On the Exponential of Right Circulant Matrices

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***Abstract*—In this paper, we investigate the exponential of a right circulant matrix. We also derive the determinant, eigenvalues, spectral norm and the bound for the Euclidean norm of the matrix.**

***Index Terms*—determinant, eigenvalues, Euclidean norm, right circulant matrix, matrix exponential, spectral norm**

***MSC 2010 Codes* – 05C50, 05B20**

I. Introduction

THE Maclaurin series expansion of e^x is given by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

If $x = A$ and A is an nxn matrix, then we have

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \qquad (1)$$

The equation above is called the exponential of the matrix A. It is used to solve systems of linear differential equations.

In this paper, we investigate e^A where A is a right circulant matrix.

Definition 1.1: A matrix $RCIRC_n(\vec{c}) \in M_{nxn}(\mathbb{C})$ is said to be a **right circulant matrix** if it is of the form

$$RCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix} \qquad (2)$$

Each row of this matrix is a right cyclic shift of the row above it and $c_{i,j} = c_{k,l}$ whenever $j - i = l - k \ (mod\ n)$.

For the rest of the paper we will use $|A|_{det}$, $\|A\|_E$, $\|A\|_2$, and $\exp[A]$ for the determinant, Euclidean norm, spectral norm and exponential of the matrix A, respectively. We will also use the notation $\text{diag}(c_0, c_1, \ldots, c_{n-1})$ to denote the diagonal matrix

$$\begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & c_{n-1} \end{pmatrix}$$

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II. Preliminaries

The following will be used to prove the main results:

- $RCIRC_n(\vec{c}) = FDF^{-1}$ where
$$D = \text{diag}(\lambda_0, \lambda_1, ..., \lambda_{n-1})$$
and λ_m are the eigenvalues of $RCIRC_n(\vec{c})$
- $RCIRC_n(\vec{a}) + RCIRC_n(\vec{b}) = F(D_1 + D_2)F^{-1}$
- $|\exp[a + ib]| = \exp[a]$
- $\|A + B\|_E \leq \|A\|_E + \|B\|_E$
- $\|AB\|_E \leq \|A\|_E \|B\|_E$

Lemma 2.1: $\|F\|_E = \sqrt{n}$ where F is the Fourier matrix given by

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{(n-2)} & \omega^{2(n-2)} & \cdots & \omega^{(n-1)(n-2)} \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

where $\omega = \exp\left[\frac{2\pi i}{n}\right]$

Proof:

$$\begin{aligned} \|F\|_E &= \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} \\ &= \sqrt{\sum_{k=1}^{n^2} \frac{1}{n}}, \text{ since } |a_{ij}| = 1 \text{ for all } i \text{ and } j \\ &= \sqrt{n} \end{aligned}$$

III. Main Results

Theorem 3.1: $exp\left[RCIRC_n(\vec{c})\right]$ is right circulant and it is given by

$$\exp\left[RCIRC_n(\vec{c})\right] = F \exp\left[D\right] F^{-1}$$

where $D = \text{diag}(\lambda_0, \lambda_1, ..., \lambda_{n-1})$

Proof:

$$\begin{aligned} \exp\left[RCIRC_n(\vec{c})\right] &= \exp\left[FDF^{-1}\right] \\ &= \sum_{k=0}^{\infty} \frac{\left[FDF^{-1}\right]^k}{k!} \\ &= F\left[\sum_{k=0}^{\infty} \frac{D^k}{k!}\right] F^{-1} \\ &= F \exp[D] F^{-1} \end{aligned}$$

Note that $\exp[D]$ is a diagonal matrix. Since it is conjugated with the Fourier matrix, it follows that it $\exp\left[RCIRC_n(\vec{c})\right]$ is right circulant.

Theorem 3.2: The eigenvalues of $\exp[RCIRC_n(\vec{c})]$ are $\exp[\lambda_m]$ where λ_m are the eigenvalues of $RCIRC_n(\vec{c})$.

Proof:

$exp[D] = \exp[\text{diag}(\lambda_0, \lambda_1, ..., \lambda_{n-1})]$ contains the eigenvalues of $\exp[RCIRC_n(\vec{c})]$ in the diagonal and so we need to find the values in the diagonal.

$$\begin{aligned} \exp[D] &= \exp[\text{diag}(\lambda_0, \lambda_1, ..., \lambda_{n-1})] \\ &= \sum_{k=0}^{\infty} \frac{[\text{diag}(\lambda_0, \lambda_1, ..., \lambda_{n-1})]^k}{k!} \\ &= \sum_{k=0}^{\infty} \left[\text{diag}\left(\frac{\lambda_0^k}{k!}, \frac{\lambda_1^k}{k!}, ..., \frac{\lambda_{n-1}^k}{k!}\right)\right] \\ &= \text{diag}\left(\sum_{k=0}^{\infty} \frac{\lambda_0^k}{k!}, \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!}, ..., \sum_{k=0}^{\infty} \frac{\lambda_{n-1}^k}{k!}\right) \\ &= \text{diag}\left(\exp[\lambda_0], \exp[\lambda_1], ..., \exp[\lambda_{n-1}]\right) \end{aligned}$$

□

Theorem 3.3:

$$|\exp[RCIRC_n(\vec{c})]|_{det} = \exp\left[\sum_{m=0}^{n-1} \lambda_m\right]$$

Proof:

$$\begin{aligned} |\exp[RCIRC_n(\vec{c})]|_{det} &= \prod_{m=0}^{n-1} \exp[\lambda_m] \\ &= \exp\left[\sum_{m=0}^{n-1} \lambda_m\right] \end{aligned}$$

□

Theorem 3.4:

$$\begin{aligned} &\exp[RCIRC_n(\vec{a})]\exp[RCIRC_n(\vec{b})] \\ &= \exp[RCIRC_n(\vec{b})]\exp[RCIRC_n(\vec{a})] \\ &= \exp\left[RCIRC_n(\vec{a}) + RCIRC_n(\vec{b})\right] \end{aligned}$$

Proof:

$$\begin{aligned} &\exp[RCIRC_n(\vec{a})]\exp[RCIRC_n(\vec{b})] \\ &= \left[F\exp[D_1]F^{-1}\right]\left[F\exp[D_2]F^{-1}\right] \\ &= F\exp[D_1]\exp[D_2]F^{-1} \\ &= F\exp[D_2]\exp[D_1]F^{-1} \\ &= \left[F\exp[D_2]F^{-1}\right]\left[F\exp[D_1]F^{-1}\right] \end{aligned}$$

Furthermore,

$$\begin{aligned} &\exp[RCIRC_n(\vec{a})]\exp[RCIRC_n(\vec{b})] \\ &= \left[F\exp[D_1]F^{-1}\right]\left[F\exp[D_2]F^{-1}\right] \\ &= F\exp[D_1]\exp[D_2]F^{-1} \\ &= F\exp[D_1 + D_2]F^{-1} \\ &= \exp\left[RCIRC_n(\vec{a}) + RCIRC_n(\vec{b})\right] \end{aligned}$$

□

Theorem 3.5:

$$\|\exp[RCIRC_n(\vec{c})]\|_2 = \exp[\alpha]$$

where $\alpha = \max\{|\alpha_m|\}$ and $\alpha_m = Re[\lambda_m]$.

Proof:

$$\begin{aligned} \|\exp[RCIRC_n(\vec{c})]\|_2 &= \max\{|\exp[\lambda_m]|\} \\ &= \max\{|\exp[\alpha_m + i\beta_m]|\} \\ &= \max\{|\exp[\alpha_m]|\} \\ &= \exp[\alpha \end{aligned}$$

where $\alpha = \max\{\alpha_m\}$. □

Theorem 3.6:

$$\|\exp[RCIRC_n(\vec{c})]\|_E \leq n\exp[\|D\|_E]$$

Proof:

$$\begin{aligned} \|\exp[RCIRC_n(\vec{c})]\|_E &\leq \|F\|_E\|\exp[D]\|_E\left\|F^{-1}\right\|_E \\ &= n\|\exp[D]\|_E \\ &\leq n\exp[\|D\|_E] \end{aligned}$$

□

IV. Conclusion

In summary, we have obtained the determinant, eigenvalues, spectral norm and the bound for the Euclidean norm of the exponential of a right circulant matrix. We have also established the commutativity in the product of the exponential of these matrices.

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