in the set $\overline{A} \subset\ X$. Hence A is compact in X.
Conversely, suppose that A is compact in X. Let $\{x_n\}$ be a sequence in $\overline{A}$. From the definition of $\overline{A}$, it follows that for each x_n there exists $y_n \in\ A$ such that
$\rho_q(x_n,\ y_n) < \frac{1}{n} \quad \forall\ n \in\ N$.
Since A is compact in X, the sequence $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}$ which converges to an element y in X. clearly $y \in \overline{A}$ and
$\rho_q(x_{n_k},\ y) \leq\ K(\rho_q(x_{n_k},\ y_{n_k}) + \rho_q(y_{n_k},\ y))$ for some $K \geq\ 1$;
$\Rightarrow \lim_{n\to\infty} \rho_q(x_{n_k},\ y) = 0$.
So the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to a point $y \in\ \overline{A}$ and hence $\overline{A}$ is compact.

Definition 3.1(ϵ-Net) Let $(X,\ \rho_q)$ be a quasi-metric space, $A \subset\ X$ and $\epsilon > 0$. A set $B \subset\ X$ is said to be an ϵ-net for the set A if for any $x \in\ A$ there exists $y \in\ B$ such that $\rho_q(x,\ y) < \epsilon$.

Definition 3.2 Let $(X,\ \rho_q)$ be a quasi-metric space and $A \subset\ X$. Then A ia said to be totally bounded if for any $\epsilon > 0$ there exists a finite ϵ-net for A.

Theorem 3.4 Any totally bounded subset in a quasi-metric space is bounded.

Proof. Let A be any totally bounded subset in a quasi-metric space $(X,\ \rho_q)$ and $\epsilon > 0$. Then there is a finite ϵ-net N for the set A. Let $x_1,\ x_2 \in\ A$, then there $\exists\ y_1,\ y_2 \in\ N$ such that $\rho_q(x_1,\ y_1) < \epsilon$ and $\rho_q(x_2,\ y_2) < \epsilon$.
Now
$\rho_q(x_1,\ x_2) \leq\ K\{\rho_q(x_1,\ y_1) + \rho_q(y_1,\ x_2)\}$
$\Rightarrow \rho_q(x_1,\ x_2) \leq K\rho_q(x_1,\ y_1) + K^2\rho_q(y_1,\ y_2) + K^2\rho_q(y_2,\ x_2)$
$\Rightarrow\ \rho_q(x_1,\ x_2) <\ K\epsilon + K^2\delta(N) + K^2\epsilon$;
$\Rightarrow\ \delta(A) = \bigvee_{x_1,\ x_2 \in\ A} \{\rho_q(x_1,\ x_2)\} \leq\ K\epsilon + K^2\delta(N) + K^2\epsilon$.
Since N is finite $\delta(N)$ is finite.
So $\delta(A)$ is finite and hence bounded.

Note 3.2 The converse of the Theorem 3.4 is not always true because it does not hold in metric spaces. It is clear that every metric space is a quasi-metric space with quasi index $K = 1$. So all the results of quasi-metric space are immediately hold in metric space. But if a particular property does not holds in metric space, it never holds in quasi-metric space.

Theorem 3.5 Let A be any subset in a quasi-metric space $(X,\ \rho_q)$.
(a) If A is compact in X, then A is totally bounded.
(b) If X is complete and A is totally bounded then A is compact in X.

proof(a). We assume that A is compact in X. Let $\epsilon > 0$ be arbitrary and $x_1 \in\ X$. If $\rho_q(x,\ x_1) < \epsilon \quad \forall\ x \in\ A$ then a finite ϵ-net B exists for A, i.e. $B = \{x_1\}$. If not, there exists a point $x_2 \in\ A$ such that $\rho_q(x_1,\ x_2) \geq \epsilon$. If for every $x \in\ A$ either $\rho_q(x,\ x_1) < \epsilon$ or $\rho_q(x,\ x_2) < \epsilon$ then a finite ϵ-net B exists for A, i.e. $B = \{x_1,\ x_2\}$. If not, there exists a point $x_3 \in\ A$ such that $\rho_q(x_1,\ x_3) \geq \epsilon$ and $\rho_q(x_2,\ x_3) \geq \epsilon$. If this process stopes after finite number of steps, then we get a finite ϵ-net for A. If it does not stope, then we get a sequence $x_2,\ x_3,\ x_4, ...$ of points in A such that $\rho_q(x_i,\ x_j) \geq \epsilon$ if $i \neq\ j$. Clearly this sequence has not any subsequence which is convergent. This contradict the fact that A is compact in X. Hence the theorem.

proof(b). Let X be complete and A be totally bounded i.e. for each $\epsilon > 0$ there exists a finite ϵ-net for A. We choose a sequence $\{\epsilon_n\},\ \ \epsilon_n > 0,\ \ \epsilon_{n+1} < \epsilon_n$ and construct for each $n = 1,\ 2, ...$ a finite ϵ_n-net

$$[x_1^{(n)},\ x_2^{(n)},\ x_3^{(n)},\ ....,\ x_{k_n}^{(n)}]$$

for the set A. Let $T = \{x_n\}$ be an arbitrary sequence in A. We assume that T is infinite and x_n's are distinct. Since T is totally bounded, $\exists$ a point $x_i^{(1)}$ in ϵ_1-net such as the open ball $S(x_i^{(1)},\ \epsilon_1) = \{x \in\ X : \rho_q(x,\ x_i^{(1)}) < \epsilon_1\}$ contains an infinite subset $T_1 \subset\ T$. Then $\delta(T_1) \leq\ 2K\epsilon_1$, where K is the index of the quasi-metric ρ_q. Since T_1 is totally bounded, $\exists$ a point $x_j^{(2)}$ in ϵ_2-net such that the open ball $S(x_j^{(2)},\ \epsilon_2)$ contains an infinite subset $T_2 \subset\ T_1$ and $\delta(T_2) \leq\ 2K\epsilon_2$. Continuing this process we obtain a sequence of infinite subsets

$$T \supset\ T_1 \supset\ T_2 \supset\ T_3 \supset\ .... \supset\ T_n \supset\ ....$$

with $\delta(T_n) \leq\ 2K\epsilon_n\ \ \ \forall\ n \in\ N$. We choose a sequence of distinct points such that $x_{p_i} \in\ T_i$, then for $n > m,\ \ \rho_q(x_{p_m},\ x_{p_n}) \leq\ \delta(T_m) \leq\ 2K\epsilon_m \rightarrow\ 0$ as $m \rightarrow\ \infty$. Clearly the sequence $\{x_{p_i}\}$ is a Cauchy sequence. Completeness of X implies that this sequence converges to some $x \in\ X$. Hence A is compact in X.

IV. CONTINUOUS FUNCTION

A definition of continuous functions on a quasi-metric space is given here and some properties of this function are studied. Finally we prove generalized Weierstrass theorem in this space.

Definition 4.1 Suppose $(X,\ \rho_q)$ and $(Y,\ \rho'_q)$ be two quasi-metric spaces and $f:\ X \rightarrow\ Y$ be a mapping. Then f is said to be continuous at $x \in\ X$ if for every sequence $\{x_n\}$ converging to x, we have the sequence $\{f(x_n)\}$ converges to $f(x)$ i.e. $\lim\limits_{n\rightarrow\infty} \rho_q(x_n,\ x) = 0\ \ \Rightarrow\ \ \lim\limits_{n\rightarrow\infty} \rho'_q(f(x_n),\ f(x)) = 0.$ The function f is said to be continuous function if it is continuous at every point of X.

Theorem 4.1 The continuous image of a compact quasi-metric space is a compact quasi-metric space.

proof. Consider two quasi-metric spaces $(X,\ \rho_q)$ and $(Y,\ \rho'_q)$ and a mapping $f\ :\ \ X \rightarrow\ \ Y$ which is continuous. Let $f(X) = Y' \subset\ Y$. We have to prove that $(Y',\ \rho'_q)$ is a compact quasi-metric space , where it is given that $(X,\ \rho_q)$ is compact. Let $\{y_n\}$ be any sequence in Y'. Then there exists a sequence $\{x_n\}$ in X such that $f(x_n) = y_n\ \ \forall\ n \in\ N$. Since $(X,\ \rho_q)$ is compact there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some $x \in\ X$. Since f is a continuous function, $\{f(x_{n_k})\}$ converges to $f(x) \in\ Y'$ i.e. the subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converges to $y = f(x) \in\ Y'$. Therefore $(Y',\ \rho'_q)$ is a compact quasi-metric space.

Theorem 4.2(Weierstrass) Every continuous real valued function f defined on a compact quasi-metric space is bounded and attains its least upper and greatest lower bounds.

proof. Let $(X,\ \rho_q)$ be a compact quasi-metric space and $f :\ X \rightarrow\ R$ be a continuous function. By Theorem 4.1, $f(X) = \{f(x)\ :\ x \in\ \ X\}$ is a compact subset of R and so it is closed and bounded set in R. So the least upper bound y_1 and the greatest lower bound y_2 of $f(X)$ exists in $f(X)$. Hence there exists $x_1,\ x_2 \in\ X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Hence the theorem.

CONCLUSIONS

A huge number of work have been done in metric and norm spaces. The concept of fuzzy metric space and fuzzy normed linear space are also not a very recent subject. But, except a few work, there are no remarkable one in quasi-metric space. Knowing this, we think that there may be some possibility of success in this space. The concepts of Cauchy sequence, Convergent sequence, Open set, Closed set etc. in a quasi-metric space are introduced by us [11]. We established some basic theorems like Cantor's intersection theorem, Baire's category theorem etc. in complete quasi-metric spaces. Using the concept of contraction mapping we prove some fixed point theorems in complete quasi-metric spaces and uniqueness of this theorems are studied. In this paper, we gave a definition of compact quasi-metric space and proved some basic properties of this space. We defined continuous function and proved generalized Weierstrass theorem in this space. So there is a scope to develop quasi-metric space in a big extent.

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