Some Fixed Point Theorems in Generating Spaces of Semi-Norm Family

G. Rano* and T. Bag

***Abstract*—In this paper, we use the concept of modified Ishikawa iteration process and Mann iteration process to get fixed points of a Zamfirescu type contractive mapping in generating space of semi-norm family (G.S.S-N.F). We have shown that both iteration method give a unique fixed point of the operator. Finally we establish a common fixed point theorem for two self mappings by using two-step iteration process with errors in this setting.**

***Index Terms*—Generating space of quasi-norm family, Ishikawa iterative process, contractive mapping, common fixed point theorems.**

MSC 2010 Codes – **47H10, 54H25, 46S40.**

I. Introduction

In 1997, Chang et al.([1], [2]) dealt with the generating spaces of quasi-metric family and by studying the common characteristic properties of both fuzzy metric spaces in the sense of Kaleva & Seikkala [3] and Menger probabilistic metric spaces [4] established that the generating spaces of quasi-metric family possess the unification characterization of the fuzzy and probabilistic situations. In 2006, Xiao & Zhu [5] introduced a concept of generating spaces of quasi-norm family and studied linear topological structure and established some fixed point theorems, specially, Schauder-type fixed point theorem in such spaces. In 2010, A.O.Bosede [6] established some generalization to approximate common fixed points for self-mappings in a normed linear space using the modified Ishikawa iteration ([7], [8]) process with errors in the sense of Liu [9] and Rafiq [10]. They use a more general contractive condition than that of Rafiq [10] to establish these results. In this paper, we introduce the concept of modified Ishikawa iterative process and define contractive mapping in generating space of quasi-norm family (G.S.Q-N.F). Finally we establish some fixed point theorems for self mappings by using this iteration in G.S.S-N.F. In 2012, G.Rano, T.Bag and S.K.Samanta [11] prove some fixed point theorems in generating spaces of quasi-metric family. They also introduce the concept of Bounded linear operators [12] and dual space for such type of operators in G.S.Q-N.F.

The organization of the paper is as follows:
Section II, comprises some preliminary results.
We establish some fixed point theorems in Section III.

* Corresponding Author

G. Rano is a research scholar in the department of Mathematics, Visva-Bharati University, Shantiniketan-731235, West Bengal, India (e-mail: gobardhanr@gmail.com)

T. Bag is a Reader in the department of Mathematics, Visva-Bharati University, Shantiniketan-731235, West Bengal, India (e-mail: tarapadavb@gmail.com)

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II. Some preliminary results.

In this section some preliminary results are given which are related to this paper.

Definition 2.1 [12] Let X be a linear space over E(Real or Complex) and θ be the origin of X. Let

$$Q = \{ |.|_\alpha : \alpha \in (0,\ 1)\}$$

be a family of mappings from X into $[0,\ \infty)$. $(X,\ Q)$ is called a generating space of quasi-norm family and Q, a quasi-norm family, if the following conditions are satisfied:

(QN1) $|x|_\alpha = 0 \ \ \forall \alpha \in (0,\ 1)$ iff $x = \theta$;

(QN2) $|ex|_\alpha = |e||x|_\alpha \quad \forall\ x \in X, \forall \alpha \in (0,\ 1)$ and $\forall\ e \in E$;

(QN3) for any $\alpha \in (0,\ 1)$ there exists a $\beta \in (0,\ \alpha]$ such that

$$|x+y|_\alpha \leq |x|_\beta + |y|_\beta \quad \text{for } x,\ y \in X;$$

(QN4) for any $x \in X$, $|x|_\alpha$ is non-increasing for $\alpha \in (0,\ 1)$.

$(X,\ Q)$ is called a generating space of sub-strong quasi-norm family, strong quasi-norm family, and semi-norm family respectively, if (QN-3) is strengthened to (QN-3u), (QN-3t) and (QN-3e), where

(QN-3u) for any $\alpha \in (0,\ 1)$ there exists $\beta \in (0,\ \alpha]$ such that $|\sum_{i=1}^{n} x_i|_\alpha \leq \sum_{i=1}^{n} |x_i|_\beta$ for any $n \in Z^+$, $x_i \in X(i = 1,\ 2, ...,\ n)$;

(QN-3t) for any $\alpha \in (0,\ 1)$ there exists a $\beta \in (0,\ \alpha]$ such that

$$|x+y|_\alpha \leq |x|_\alpha + |y|_\beta \quad \text{for } x,\ y \in X;$$

(QN-3e) for any $\alpha \in (0,\ 1)$, it holds that $|x+y|_\alpha \leq |x|_\alpha + |y|_\alpha$ for $x, y \in X$.

Definition 2.2 [12] Let (X, Q) be a generating space of quasi-norm family(G.S.Q-N.F).

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said

(a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} |x_n - x|_\alpha = 0 \quad$ for each $\alpha \in (0,\ 1)$;

(b) to be a Cauchy sequence if $\lim_{m,\ n\to\infty} |x_n - x_m|_\alpha = 0 \quad$ for each $\alpha \in (0,\ 1)$.

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.

Definition 2.3 [12] Let (X, Q) be a generating space of quasi-norm family.

(a) A subset A of X is said to be bounded if for each $\alpha \in (0,\ 1)$ there exists a real number $M(\alpha)$ such that $|x|_\alpha \leq\ M(\alpha) \ \ \forall\ x \in\ A$.

(b) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim_{n\to\infty} x_n = x$ implies $x \in\ A$.

(c) A subset A of X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A.

Lemma 2.1 [9] Let $\{\rho_n\}$, $\{s_n\}$, $\{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying
$\rho_{n+1} \leq (1 - s_n)\rho_n + s_n t_n + k_n \quad \forall n \in N$.
If $\sum_{n=0}^{\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = 0$, $\sum_{n=0}^{\infty} k_n < \infty$ hold, then $\lim_{n\to\infty} \rho_n = 0$.

Following are the different types of iteration introduced by different authors.

Iteration 2.1 [6] Let K be a nonempty closed convex subset of a normed linear space E and $T : K \to K$ a self mapping. For arbitrary $x_0 \in K$, the Mann [13] iteration process for the sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - b_n)x_n + b_n T x_n, \quad n = 0, 1, 2, ... \tag{2.1}$$

Iteration 2.2 Ishikawa [6] iteration process for the sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - b_n)x_n + b_n T y_n$$
$$y_n = (1 - b'_n)x_n + b'_n T x_n, \quad n = 0, 1, 2, ... \tag{2.2}$$

where $\{b_n\}$ and $\{b'_n\}$ are sequences of real numbers in [0, 1].

Iteration 2.3 The concept of Mann iteration process with errors was introduced by Liu [9] by using the following sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - b_n)x_n + b_n T x_n + u_n \tag{2.3}$$

where $\{b_n\}$ is a sequence of real number in [0, 1] and $\sum_{n=0}^{\infty} ||u_n|| < \infty$.

Iteration 2.4 The concept of Ishikawa iteration process with errors was introduced by Liu [9] by using the following sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - b_n)x_n + b_n T x_n + u_n$$
$$y_n = (1 - b'_n)x_n + b'_n\ T x_n + v_n, \quad n = 0, 1, 2, ... \tag{2.4}$$

where $\{b_n\}$ and $\{b'_n\}$ are two sequences of real numbers in [0, 1] and
$\sum_{n=0}^{\infty} ||u_n|| < \infty$, $\sum_{n=0}^{\infty} ||v_n|| < \infty$.

Iteration 2.5 Rafiq [10] studied the two-step iteration process with errors in the sense of Liu [9] by using the two self mappings S and T of K and by the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - b_n)x_n + b_n S y_n + u_n$$
$$y_n = (1 - b'_n)x_n + b'_n\ T x_n + v_n, \quad n = 0, 1, 2, ... \tag{2.5}$$

where $\{b_n\}$ and $\{b'_n\}$ are two sequence of real number in [0, 1] and $\{u_n\}$, $\{v_n\}$ are two summable sequence in K.

Definition 2.5(Zamfirescu Contraction) [14] Let $(E, ||.||)$ be a normed linear space and $T : E \to E$. The operator T is said to be contractive if it satisfies the following condition:

$$||Tx - Ty|| \leq (2\delta\ ||x - Tx|| + \delta\ ||x - y||)\ ,\ \forall x, y \in X \text{ for some } \delta \in (0, 1) \tag{2.6}$$

III. Fixed Point Theorems

In this section we establish some fixed point theorems for contractive mappings by using different iterative [15] process.

Definition 3.1 Let (X, Q) be a G.S.Q-N.F and $T : X \to X$. The operator T is said to be continuous if for any sequence $\{x_n\}$ and $x \in X$ with $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} T(x_n) = Tx$.

Theorem 3.1 Let (X, Q) be a complete generating space of semi-norm family and $T : X \to X$ be a continuous mapping on X satisfying at least one of the following condition:
(i) $|Tx - Ty|_\alpha \leq \delta\ |x - y|_\alpha \quad \forall\alpha \in (0, 1), \forall x, y \in X$ for some fixed $\delta \in (0, 1)$;
(ii) $|Tx - Ty|_\alpha \leq \delta\ (|x - Tx|_\alpha + |y - Ty|_\alpha) \quad \forall\alpha \in (0, 1), \forall x, y \in X$ for some fixed $\delta \in (0, \frac{1}{2})$.
Then by the iterative sequence $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$
converges to the unique fixed point of T.

Proof. Let $x_0 \in X$ and $x_1 = T(x_0)$, $x_2 = T(x_1) = T^2(x_0), ..., x_n = T(x_{n-1}) = T^n(x_0)$ and so on. If T satisfies (i) then for any $\alpha \in (0, 1)$ and $n, p \in N$ we get
$|x_{n+p} - x_n|_\alpha = |T(x_{n+p-1}) - T(x_{n-1})|_\alpha \leq \delta\ |x_{n+p-1} - x_{n-1}|_\alpha$
$\Rightarrow |x_{n+p} - x_n|_\alpha \leq \delta^n\ |x_p - x_0|_\alpha$
$\Rightarrow \lim_{n\to\infty} |x_{n+p} - x_n|_\alpha = 0$.

If T satisfies (ii) then for any $\alpha \in (0, 1)$ and $n, p \in N$ we get
$|x_1 - x_2|_\alpha = |T(x_0) - T(x_1)|_\alpha \leq \delta\ [|x_0 - T(x_0)|_\alpha + |x_1 - T(x_1)|_\alpha$
$\Rightarrow |x_1 - x_2|_\alpha \leq \frac{\delta}{1-\delta}|x_0 - x_1|_\alpha$.
Similarly we can show that
$|x_2 - x_3|_\alpha \leq (\frac{\delta}{1-\delta})^2|x_0 - x_1|_\alpha$,
$|x_3 - x_4|_\alpha \leq (\frac{\delta}{1-\delta})^3|x_0 - x_1|_\alpha$,
....
$|x_n - x_{n+1}|_\alpha \leq (\frac{\delta}{1-\delta})^n|x_0 - x_1|_\alpha$.
Now
$|x_n - x_{n+p}|_\alpha \leq |x_n - x_{n+1}|_\alpha + |x_{n+1} - x_{n+2}|_\alpha + ... + |x_{n+p-1} - x_{n+p}|_\alpha$
$\leq (r^n + r^{n+1} + r^{n+2} + ... + r^{n+p-1})\ |x_0 - x_1|_\alpha$, where $r = \frac{\delta}{1-\delta}$.
$= \frac{r^n(1-r^p)}{1-r}\ |x_0 - x_1|_\alpha$.
Since $0 < \delta < \frac{1}{2}$, we have $0 < r < 1$ and
$\lim_{n\to\infty} |x_n - x_{n+p}|_\alpha = 0$.
Since $\alpha \in (0, 1)$ is arbitrary
$\lim_{m,n\to\infty} |x_n - x_{n+p}|_\alpha = 0 \quad \forall\alpha \in (0, 1)$.
Hence in both cases $\{x_n\}$ is a Cauchy sequence in (X, Q).
Since (X, Q) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.
Since T is continuous, so $\lim_{n\to\infty} T(x_n) = Tx$.
Now $Tx = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x$.
Hence x is a fixed point of T.

Uniqueness: Let $x, y \in X$ be any two fixed points of T. Then from (i) for any $\alpha \in (0, 1)$
$|x - y|_\alpha = |Tx - Ty|_\alpha \leq \delta|x - y|\alpha$
$\Rightarrow (1 - \delta)|x - y|_\alpha = 0$
$\Rightarrow |x - y|_\alpha = 0 \quad \forall\alpha \in (0, 1)$.

Hence $x = y$. From (ii), similarly we can prove the same results.

Definition 3.2(Contraction) Let $(X,\ Q)$ be a G.S.Q-N.F and $T : X \to X$. The operator T is said to be contractive if it satisfies the following condition:
$|Tx - Ty|_\alpha \leq (2\delta\ |x - Tx|_\alpha + \delta\ |x - y|_\alpha)$ $\forall \alpha \in (0,\ 1),\ \forall\ x,\ y \in X$ and for some fixed $\delta \in (0,\ 1)$ (3.1)

Theorem 3.2 Let K be a nonempty closed convex subset of a generating space of semi-norm family $(X,\ Q)$. Suppose that $T : K \to K$ is a self mapping on K satisfying the contractive condition (7). Suppose also that for arbitrary $x_0 \in K$, $\{x_n\}_{n=0}^{\infty}$ is a sequence defined by
$x_{n+1} = (1 - b_n)x_n + b_nTx_n + u_n$
where $\{b_n\}$ is a sequence of real number in $[0,\ 1]$ with
$\sum_{n=0}^{\infty} b_n = \infty$ and
$\sum_{n=0}^{\infty} |u_n|_\alpha < \infty\ \forall\ \alpha \in (0,\ 1)$.
Let $F_T \neq \phi$, where F_T is the set of fixed points of T. Then sequence $\{x_n\}$ converges to the unique fixed point of T.

Proof. Since T satisfies the contractive condition (3.1), so
$|Tx - Ty|_\alpha \leq (2\delta\ |x - Tx|_\alpha + \delta\ |x - y|_\alpha)$ $\forall \alpha \in (0,\ 1),\ \forall\ x,\ y \in X$ and for some $\delta \in (0,\ 1)$.
By assumption, $F_T \neq \phi$. Let $p \in F_T$.
Therefore, for arbitrary $x_0 \in K$ and by using iteration (2.1), we get

$$\begin{aligned} x_{n+1} - p &= (1 - b_n)x_n + b_nTx_n + u_n - p \\ &= (1 - b_n)(x_n - p) + b_n(Tx_n - p) + u_n \\ &= (1 - b_n)(x_n - p) + b_n(Tx_n - Tp) + u_n \end{aligned}$$

$$\Rightarrow |x_{n+1} - p|_\alpha = |(1 - b_n)(x_n - p) + b_n(Tx_n - Tp) + u_n|_\alpha \leq (1 - b_n)|(x_n - p)|_\alpha + b_n|(Tx_n - Tp)|_\alpha + |u_n|_\alpha.$$

By using contractive property of T we get
$|x_{n+1}-p|_\alpha \leq (1-b_n)|(x_n-p)|_\alpha+b_n(2\delta\ |p-Tp|_\alpha + \delta\ |x_n-p|_\alpha) + |u_n|_\alpha$
By observing $Tp = p$, we get
$|x_{n+1} - p|_\alpha \leq (1 - b_n)|(x_n - p)|_\alpha + b_n\ \delta\ |x_n - p|_\alpha + |u_n|_\alpha$
and hence,
$|x_{n+1} - p|_\alpha \leq (1 - b_n + b_n\delta)|(x_n - p)|_\alpha + |u_n|_\alpha$.
By applying Lemma 2.1 and using the fact that
$0 \leq b_n \leq 1,\ \ 0 < \delta_1 < 1,\ \ 0 < (1 - b_n + b_n\delta_1) < 1$,
$\sum_{n=0}^{\infty} b_n = \infty,\ \ \sum_{n=0}^{\infty} |u_n|_\alpha < \infty,\ \ \forall\ \alpha \in (0,\ 1)$,
we obtain,
$\lim_{n\to\infty} |x_{n+1} - p|_\alpha = 0$ for each $\alpha \in (0,\ 1)$.
Hence the sequence $\{x_n\}$ converges to a fixed point of T.
To prove the uniqueness, we take $p_1,\ p_2 \in F_T$ and $p_1 \neq p_2$.
By using the contractive condition (7) and $0 < \delta < 1$, we get
$|p_1 - p_2|_\alpha = |Tp_1 - Tp_2|_\alpha \leq (2\delta\ |p_1 - Tp_1|_\alpha + \delta\ |p_1 - p_2|_\alpha)$ $\forall\alpha \in (0,\ 1)$
$\Rightarrow |p_1 - p_2|_\alpha \leq \delta\ |p_1 - p_2|_\alpha < |p_1 - p_2|_\alpha$ $\forall\alpha \in (0,\ 1)$
which is a contradiction. This completes the proof.

Theorem 3.3 Let K be a nonempty closed convex subset of a generating space of semi-norm family $(X,\ Q)$. Suppose that $T : K \to K$ is a self mapping on K satisfying the contractive condition (7). Suppose also that for arbitrary $x_0 \in K$, $\{x_n\}_{n=0}^{\infty}$ is a sequence defined by
$x_{n+1} = (1 - b_n)x_n + b_nTy_n$,
$y_n = (1 - b'_n)x_n + b'_n\ Tx_n,\ \ n = 0, 1, 2, \ldots\ldots\ldots\ldots$
where $\{b_n\}$ and $\{b'_n\}$ are sequences of real number in $[0,\ 1]$ with $\sum_{n=0}^{\infty} b_n = \infty$.
Let $F_T \neq \phi$, where F_T is the set of fixed points of T. Then sequence $\{x_n\}$ converges to the unique fixed point of T.

Proof. Since T satisfies the contractive condition (7), so
$|Tx - Ty|_\alpha \leq (2\delta\ |x - Tx|_\alpha + \delta\ |x - y|_\alpha)$ $\forall \alpha \in (0,\ 1),\ \forall\ x,\ y \in X$ and for some $\delta \in (0,\ 1)$.
By assumption, $F_T \neq \phi$. Let $p \in F_T$.
Therefore, for arbitrary $x_0 \in K$ and by using iteration (1), we get

$$\begin{aligned} x_{n+1} - p &= (1 - b_n)x_n + b_nTy_n - (1 - b_n + b_n)p \\ &= (1 - b_n)(x_n - p) + b_n(Ty_n - p) \end{aligned}$$

$$\begin{aligned} \Rightarrow |x_{n+1} - p|_\alpha &= |(1 - b_n)(x_n - p) + b_n(Ty_n - p)|_\alpha \\ &\leq (1 - b_n)|(x_n - p)|_\alpha + b_n|(Ty_n - p)|_\alpha \end{aligned} \quad (3.2)$$

Putting $x = p$ and $y = y_n$ in (7) and observing $Tp = p$ we get

$$|Ty_n - p|_\alpha \leq \delta\ |y_n - p|_\alpha. \quad (3.3)$$

Further we have

$$\begin{aligned} |y_n - p|_\alpha &= |(1 - b'_n)(x_n - p) + b'_n(Tx_n - p)|_\alpha \\ &\leq (1 - b'_n)|(x_n - p)|_\alpha + b'_n|(Tx_n - p)|_\alpha \end{aligned} \quad (3.4)$$

Putting $x = p$ and $y = x_n$ in (7) and observing $Tp = p$ we get

$$|Tx_n - p|_\alpha \leq \delta\ |x_n - p|_\alpha. \quad (3.5)$$

and hence, by (3.2)-(3.5) we obtain
$|x_{n+1} - p|_\alpha \leq [1 - b_n + \delta\ b_n(1 - b'_n) + \delta^2 b_n\ b'_n]|(x_n - p)|_\alpha$
$\Rightarrow |x_{n+1} - p|_\alpha \leq [1 - (1 - \delta)b_n(1 + \delta\ b'_n)]|(x_n - p)|_\alpha$,
which, by the inequality
$1 - (1 - \delta)b_n(1 + \delta\ b'_n) \leq 1 - (1 - \delta)^2 b_n$
implies
$|x_{n+1} - p|_\alpha \leq [1 - (1 - \delta)^2 b_n]|(x_n - p)|_\alpha,\ n = 0,\ 1,\ 2, \ldots\ldots$
$\Rightarrow |x_{n+1} - p|_\alpha \leq \prod_{k=0}^{n}[1 - (1 - \delta)^2 b_k]|(x_0 - p)|_\alpha,\ n = 0,\ 1,\ 2, \ldots\ldots.$
Using the fact that
$0 \leq b_n \leq 1,\ \ 0 < \delta < 1$ and $\sum_{n=0}^{\infty} b_n = \infty$,
we obtain,
$\lim_{n\to\infty} \prod_{k=0}^{n}[1 - (1 - \delta)^2 b_k] = 0$
$\Rightarrow |x_{n+1} - p|_\alpha = 0$ for each $\alpha \in (0,\ 1)$
Hence the sequence $\{x_n\}$ converges to a fixed point p of T.
To prove the uniqueness, we take $p_1,\ p_2 \in F_T$ and $p_1 \neq p_2$.
By using the contractive condition (3.1) and $0 < \delta < 1$, we get
$|p_1 - p_2|_\alpha = |Tp_1 - Tp_2|_\alpha \leq (2\delta\ |p_1 - Tp_1|_\alpha + \delta\ |p_1 - p_2|_\alpha)$ $\forall\alpha \in (0,\ 1)$

$\Rightarrow \ |p_1 - p_2|_\alpha \leq \ \delta \ |p_1 - p_2|_\alpha < \ |p_1 - p_2|_\alpha \quad \forall \alpha \in (0,\ 1)$ which is a contradiction. This completes the proof.

Theorem 3.4 Let K be a nonempty closed convex subset of a generating space of semi-norm family $(X,\ Q)$. Suppose that $S,\ T : K \to \ K$ are two self-mappings on K satisfying the contractive condition (3.1). Suppose also that for arbitrary $x_0 \in \ K,\ \{x_n\}_{n=0}^{\infty}$ is a sequence defined by
$x_{n+1} = (1 - b_n)x_n + b_n S y_n + u_n,$
$y_n = (1 - b'_n)x_n + b'_n \ T x_n + v_n, \quad n = 0, 1, 2, ...$
where $\{b_n\}$ and $\{b'_n\}$ are two sequence of real number in $[0,\ 1]$ with $\sum_{n=0}^{\infty} \ b_n \ = \infty$ and
$\sum_{n=0}^{\infty} \ |u_n|_\alpha \ < \infty, \quad \sum_{n=0}^{\infty} \ |v_n|_\alpha \ = 0 \ \ \forall \ \alpha \in \ (0,\ 1).$
Let $F_S \bigcap \ F_T \neq \ \phi$, where F_S and F_T are the sets of fixed points of S and T respectively.Then sequence $\{x_n\}$ converges to a common fixed point of S and T.

Proof. Since S and T satisfy the contractive condition, so
$|Sx - Sy|_\alpha \ \leq \ (2\delta_1 \ |x - Sx|_\alpha \ + \ \delta_1 \ |x - y|_\alpha) \quad \forall \alpha \in (0,\ 1),\ \forall \ x,\ y \in \ X$ for some $\delta_1 \in \ (0,\ 1)$.
$|Tx - Ty|_\alpha \ \leq \ (2\delta_2 \ |x - Tx|_\alpha \ + \ \delta_2 \ |x - y|_\alpha) \quad \forall \alpha \in (0,\ 1),\ \forall \ x,\ y \in \ X$ for some $\delta_2 \in \ (0,\ 1)$.
By assumption, $F_S \bigcap \ F_T \neq \phi$. Let $p \in \ F_S \bigcap \ F_T$.
Therefore, for arbitrary $x_0 \in \ K$ and by using Iteration (2), we get

$$\begin{aligned} x_{n+1} - p &= (1 - b_n)x_n + b_n S y_n + u_n - p \\ &= (1 - b_n)(x_n - p) + b_n(S y_n - p) + u_n \\ &= (1 - b_n)(x_n - p) + b_n(S y_n - Sp) + u_n \end{aligned}$$

$$\begin{aligned} \Rightarrow \ |x_{n+1} - p|_\alpha &= |(1 - b_n)(x_n - p) + b_n(S y_n - Sp) + u_n|_\alpha \\ &\leq \ (1 - b_n)|(x_n - p)|_\alpha + b_n|(S y_n - Sp)|_\alpha + |u_n|_\alpha. \end{aligned}$$

By using contractive property of S we get
$|x_{n+1} - p|_\alpha \ \leq \ (1 - b_n)|(x_n - p)|_\alpha + b_n(2\delta_1 \ |p - Sp|_\alpha \ + \ \delta_1 \ |y_n - p|_\alpha) + |u_n|_\alpha$
By observing $Sp = p$, we get

$$|x_{n+1} - p|_\alpha \leq \ (1 - b_n)|(x_n - p)|_\alpha + b_n \ \delta_1 \ |y_n - p|_\alpha + |u_n|_\alpha \tag{3.6}$$

Similarly we get

$$\begin{aligned} |y_n - p|_\alpha &\leq \ (1 - b'_n)|(x_n - p)|_\alpha + b'_n \ \delta_2 \ |x_n - p|_\alpha + |v_n|_\alpha \\ &\leq \ (1 - b'_n + \delta_2 b'_n)|(x_n - p)|_\alpha + |v_n|_\alpha. \end{aligned}$$

By observing that $(1 - b'_n + \delta_2 b'_n) \leq \ 1$, we obtain

$$|y_n - p|_\alpha \leq \ |(x_n - p)|_\alpha + |v_n|_\alpha \tag{3.7}$$

From (3.6) and (3.7) we get
$|x_{n+1} - p|_\alpha \leq \ (1 - b_n + b_n\delta_1)|(x_n - p)|_\alpha + b_n \ \delta_1 \ |v_n|_\alpha + |u_n|_\alpha$.
By applying Lemma 1.1 and using the fact that
$0 \leq \ b_n \ \leq \ 1, \quad 0 < \delta_1 < 1, \quad 0 < (1 - b_n + b_n\delta_1) < 1,$
$\sum_{n=0}^{\infty} \ b_n \ = \ \infty, \quad \sum_{n=0}^{\infty} \ |u_n|_\alpha \ < \ \infty, \quad \lim_{n \to \infty} |v_n|_\alpha \ = 0 \ \ \forall \ \alpha \in \ (0,\ 1),$
we obtain
$\lim_{n \to \infty} |x_{n+1} - p|_\alpha = 0 \quad$ for each $\alpha \in (0,\ 1)$.
Hence the sequence $\{x_n\}$ converges to a common fixed point of S and T.

CONCLUSIONS

Fixed points and fixed point theorems have always been a major technical tool of many branches of mathematics, like as differential equation, economics, game theory, dynamics, control theory, functional analysis etc. The application of this theorems are increasing day by day with the development of techniques for computing and making fixed point method which is a major of applied mathematics. In this paper, we use the concept of modified Ishikawa iteration process and Mann iteration process to get fixed points of a Zamfirescu type contractive mapping in generating space of semi-norm family (G.S.S-N.F). Since our field of research is relatively new, there are many scope to develop many new type fixed theorems in this setting.

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