Some Fixed Point Theorems in Generating Spaces of Semi-Norm Family

G. Rano* and T. Bag

Abstract—In this paper, we use the concept of modified Ishikawa iteration process and Mann iteration process to get fixed points of a Zamfirescu type contractive mapping in generating space of semi-norm family (G.S.S.N.F). We have shown that both iteration method give a unique fixed point of the operator. Finally we establish a common fixed point theorem for two self mappings by using two-step iteration process with errors in this setting.

Index Terms—Generating space of quasi-norm family, Ishikawa iterative process, contractive mapping, common fixed point theorems.

MSC 2010 Codes – 47H10, 54H25, 46S40.

I. INTRODUCTION

In 1997, Chang et al.([1], [2]) dealt with the generating spaces of quasi-metric family and by studying the common characteristic properties of both fuzzy metric spaces in the sense of Kaleva & Seikkala [3] and Menger probabilistic metric spaces [4] established that the generating spaces of quasi-metric family possess the unification characterization of the fuzzy and probabilistic situations. In 2006, Xiao & Zhu [5] introduced a concept of generating spaces of quasi-norm family and studied linear topological structure and established some fixed point theorems, specially, Schauder-type fixed point theorem in such spaces. In 2010, A.O.Bosede [6] established some generalization to approximate common fixed points for self-mappings in a normed linear space using the modified Ishikawa iteration ([7], [8]) process with errors in the sense of Liu [9] and Rafiq [10]. They use a more general contractive condition than that of Rafiq [10] to establish these results. In this paper, we introduce the concept of modified Ishikawa iterative process and define contractive mapping in generating space of quasi-norm family (G.S.Q-N.F). Finally we establish some fixed point theorems for self-mappings by using this iteration in G.S.S-N.F. In 2012, G.Rano, T.Bag and S.K.Samanta [11] prove some fixed point theorems in generating spaces of quasi-metric family. They also introduce the concept of Bounded linear operators [12] and dual space for such type of operators in G.S.N-Q.F.

The organization of the paper is as follows:

Section II, comprises some preliminary results.

II. SOME PRELIMINARY RESULTS.

In this section some preliminary results are given which are related to this paper.

Definition 2.1 [12] Let X be a linear space over $E$(Real or Complex) and $\theta$ be the origin of $X$. Let $$Q = \{ |x| : \alpha \in (0, 1) \}$$ be a family of mappings from $X$ into $[0, \infty)$. $(X, Q)$ is called a generating space of quasi-norm family and $Q$, a quasi-norm family, if the following conditions are satisfied:

(QN1) $|x| = 0 \iff x = \theta$;

(QN2) $|\alpha x| = |\alpha||x|$ for any $\alpha \in (0, 1)$ and $\forall \epsilon \in E$;

(QN3) for any $\alpha \in (0, 1)$ there exists a $\beta \in (0, \alpha]$ such that $|x + \beta \alpha| \leq |x| + |\beta\alpha|$ for $x, \beta \in X$;

(QN4) for any $x \in X$, $|x| \leq 1$.

$(X, Q)$ is called a generating space of sub-strong quasi-norm family, strong quasi-norm family, and semi-norm family respectively, if (QN-3) is strengthened to (QN-3u), (QN-3i) and (QN-3e), where

(QN-3u) for any $\alpha \in (0, 1)$ there exists a $\beta \in (0, \alpha]$ such that

$$|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|$$

for any $\alpha \in (0, 1)$.

(QN-3i) for any $\alpha \in (0, 1)$ there exists a $\beta \in (0, \alpha]$ such that

$$|x + \beta \alpha| \leq |x| + |\beta\alpha|$$

for $x, \beta \in X$.

Definition 2.2 [12] Let $(X, Q)$ be a generating space of quasi-norm family(G.S.Q-N.F).\n
(i) A sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ is said (a) to converge to $x \in X$ denoted by $\lim_{n \to \infty} x_n = x$ if $\lim_{n \to \infty} |x_n - x| = 0$ for each $\alpha \in (0, 1)$;

(b) to be a Cauchy sequence if $\lim_{m, n \to \infty} |x_n - x_m| = 0$ for each $\alpha \in (0, 1)$.

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in $B$ converges in $B$.

Definition 2.3 [12] Let $(X, Q)$ be a generating space of quasi-norm family.\n
(a) A subset $A$ of $X$ is said to be bounded if for each $\alpha \in (0, 1)$ there exists a real number $M(\alpha)$ such that $|x| \leq M(\alpha)$ for all $x \in A$.

(b) A subset $A$ of $X$ is said to be closed if for any sequence $\{x_n\}$ of points of $A$ with $\lim_{n \to \infty} x_n = x$ implies $x \in A$.

(c) A subset $A$ of $X$ is said to be compact if for any sequence $\{x_n\}$ of points of $A$ has a convergent subsequence which converges to a point in $A$. 

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The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F. 5104/DRS/2009(SAP-I)]
Lemma 2.1 [9] Let \( \{\rho_n\}, \{s_n\}, \{t_n\} \) and \( \{k_n\} \) be sequences of nonnegative numbers satisfying \( \rho_{n+1} \leq (1-s_n)\rho_n + s_n t_n + k_n \) \( \forall n \in \mathbb{N} \).

If \( \sum_{n=0}^{\infty} s_n = \infty, \lim_{n \to \infty} t_n = 0, \sum_{n=0}^{\infty} k_n < \infty \) hold, then \( \lim_{n \to \infty} \rho_n = 0. \)

Following are the different types of iteration introduced by different authors.

Iteration 2.1 [6] Let \( K \) be a nonempty closed convex subset of a normed linear space \( E \) and \( T : K \to K \) a self mapping. For arbitrary \( x_0 \in K \), the Mann [13] iteration process for the sequence \( \{x_n\} \) is defined by

\[
x_{n+1} = (1 - b_n)x_n + b_n Tx_n, \quad n = 0, 1, 2, \ldots
\]

(2.1)

Iteration 2.2 Ishikawa [6] iteration process for the sequence \( \{x_n\} \) is defined by

\[
x_{n+1} = (1 - b_n)x_n + b_n Ty_n \\
y_n = (1 - b'_n)x_n + b'_n Tx_n, \quad n = 0, 1, 2, \ldots
\]

(2.2)

where \( \{b_n\} \) and \( \{b'_n\} \) are sequences of real numbers in [0, 1].

Iteration 2.3 The concept of Mann iteration process with errors was introduced by Liu [9] by using the following sequence \( \{x_n\} \) defined by

\[
x_{n+1} = (1 - b_n)x_n + b_n Tx_n + u_n
\]

(2.3)

where \( \{b_n\} \) is a sequence of real number in \([0, 1]\) and \( \sum_{n=0}^{\infty} ||u_n|| < \infty. \)

Iteration 2.4 The concept of Ishikawa iteration process with errors was introduced by Liu [9] by using the following sequence \( \{x_n\} \) defined by

\[
x_{n+1} = (1 - b_n)x_n + b_n Ty_n + u_n \\
y_n = (1 - b'_n)x_n + b'_n Tx_n + v_n, \quad n = 0, 1, 2, \ldots
\]

(2.4)

where \( \{b_n\} \) and \( \{b'_n\} \) are two sequences of real numbers in \([0, 1]\) and \( \sum_{n=0}^{\infty} ||u_n|| < \infty, \sum_{n=0}^{\infty} ||v_n|| < \infty. \)

Iteration 2.5 Rafiq [10] studied the two-step iteration process with errors in the sense of Liu [9] by using the two self mappings \( S \) and \( T \) of \( K \) and by the sequence \( \{x_n\}_{n=0}^{\infty} \) defined by

\[
x_{n+1} = (1 - b_n)x_n + b_n Sy_n + u_n \\
y_n = (1 - b'_n)x_n + b'_n Tx_n + v_n, \quad n = 0, 1, 2, \ldots
\]

(2.5)

where \( \{b_n\} \) and \( \{b'_n\} \) are two sequence of real number in \([0, 1]\) and \( \{u_n\}, \{v_n\} \) are two summable sequence in \( K \).

Definition 2.5(Zamfirescu Contraction) [14] Let \( (E, ||.||) \) be a normed linear space and \( T : E \to E \). The operator \( T \) is said to be contractive if it satisfies the following condition:

\[ ||Tx - Ty|| \leq (2\delta ||x - Tx|| + \delta ||x - y||), \forall x, y \in X \]

(2.6)

for some \( \delta \in (0, 1) \).

III. FIXED POINT THEOREMS

In this section we establish some fixed point theorems for contractive mappings by using different iterative [15] process.

Definition 3.1 Let \( (X, Q) \) be a G.S.Q.N.F and \( T : X \to X \). The operator \( T \) is said to be continuous if for any sequence \( \{x_n\} \) and \( x \in X \) with \( \lim_{n \to \infty} x_n = x \) implies \( \lim_{n \to \infty} T(x_n) = Tx. \)

Theorem 3.1 Let \( (X, Q) \) be a complete generating space of semi-norm family and \( T : X \to X \) be a continuous mapping on \( X \) satisfying at least one of the following condition:

(i) \( |Tx - Ty|_\alpha \leq \delta (|x - y|_\alpha, \forall \alpha \in (0, 1), \forall x, y \in X \) for some fixed \( \delta \in (0, 1); \)

(ii) \( |Tx - Ty|_\alpha \leq \delta (|x - Tx|_\alpha + |y - Ty|_\alpha), \forall \alpha \in (0, 1), \forall x, y \in X \) for some fixed \( \delta \in (0, \frac{1}{4}). \)

Then by the iterative sequence \( x_{n+1} = Tx_n \) for \( n = 0, 1, 2, \ldots \)

converges to the unique fixed point of \( T. \)

Proof. Let \( x_0 \in X \) and \( x_1 = T(x_0), x_2 = T(x_1) = T^2(x_0), \ldots, x_n = T(x_{n-1}) = T^n(x_0) \) and so on. If \( T \) satisfies (i) then for any \( \alpha \in (0, 1) \) and \( n, p \in N \) we get

\[ |x_{n+p} - x_n|_\alpha = |T(x_{n+p-1}) - T(x_{n-1})|_\alpha \leq \delta |x_{n+p-1} - x_{n-1}|_\alpha \]

\[ \Rightarrow |x_{n+p} - x_n|_\alpha \leq \delta^n |x_0 - x_0|_\alpha \]

\[ \Rightarrow \lim_{n \to \infty} |x_{n+p} - x_n|_\alpha = 0. \]

If \( T \) satisfies (ii) then for any \( \alpha \in (0, 1) \) and \( n, p \in N \) we get

\[ |x_1 - x_2|_\alpha = |T(x_0) - T(x_1)|_\alpha \leq \delta |x_0 - T(x_0)|_\alpha + |x_1 - T(x_1)|_\alpha \]

\[ \Rightarrow |x_1 - x_2|_\alpha \leq \frac{\delta}{1 - \delta} |x_0 - x_1|_\alpha. \]

Similarly we can show that

\[ |x_2 - x_3|_\alpha \leq \left( \frac{\delta}{1 - \delta} \right)^2 |x_0 - x_1|_\alpha. \]

\[ |x_3 - x_4|_\alpha \leq \left( \frac{\delta}{1 - \delta} \right)^3 |x_0 - x_1|_\alpha. \]

\[ \ldots \]

\[ |x_n - x_{n+1}|_\alpha \leq \left( \frac{\delta}{1 - \delta} \right)^n |x_0 - x_1|_\alpha. \]

Now

\[ |x_n - x_{n+p}|_\alpha \leq |x_n - x_{n+1}|_\alpha + |x_{n+1} - x_{n+2}|_\alpha + \ldots + |x_{n+p-1} - x_{n+p}|_\alpha \]

\[ \leq \left( r^n + r^{n+1} + r^{n+2} + \ldots + r^{n+p-1} \right) |x_0 - x_{n+1}|_\alpha, \]

where \( r = \frac{\delta}{1 - \delta}. \)

Since \( 0 < \delta < \frac{1}{2}, \) we have \( 0 < r < 1 \) and

\[ \lim_{n \to \infty} |x_n - x_{n+p}|_\alpha = 0. \]

Since \( \alpha \in (0, 1) \) is arbitrary

\[ \lim_{n \to \infty} |x_n - x_{n+p}|_\alpha = 0 \quad \forall \alpha \in (0, 1). \]

Hence in both cases \( \{x_n\} \) is a Cauchy sequence in \( (X, Q). \)

Since \((X, Q)\) is complete, there exists \( x \in X \) such that \( \lim_{n \to \infty} x_n = x. \)

Since \( T \) is continuous, so \( \lim_{n \to \infty} T(x_n) = Tx. \)

Now \( Tx = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = x. \)

Hence \( x \) is a fixed point of \( T. \)

Uniqueness: Let \( x, y \in X \) be any two fixed points of \( T. \) Then from (i) for any \( \alpha \in (0, 1) \)

\[ |x - y|_\alpha = |T(x) - T(y)|_\alpha \leq \delta|x - y|_\alpha \]

\[ \Rightarrow (1 - \delta)|x - y|_\alpha = 0 \]

\[ \Rightarrow x - y|_\alpha = 0 \quad \forall \alpha \in (0, 1). \]
Hence \( x = y \). From (ii), similarly we can prove the same results.

**Definition 3.2 (Contraction)** Let \((X, Q)\) be a G.S.Q-N.F and \(T : X \to X\). The operator \(T\) is said to be contractive if it satisfies the following condition: 
\[
|Tx - Ty|_\alpha \leq (2\alpha |x - T x|_\alpha + \delta |x - y|_\alpha) \quad \forall \alpha \in (0, 1), \forall x, y \in X \text{ and for some fixed } \delta \in (0, 1)
\]

(3.1)

**Theorem 3.2** Let \(K\) be a nonempty closed convex subset of a generating space of semi-norm family \((X, Q)\). Suppose that \(T : K \to K\) is a self-mapping on \(K\) satisfying the contractive condition (7). Suppose also that for arbitrary \(x_0 \in K\), \(\{x_n\}_{n=0}^\infty\) is a sequence defined by \(x_{n+1} = (1-b_n)x_n + b_nTx_n + u_n\) where \(\{b_n\}\) is a sequence of real number in \([0, 1]\) with 
\[
\sum_{n=0}^\infty b_n = \infty \quad \text{and} \quad \sum_{n=0}^\infty |u_n|_\alpha < \infty \forall \alpha \in (0, 1).
\]

Let \(K_T \neq \emptyset\), where \(K_T\) is the set of fixed points of \(T\). Then sequence \(\{x_n\}\) converges to the unique fixed point of \(T\).

**Proof.** Since \(T\) satisfies the contractive condition (3.1), so 
\[
|Tx - Ty|_\alpha \leq (2\alpha |x - T x|_\alpha + \delta |x - y|_\alpha) \quad \forall \alpha \in (0, 1), \forall x, y \in X \text{ and for some } \delta \in (0, 1).
\]

By assumption, \(T_T \neq \emptyset\). Let \(p \in K_T\). Therefore, for arbitrary \(x_0 \in K\) and by iteration (2.1), we get 
\[
\begin{align*}
x_{n+1} \to p & = (1-b_n)x_n + b_nTx_n + u_n - p \\
& = (1-b_n)(x_n - p) + b_n(Tx_n - p) + u_n \\
& = (1-b_n)(x_n - p) + b_n(Tx_n - Tp) + u_n \\
\Rightarrow |x_{n+1} - p|_\alpha & = |(1-b_n)(x_n - p) + b_n(Tx_n - Tp) + u_n|_\alpha \\
& \leq (1-b_n)|x_n - p|_\alpha + b_n|Tx_n - Tp|_\alpha + |u_n|_\alpha.
\end{align*}
\]

By using contraction property of \(T\) we get 
\[
|x_{n+1} - p|_\alpha \leq (1-b_n)|x_n - p|_\alpha + b_n(2\alpha |p - Tp|_\alpha + \delta |x_n - p|_\alpha) + |u_n|_\alpha.
\]

By observing \(T_T = p\), we get 
\[
|x_{n+1} - p|_\alpha \leq (1-b_n)|x_n - p|_\alpha + b_n(2\alpha |p - Tp|_\alpha + \delta |x_n - p|_\alpha) + |u_n|_\alpha.
\]

and hence, 
\[
|x_{n+1} - p|_\alpha \leq (1-b_n)|x_n - p|_\alpha + b_n |u_n|_\alpha.
\]

By applying Lemma 2.1 and using the fact that 
\[
\begin{align*}
0 \leq b_n & \leq 1, \\ 0 < d & < 1, \\ 0 < (1-b_n + b_n \delta) & < 1,
\end{align*}
\]

we obtain, 
\[
\lim_{n \to \infty} |x_{n+1} - p|_\alpha = 0 \quad \text{for each } \alpha \in (0, 1).
\]

Hence the sequence \(\{x_n\}\) converges to a fixed point of \(T\). To prove the uniqueness, we take \(p_1, p_2 \in K_T\) and \(p_1 \neq p_2\).

By using the contractive condition (7) and \(0 < \delta < 1\), we get 
\[
\begin{align*}
|p_1 - p_2|_\alpha & = |T_{p_1} - T_{p_2}|_\alpha \leq (2\alpha |p_1 - Tp_1|_\alpha + \delta |p_1 - p_2|_\alpha) \quad \forall \alpha \in (0, 1) \\
\Rightarrow |p_1 - p_2|_\alpha & \leq \delta |p_1 - p_2|_\alpha < |p_1 - p_2|_\alpha, \forall \alpha \in (0, 1)
\end{align*}
\]

which is a contradiction. This completes the proof.

**Theorem 3.3** Let \(K\) be a nonempty closed convex subset of a generating space of semi-norm family \((X, Q)\). Suppose that \(T : K \to K\) is a self-mapping on \(K\) satisfying the contractive condition (7). Suppose also that for arbitrary \(x_0 \in K\), \(\{x_n\}_{n=0}^\infty\) is a sequence defined by 
\[
x_{n+1} = (1-b_n)x_n + b_nTy_n + n = 0, 1, 2, \ldots,
\]

where \(\{b_n\}\) and \(\{b'_n\}\) are sequences of real number in \([0, 1]\) with 
\[
\sum_{n=0}^\infty b_n = \infty.
\]

Let \(K_T \neq \emptyset\), where \(K_T\) is the set of fixed points of \(T\). Then sequence \(\{x_n\}\) converges to the unique fixed point of \(T\).

**Proof.** Since \(T\) satisfies the contractive condition (7), so 
\[
|Tx - Ty|_\alpha \leq (2\alpha |x - T x|_\alpha + \delta |x - y|_\alpha) \quad \forall \alpha \in (0, 1), \forall x, y \in X \text{ and for some } \delta \in (0, 1).
\]

By assumption, \(T_T \neq \emptyset\). Let \(p \in K_T\). Therefore, for arbitrary \(x_0 \in K\) and by iteration (1), we get 
\[
\begin{align*}
x_{n+1} - p & = (1-b_n)x_n + b_nTy_n - (1-b_n + b_n)p \\
& = (1-b_n)(x_n - p) + b_n(Ty_n - p) + u_n \\
\Rightarrow |x_{n+1} - p|_\alpha & = |(1-b_n)(x_n - p) + b_n(Ty_n - p) + u_n|_\alpha \\
& \leq (1-b_n)|x_n - p|_\alpha + b_n|Ty_n - p|_\alpha + |u_n|_\alpha.
\end{align*}
\]

(3.2)

Putting \(x = p\) and \(y = y_n\) in (7) and observing \(Tp = p\) we get 
\[
|Ty_n - p|_\alpha \leq \delta |y_n - p|_\alpha.
\]

(3.3)

Further we have 
\[
|y_n - p|_\alpha = |(1-b'_n)(x_n - p) + b'_n(Tx_n - p)|_\alpha \\
\leq (1-b'_n)|x_n - p|_\alpha + b'_n|(Tx_n - p)|_\alpha.
\]

(3.4)

Putting \(x = p\) and \(y = x_n\) in (7) and observing \(Tp = p\) we get 
\[
|Tx_n - p|_\alpha \leq \delta |x_n - p|_\alpha.
\]

(3.5)

and hence, by (3.2)-(3.5) we obtain 
\[
|x_{n+1} - p|_\alpha \leq \delta b_n(1 + b'_n) \leq 1 - (1 - \delta)^2 b_n
\]

implies 
\[
|x_{n+1} - p|_\alpha \leq \delta (1 - \delta)^2 b_k(x_n - p)|_\alpha, \quad n = 0, 1, 2, \ldots
\]

\[
\Rightarrow |x_{n+1} - p|_\alpha \leq \prod_{k=0}^\infty (1 - \delta)^2 b_k(x_0 - p)|_\alpha, \quad n = 0, 1, 2, \ldots
\]

Using the fact that 
\[
0 \leq b_n \leq 1, \quad 0 < \delta < 1 \text{ and } \sum_{n=0}^\infty b_n = \infty,
\]

we obtain, 
\[
\lim_{n \to \infty} \prod_{k=0}^\infty (1 - \delta)^2 b_k(x_0 - p)|_\alpha = 0
\]

\[
\Rightarrow |x_{n+1} - p|_\alpha = 0 \quad \text{for each } \alpha \in (0, 1).
\]

Hence the sequence \(\{x_n\}\) converges to a fixed point of \(T\). To prove the uniqueness, we take \(p_1, p_2 \in K_T\) and \(p_1 \neq p_2\).

By using the contractive condition (3.1) and \(0 < \delta < 1\), we get 
\[
|p_1 - p_2|_\alpha = |Tp_1 - Tp_2|_\alpha \leq (2\alpha |p_1 - Tp_1|_\alpha + \delta |p_1 - p_2|_\alpha) \quad \forall \alpha \in (0, 1)
\]

\[
\Rightarrow |p_1 - p_2|_\alpha \leq \delta |p_1 - p_2|_\alpha < |p_1 - p_2|_\alpha, \forall \alpha \in (0, 1)
\]

which is a contradiction. This completes the proof.
\[ |p_1 - p_2| \leq \delta \quad |p_1 - p_2| < |p_1 - p_2| \quad \forall \alpha \in (0, 1) \]
which is a contradiction. This completes the proof.

**Theorem 3.4** Let \( K \) be a nonempty closed convex subset of a generating space of semi-norm family \((X, Q)\). Suppose that \( S, T: K \to K \) are two self-mappings on \( K \) satisfying the contractive condition (3.1). Suppose also that for arbitrary \( x_0 \in K \), \( \{x_n\}_{n=0}^{\infty} \) is a sequence defined by

\[
x_{n+1} = (1 - b_n)x_n + b_n Sy_n + u_n,
\]

\[
y_n = (1 - b_n')x_n + b_n'Tx_n + v_n, \quad n = 0, 1, 2, \ldots \]

where \( \{b_n\} \) and \( \{b_n'\} \) are two sequence of real number in \([0, 1]\) with \( \sum_{n=0}^{\infty} b_n = \infty \) and

\[
\sum_{n=0}^{\infty} |u_n|_\alpha < \infty, \quad \sum_{n=0}^{\infty} |v_n|_\alpha = 0 \quad \forall \alpha \in (0, 1).
\]

Let \( F_S \cap F_T \neq \emptyset \), where \( F_S \) and \( F_T \) are the sets of fixed points of \( S \) and \( T \) respectively. Then sequence \( \{x_n\} \) converges to a common fixed point of \( S \) and \( T \).

**Proof.** Since \( S \) and \( T \) satisfy the contractive condition, so

\[
|Sx - Sy|_\alpha \leq (2\delta_1 |x - y|_\alpha + \delta_1 |x - y|_\alpha) \quad \forall \alpha \in (0, 1), \quad \forall x, \ y \in X \quad \text{for some} \quad \delta_1 \in (0, 1).
\]

\[
|Tx - Ty|_\alpha \leq (2\delta_2 |x - y|_\alpha + \delta_2 |x - y|_\alpha) \quad \forall \alpha \in (0, 1), \quad \forall x, \ y \in X \quad \text{for some} \quad \delta_2 \in (0, 1).
\]

By assumption, \( F_S \cap F_T \neq \emptyset \). Let \( p \in F_S \cap F_T \).

Therefore, for arbitrary \( x_0 \in K \) and by using Iteration (2), we get

\[
x_{n+1} - p = (1 - b_n)x_n + b_n Sy_n + u_n - p = (1 - b_n)(x_n - p) + b_n(Sy_n - p) + u_n
\]

\[
\Rightarrow |x_{n+1} - p|_\alpha = |(1 - b_n)(x_n - p) + b_n(Sy_n - Sp) + u_n|_\alpha \leq (1 - b_n)|(x_n - p)|_\alpha + b_n|Sy_n - Sp|_\alpha + |u_n|_\alpha.
\]

By using contractive property of \( S \) we get

\[
|x_{n+1} - p|_\alpha \leq (1 - b_n)|(x_n - p)|_\alpha + b_n \delta_1 |y_n - p|_\alpha + |u_n|_\alpha.
\]

By observing \( Sp = p \), we get

\[
|x_{n+1} - p|_\alpha \leq (1 - b_n)|(x_n - p)|_\alpha + b_n \delta_1 |y_n - p|_\alpha + |v_n|_\alpha.
\]

Similarly we get

\[
|y_n - p|_\alpha \leq (1 - b_n')|(x_n - p)|_\alpha + b_n' \delta_2 |x_n - p|_\alpha + |u_n|_\alpha \leq (1 - b_n' + \delta_2 b_n')|(x_n - p)|_\alpha + |v_n|_\alpha.
\]

By observing that \( (1 - b_n' + \delta_2 b_n') \leq 1 \), we obtain

\[
|y_n - p|_\alpha \leq |(x_n - p)|_\alpha + |v_n|_\alpha.
\]

From (3.6) and (3.7) we get

\[
|x_{n+1} - p|_\alpha \leq (1 - b_n + b_n \delta_1)|(x_n - p)|_\alpha + b_n \delta_1 |v_n|_\alpha + |u_n|_\alpha.
\]

By applying Lemma 1.1 and using the fact that

\[
0 \leq b_n \leq 1, \quad 0 < \delta_1 < 1, \quad 0 < (1 - b_n + b_n \delta_1) < 1,
\]

\[
\sum_{n=0}^{\infty} b_n = \infty, \quad \sum_{n=0}^{\infty} |u_n|_\alpha < \infty, \quad \lim_{n \to \infty} |v_n|_\alpha = 0 \quad \forall \alpha \in (0, 1),
\]

we obtain

\[
\lim_{n \to \infty} |x_{n+1} - p|_\alpha = 0 \quad \text{for each} \quad \alpha \in (0, 1).
\]

Hence the sequence \( \{x_n\} \) converges to a common fixed point of \( S \) and \( T \).