Uni-Soft Ideals of Ternary Semigroups

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***Abstract*—In this paper we introduce the notions of uni-soft ternary semigroups, uni-soft left(right, lateral) ideals and uni-soft quasi ideals of ternary semigroups and also investigate some related properties of these terms. We also discuss some characterizations of uni-soft ternary semigroups and uni-soft left(right, lateral) ideals of ternary semigroups.**

***Index Terms*—Uni-soft ternary semigroup, uni-soft left(right, lateral) ideal, uni-soft product, uni-soft quasi ideal.**

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I. Introduction

NOWADAYS we are facing so many problems in practical life which involves uncertainties. Specially, these kinds of problems arise in the field of medical sciences, physics, engineering and computer science etc. The classical set theory has very limited resources regarding these kinds of problems and therefore not fully suitable for conducting such types of problems. In the last fifty years, many mathematical theories have been developed for dealing such kinds of problems. But due to the inadequacy of the parametrization tool, these theories were not fully suitable to conduct such problems of uncertainties.

To solve this problem, in 1999 Molodtsoy [1] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. He successfully applied his theory into several directions such as smoothness of functions, integration, theory of probability etc. After Molodtsov, Maji et al. have defined some binary operations on soft sets in [2]. These binary operations were corrected by Ali et al. in [3]. In [4], Feng et al. established a connection between soft sets and rough sets, they initiated the concept of soft rough approximations, soft rough sets and some related notions. Cagman and Enginolu [5] redefined some operations of soft set theory and also constructed uni-int soft decision making method and applied it to decision making problems.

Algebraic structures play an important role in mathematics with wide applications in the field of computer sciences, information sciences, engineering, social sciences etc. For this purpose the idea of soft set theory is injected in many algebraic structure. For example, Akas and Cagman [6] introduce the concept of soft groups. Feng et al. [7] introduced the notion of soft semirings. In 2009, Shabir at al. [8] studied soft semigroups and soft ideals over semigroup and discussed their related properties. In 2013, Kim et al. [9] initiated the notions of uni-soft semigroups, uni-soft left(right) ideals and uni-soft quasi ideals of semigroups. Tariq Shah [10] initiated the concept of soft ordered AG-groupoid and injected the idea of soft set theory in AG-groupoids. Recently in 2014, the concept of uni-soft ideals is also extended to the field of AG-groupoids [11].

The concept of ternary algebraic system was introduced by Lehmer [12] in 1932. But before Lehmer, such structures was studied by Kasner [13] who give the idea of n-ary algebras. Lehmar investigated some algebraic systems called triplexes which turned to be commutative ternary groups. The notion of ternary semigroup was introduced by Banach. Any semigroup can be reduced to a ternary semigroups. However, Banach shows by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. Sioson [14] worked on the ideal theory in ternary semigroups. Dixit and Dewan [15] worked on quasi-ideals and bi-ideals in ternary semigroups. In [16], Shabir and Bano initiated the concept of prime, semi prime and strongly prime bi-ideals in ternary semigroups. In 2012, Shabir and Ali [17] initiated the concept of soft ternary semigroups. In this paper we aim to extend the concept of soft set theory in ternary semigroups by applying the notions initiated by Kim et al. [9], to ternary semigroups.

II. Preliminaries

From a ternary semigroup, we mean an algebraic structure (S, F) such that S is a non-empty set and $F : S^3 \to S$ is a ternary operation satisfying the following associative law: $F(F(a, b, c), d, e) = F(a, F(b, c, d), e) = F(a, b, F(c, d, e))$, $\forall\, a, b, c, d, e \in S$. For simplicity we write $F(a, b, c)$ as abc and take the ternary operation F as ".". If A, B, C are non-empty subsets of a ternary semigroup S then their product ABC is defined as:

$$ABC = \{abc \in S \mid a \in A, b \in B \text{ and } c \in C\}.$$

A non-empty subset A of a ternary semigroup S is called a ternary subsemigroup of S if, $AAA = A^3 \subseteq A$. A non-empty subset A of a ternary semigroup S is called a left(right, lateral) ideal of S, if $SSA \subseteq A(ASS \subseteq A, SAS \subseteq A)$. By a two sided ideal, we mean a subset A of S which is both left and right ideal of S. A non-empty subset A of S is called an ideal of S, if it is left, right and a lateral ideal of S. Also a left (right, lateral, two sided) ideal I of a ternary semigroup S is idempotent if $I^3 = I$. A non-empty subset Q of a ternary semigroup S is called a quasi-ideal of S if $(QSS) \cap (SQS) \cap (SSQ) \subseteq Q$, $(QSS) \cap (SSQSS) \cap (SSQ) \subseteq Q$. It should be noted that every left, right and lateral ideal in a ternary semigroup is a quasi-ideal but the converse is not true in general. Similarly, a non-empty subset A of a ternary semigroup S is called a bi-ideal of S if $AAA \subseteq A$ and $ASASA \subseteq A$. Every quasi-ideal of a ternary semigroup is a bi-ideal. An element $a \in S$

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is called regular if there exists some $x \in S$ such that $a = axa$. A ternary semigroup S is called regular if all the elements of S are regular.

Definition 2.1 [1] Let U be an initial universe and E be a set of parameters. Let A be a non-empty subset of E and $P(U)$ denotes the power set of U. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$. We can say that a soft set is a parameterized family of subset of the universe set U. The function F is called approximate function of the soft set (F, A).

Definition 2.2 [9] For any two soft sets (F, S) and (G, S) over a common universe U, we say that (F, S) is the soft subset of (G, S) denoted by $(F, S) \subseteq (G, S)$ if $F(x) \subseteq G(x), \forall x \in S$.

Definition 2.3 [9] The soft union of (F, S) and (G, S) over a common universe U is the soft set $(F \cup G, S)$, where $F \cup G$ is define by:

$$(F \cup G)(x) = F(x) \cup G(x), \forall x \in S.$$

Definition 2.4 [9] The soft intersection of (F, S) and (G, S) over U is the soft set $(F \cap G, S)$ over U in which $F \cap G$ is defined by:

$$(F \cap G)(x) = F(x) \cap G(x), \forall x \in S.$$

Definition 2.5 [9] Let (F, A) be a soft set over a universe U and η be any subset of U, then the η-exclusive set of (F, A), denoted by $e_A(F; \eta)$ is defined to be the set:

$$e_A(F; \eta) = \{x \in A \mid F(x) \subseteq \eta\}.$$

III. Uni-soft ideals of ternary semigroups

In what follows, we take $E = S$, where E is the set of parameters and S is a ternary semigroup, unless otherwise specified. We need some definitions which will be useful to derive the main results of the paper.

Definition 3.1 Let (F, S) be a soft set over U then (F, S) is called a uni-soft ternary semigroup over U if $F(xyz) \subseteq F(x) \cup F(y) \cup F(z)\ \forall\ x, y, z \in S$.

Definition 3.2 Let (F, S) be a soft set over U, then (F, S) is called a uni-soft left(right, lateral) ideal over U if $F(xyz) \subseteq F(z)((F(xyz) \subseteq F(x), (F(xyz) \subseteq F(y))\ \forall\ x, y, z \in S$.

Definition 3.3 A soft set (F, S) over U is called a uni-soft two-sided ideal over U if it is both a uni-soft left ideal and a uni-soft right ideal over U. Similarly, (F, S) is called a uni-soft ideal over U if it is uni-soft left ideal, uni-soft right ideal and uni-soft lateral ideal over U.

Example 3.1. Let $S = \{a, b, c, d\}$ be a semigroup under the operation ( ), given below:

( )	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define ternary operator [ ], as $[xyz] = (xy)z = x(yz)$. Then (S,[ ]) is a ternary semigroup. Now let (F, S) be a soft set over U defined by:

$$F : S \to P(U),\ x \mapsto \begin{cases} \eta_1 & \text{if } x = a, \\ \eta_2 & \text{if } x = b, \\ \eta_3 & \text{if } x = d, \\ \eta_4 & \text{if } x = c, \end{cases}$$

where $\eta_1, \eta_2, \eta_3, \eta_4 \in P(U)$ with $\eta_1 \subset \eta_2 \subset \eta_3 \subset \eta_4$. Then (F, S) is a uni-soft left(right, lateral) ideal over U.

Clearly every uni-soft left(right, lateral) ideal over U is a uni-soft ternary semigroup, but the converse does not hold in general as we see in the following example.

Example 3.2. Consider $S = \{-i, 0, i\}$, where S is a ternary semigroup under the usual multiplication of complex number. Let (F, S) be a soft set over U defined by:

$$F : S \to P(U),\ x \mapsto \begin{cases} \eta_1 & \text{if } x = 0, \\ \eta_2 & \text{if } x = i, \\ \eta_3 & \text{if } x = -i, \end{cases}$$

where $\eta_1, \eta_2, \eta_3 \in P(U)$ with $\eta_1 \subset \eta_2 \subset \eta_3$. Then (F, S) is a uni-soft ternary semigroup over U but it is not a uni-soft left(right, lateral) ideal over U. Since $\eta_3 = F(-i) = F(-i. - i.i) \nsubseteq F(i) = \eta_2$, shows that (F, S) is not a uni-soft left ideal over U. Also $\eta_3 = F(-i) = F(i. -i. -i) \nsubseteq F(i) = \eta_2$, implies that (F, S) is not a uni-soft right ideal over U. Similarly $\eta_3 = F(-i) = F(-i.i. - i) \nsubseteq F(i) = \eta_2$, shows that (F, S) is not a uni-soft lateral ideal over U.

Theorem 3.1. If (F, S) and (G, S) are any two uni-soft ternary semigroups over U, then their soft union $(F \cup G, S)$ is also a uni-soft ternary semigroup over U.

Proof: Since (F, S) and (G, S) are uni-soft ternary semigroups over U, then $F(xyz) \subseteq F(x) \cup F(y) \cup F(z)$ and $G(xyz) \subseteq G(x) \cup G(y) \cup G(z); \forall x, y, z \in S$. Now,

$$(F \cup G)(xyz) = F(xyz) \cup G(xyz) \subseteq (F(x) \cup F(y) \cup F(z)) \cup$$

$$(G(x) \cup G(y) \cup G(z)) = (F(x) \cup G(x)) \cup (F(y) \cup G(y)) \cup$$

$$(F(z) \cup G(z)) = (F \cup G)(x) \cup (F \cup G)(y) \cup (F \cup G)(z)$$

Hence $(F \cup G, S)$ is a uni-soft ternary semigroup over U.

Theorem 3.2. If (F, S) and (G, S) are two uni-soft left(right, lateral) ideals over U, then their soft union $(F \cup G, S)$ is also a uni-soft left(right, lateral) ideal over U.

Proof: Suppose (F, S) and (G, S) are two uni-soft left ideals over U, then by Definition (3.2) we have $F(xyz) \subseteq F(z)$ and $G(xyz) \subseteq G(z); \forall x, y, z \in S$. Now,

$$(F \cup G)(xyz) = F(xyz) \cup G(xyz)$$

$$\subseteq (F(z) \cup G(z)) = (F \cup G)(z)$$

$$\Rightarrow (F \cup G)(xyz) \subseteq (F \cup G)(z).$$

Hence $(F \cup G, S)$ is a uni-soft left ideal over U.

Similarly we can prove that if (F, S) and (G, S) are any two uni-soft right(lateral) ideals over U, then their soft union $(F \cup G, S)$ is also a uni-soft right(lateral) ideal over U.

Corollary 3.1. If (F, S) and (G, S) are any two uni-soft ideals over U, then their soft union $(F \cup G, S)$ is also a uni-soft ideal over U.

Theorem 3.3. For a soft set (F, S) over U, the following are equivalent:

(1) (F, S) is a uni-soft ternary semigroup over U.

(2) For all $\eta \in P(U)$ with $e_S(F;\eta) \neq \phi$, $e_S(F;\eta)$ is a ternary subsemigroup of S.

Proof: Let (F,S) is a uni-soft ternary semigroup over U and $\eta \subseteq U$ such that $e_S(F;\eta) \neq \phi$. Let $x,y,z \in e_S(F;\eta)$, then $F(x) \subseteq \eta$, $F(y) \subseteq \eta$ and $F(z) \subseteq \eta$. As (F,S) is a uni-soft ternary semigroup over U, so we can write

$$F(xyz) \subseteq F(x) \cup F(y) \cup F(z) \subseteq \eta \cup \eta \cup \eta = \eta$$
$$\Rightarrow F(xyz) \subseteq \eta$$

which shows that $xyz \in e_S(F;\eta)$. Hence $e_S(F;\eta)$ is a ternary subsemigroup of S.

Conversely, assume that $\forall\ \eta \in P(U)$ with $e_S(F;\eta) \neq \phi$, $e_S(F;\eta)$ is a ternary subsemigroup of S. Let $x,y,z \in S$ such that $F(x) \subseteq \eta_1$, $F(y) \subseteq \eta_2$ and $F(z) \subseteq \eta_3$. Taking $\eta = \eta_1 \cup \eta_2 \cup \eta_3$, then $F(x) \subseteq \eta$, $F(y) \subseteq \eta$ and $F(z) \subseteq \eta$, implies that $x,y,z \in e_S(F;\eta)$. Since $e_S(F;\eta)$ is a ternary subsemigroup of S, therefore $xyz \in e_S(F;\eta)$. Further

$$F(xyz) \subseteq \eta = \eta_1 \cup \eta_2 \cup \eta_3 = F(x) \cup F(y) \cup F(z).$$

Hence (F,S) is a uni-soft ternary semigroup over U.

In similar manner, we can prove the following theorem.

Theorem 3.4. For a soft set (F,S) over U, the following are equivalent:

(1) (F,S) is a uni-soft left(right, lateral) ideal over U.

(2) For all $\eta \in P(U)$ with $e_S(F;\eta) \neq \phi$, $e_S(F;\eta)$ is a left(right, lateral) ideal of S.

The above theorem yield the following corollary.

Corollary 3.2. A soft set (F,S) over U is a uni-soft ideal over U if and only if for all $\eta \in P(U)$ with $e_S(F;\eta) \neq \phi$, $e_S(F;\eta)$ is an ideal of S.

Definition 3.4 [9] Let (F,S) be a soft set over U and η be any subset of U with $e_S(F;\eta) \neq \phi$. The soft set (F^*,S) over U is defined by:

$$F^*: S \to P(U),\ x \mapsto \begin{cases} F(x) & \text{if } x \in e_S(F;\eta), \\ \rho & \text{otherwise.} \end{cases}$$

where ρ is a subset of U with $F(x) \subset \rho$.

Theorem 3.5. If (F,S) is a uni-soft ternary semigroup over U, then (F^*,S) is also a uni-soft ternary semigroup over U.

Proof: Let (F,S) is a uni-soft ternary semigroup over U and $x,y,z \in S$. If $x,y,z \in e_S(F;\eta)$, then $xyz \in e_S(F;\eta)$ as $e_S(F;\eta)$ is a ternary subsemigroup of S (by Theorem (3.3)). Therefore

$$F^*(xyz) = F(xyz) \subseteq F(x) \cup F(y) \cup F(z)$$
$$= F^*(x) \cup F^*(y) \cup F^*(z).$$

Which implies that $F^*(xyz) \subseteq F^*(x) \cup F^*(y) \cup F^*(z)$. Now, if x,y or $z \notin e_S(F;\eta)$, then $F^*(x) = \rho$, $F^*(y) = \rho$ or $F^*(z) = \rho$. Thus

$$F^*(xyz) \subseteq \rho = \rho \cup \rho \cup \rho = F^*(x) \cup F^*(y) \cup F^*(z).$$

Hence (F^*,S) is a uni-soft ternary semigroup over U.

Theorem 3.6. If (F,S) is a uni-soft left(right, lateral) ideal over U, then (F^*,S) is also a uni-soft left(right, lateral) ideal over U.

Proof: Assume that (F,S) is a uni-soft left ideal over U and $x,y,z \in S$ with $z \in e_S(F;\eta)$. Then by Theorem (3.4), $e_S(F;\eta)$ is a left ideal of S and therefore $xyz \in e_S(F;\eta)$. Also

$$F^*(xyz) = F(xyz) \subseteq F(z) = F^*(z),$$

implies that $F^*(xyz) \subseteq F^*(z)$. Now, if $z \notin e_S(F;\eta)$, then $F^*(z) = \rho$. Thus $F^*(xyz) \subseteq \rho = F^*(z)$. Therefore (F^*,S) is a uni-soft left ideal over U. In similar way we can show that if (F,S) is a uni-soft right(lateral) ideal over U, then (F^*,S) is also a uni-soft right(lateral) ideal over U.

Corollary 3.3. If (F,S) is a uni-soft ideal over U, then (F^*,S) is also a uni-soft ideal over U.

Definition 3.5 [9] For a nonempty subset K of S, the soft set (χ_K,S) is called the uni-characteristic soft set where χ_K is defined by:

$$\chi_K: S \longrightarrow P(U),\ x \longmapsto \begin{cases} U & \text{if } x \notin K, \\ \phi & \text{if } x \in K. \end{cases}$$

The soft set (χ_S,S) is called the uni-empty soft set over U.

Theorem 3.7. For a nonempty subset K of S, the following are equivalent:

(1) K is a left(right, lateral) ideal of S.

(2) The uni-characteristic soft set (χ_K,S) over U is the uni-soft left(right, lateral) ideal over U.

Proof: Let K be a left ideal of S and $x,y,z \in S$ such that $z \in K$, then $xyz \in K$ as K is a left ideal of S. Therefore $\chi_K(xyz) = \phi = \chi_K(z)$. Now, if $z \notin K$, then $\chi_K(xyz) \subseteq U = \chi_K(z)$. Hence (χ_K,S) is a uni-soft left ideal over U.

Similarly we can prove that if K is a right(lateral) ideal of S, then (χ_K,S) is also a uni-soft right(lateral) ideal over U.

Conversely, assume that (χ_K,S) is uni-soft left ideal over U and $x,y,z \in S$ such that $z \in K$. Since (χ_K,S) is uni-soft left ideal over U so we have $\chi_K(xyz) \subseteq \chi_K(z)$. Also $\chi_K(z) = \phi$ as $z \in K$. Thus $\chi_K(xyz) \subseteq \chi_K(z) = \phi$, implies that $\chi_K(xyz) = \phi \Rightarrow xyz \in K$. Hence K is a left ideal of S.

Similarly we can prove that if (χ_K,S) is a uni-soft right(lateral) ideal over U, then K is a right(lateral) ideal of S.

Corollary 3.4. For any nonempty subset K of S, the following are equivalent:

(1) K is an ideal of S.

(2) The uni-characteristic soft set (χ_K,S) over U is the uni-soft ideal over U.

Definition 3.6 For the soft sets (F,S), (G,S) and (H,S) over U, the uni-soft product is defined by $(F \circ G \circ H, S)$ over U where $F \circ G \circ H$ is given by:

$$F \circ G \circ H : S \longrightarrow P(U),$$

$$x \longmapsto \begin{cases} \bigcap\limits_{w=xyz} \{F(x) \cup G(y) \cup H(z)\} & \text{if } \exists\ x,y,z \in S \\ & \text{such that } w = xyz, \\ U & \text{otherwise.} \end{cases}$$

Theorem 3.8. A soft set (F,S) over U is a uni-soft ternary semigroup over U if and only if $(F,S) \subseteq (F \circ F \circ F, S)$.

Proof: Suppose that (F,S) is a uni-soft ternary semigroup over U, then for all $w \in S$ with $w = xyz$ we have

$F(w) \subseteq F(x) \cup F(y) \cup F(z)$. This implies that

$$F(w) \subseteq \bigcap_{w=xyz} \{F(x) \cup F(y) \cup F(z)\}$$
$$= (F \circ F \circ F)(w), \forall w \in S,$$

Otherwise, $F(w) \subseteq U = (F \circ F \circ F)(w)$.
Hence $(F,S) \subseteq (F \circ F \circ F, S)$.

Conversely, let $(F,S) \subseteq (F \circ F \circ F, S)$ and $x,y,z \in S$. Then $F(xyz) \subseteq (F \circ F \circ F)(xyz) \subseteq F(x) \cup F(y) \cup F(z)$, shows that (F,S) is a uni-soft ternary semigroup over U.

Proposition 3.1. For the soft sets (F_1,S), (F_2,S), (G_1,S), (G_2,S), (H_1,S) and (H_2,S) over U, if $(F_1,S) \subseteq (F_2,S), (G_1,S) \subseteq (G_2,S)$ and $(H_1,S) \subseteq (H_2,S)$, then $(F_1 \circ G_1 \circ H_1, S) \subseteq (F_2 \circ G_2 \circ H_2, S)$.

Proof: Assume that $w \in S$ such that w cannot be expressed as $w = xyz$ for some $x,y,z \in S$. Then

$$(F_1 \circ G_1 \circ H_1)(w) = U = (F_2 \circ G_2 \circ H_2)(w)$$
$$\Rightarrow (F_1 \circ G_1 \circ H_1, S) \subseteq (F_2 \circ G_2 \circ H_2, S)$$

Now, if there exist some $x,y,z \in S$ such that $w = xyz$, then

$$(F_1 \circ G_1 \circ H_1)(w) = \bigcap_{w=xyz} \{F_1(x) \circ G_1(y) \circ H_1(z)\}$$
$$\subseteq \bigcap_{w=xyz} \{F_2(x) \circ G_2(y) \circ H_2(z)\} = (F_2 \circ G_2 \circ H_2)(w).$$

Hence $(F_1 \circ G_1 \circ H_1, S) \subseteq (F_2 \circ G_2 \circ H_2, S)$.

Theorem 3.9. For a soft set (K,S) and uni-empty soft set (χ_S, S) over U, the following are equivalent:

(1) $(K,S) \subseteq (\chi_S \circ \chi_S \circ K, S)$.

(2) (K,S) is a uni-soft left ideal over U.

Proof: Let $(K,S) \subseteq (\chi_S \circ \chi_S \circ K, S)$ and $x,y,z \in S$. Then

$$K(xyz) \subseteq (\chi_S \circ \chi_S \circ K)(xyz)$$
$$\subseteq \chi_S(x) \cup \chi_S(y) \cup K(z) = \phi \cup \phi \cup K(z) = K(z)$$
$$\Rightarrow K(xyz) \subseteq K(z).$$

Hence (K,S) is a uni-soft left ideal over U.

Conversely, let (K,S) is a uni-soft left ideal over U and $x,y,z \in S$. Now, If $w = xyz$ for some $x,y,z \in S$. Then

$$(\chi_S \circ \chi_S \circ K)(w) = \bigcap_{w=xyz} \{\chi_S(x) \cup \chi_S(y) \cup K(z)\}$$
$$\supseteq \bigcap_{w=xyz} \{\phi \cup \phi \cup K(xyz)\} = \bigcap_{w=xyz} \{K(xyz)\} = K(w)$$

Otherwise, $K(w) \subseteq U = (\chi_S \circ \chi_S \circ K)(w)$, for all $w \in S$. Hence $(K,S) \subseteq (\chi_S \circ \chi_S \circ K, S)$.

Theorem 3.10. For a soft set (K,S) and uni-empty soft set (χ_S, S) over U, the following are equivalent:

(1) $(K,S) \subseteq (\chi_S \circ K \circ \chi_S, S)$.

(2) (K,S) is a uni-soft lateral ideal over U.

Proof: Suppose $(K,S) \subseteq (\chi_S \circ K \circ \chi_S, S)$ and $x,y,z \in S$. Then

$$K(xyz) \subseteq (\chi_S \circ K \circ \chi_S)(xyz)$$
$$\subseteq \chi_S(x) \cup K(y) \cup \chi_S(z) = \phi \cup K(y) \cup \phi = K(y)$$
$$\Rightarrow K(xyz) \subseteq K(y).$$

Therefore (K,S) is a uni-soft lateral ideal over U.

Conversely, let (K,S) is a uni-soft lateral ideal over U and $x,y,z \in S$. Also let $w = xyz$ for some $x,y,z \in S$. Then

$$(\chi_S \circ K \circ \chi_S)(w) = \bigcap_{w=xyz} \{\chi_S(x) \cup K(y) \cup \chi_S(z)\}$$
$$\supseteq \bigcap_{w=xyz} \{\phi \cup K(xyz) \cup \phi\} = \bigcap_{w=xyz} \{K(xyz)\} = K(w)$$

Otherwise, $K(w) \subseteq U = (\chi_S \circ K \circ \chi_S)(w)$, for all $w \in S$. Hence $(K,S) \subseteq (\chi_S \circ K \circ \chi_S, S)$.

Theorem 3.11. For a soft set (K,S) and uni-empty soft set $(\chi_S\, S)$ over U, the following are equivalent:

(1) $(K,S) \subseteq (K \circ \chi_S \circ \chi_S, S)$.

(2) (K,S) is a uni-soft right ideal over U.

Proof: The proof is same as the proof of Theorem (3.9) and Theorem (3.10).

Corollary 3.5. For a soft set (K,S) and uni-empty soft set (χ_S, S) over U, the following are equivalent:

(1) $(K,S) \subseteq (\chi_S \circ \chi_S \circ K, S)$, $(K,S) \subseteq (\chi_S \circ K \circ \chi_S, S)$ and $(K,S) \subseteq (K \circ \chi_S \circ \chi_S, S)$.

(2) (K,S) is a uni-soft ideal over U.

Theorem 3.12. For the soft sets (F,S), (G,S) and (H,S) over U, if (F,S) is a uni-soft left ideal over U, then $(F \circ G \circ H, S)$ is a uni-soft left ideal over U.

Proof: Suppose that $x,y,z \in S$ and z can be expressed as $z = abc$ for some $a,b,c \in S$. Then

$$(F \circ G \circ H)(z) = \bigcap_{z=abc} \{F(a) \cup G(b) \cup H(c)\}$$
$$\supseteq \bigcap_{xyz=(xya)bc} \{F(xya) \cup G(b) \cup H(c)\}$$
$$= \bigcap_{xyz=kbc} \{F(k) \cup G(b) \cup H(c)\} = (F \circ G \circ H)(xyz)$$
$$\Rightarrow (F \circ G \circ H)(xyz) \subseteq (F \circ G \circ H)(z).$$

Hence $(F \circ G \circ H, S)$ is a uni-soft left ideal over U.

Theorem 3.13. For the soft sets (F,S), (G,S) and (H,S) over U, if (H,S) is a uni-soft right ideal over U, then $(F \circ G \circ H, S)$ is also a uni-soft right ideal over U.

Proof: Proof is same as Theorem (3.12).

Theorem 3.14. Let (F,S), (G,S) and (H,S) are uni-soft left, uni-soft lateral and uni-soft right ideals over U, respectively. Then $(F \circ G \circ H, S) \supseteq (F \cup G \cup H, S)$.

Proof: Assume that (F,S), (G,S) and (H,S) are soft sets over U such that (F,S) is a uni-soft left ideal, (G,S) is a uni-soft lateral ideal and (H,S) is a uni-soft right ideal over U, respectively. If $x \in S$ such that $x = abc$ for some $a,b,c \in S$. Then

$$(F \circ G \circ H)(x) = \bigcap_{x=abc} \{F(a) \cup G(b) \cup H(c)\}$$
$$\supseteq \bigcap_{x=abc} \{F(abc) \cup G(abc) \cup H(abc)\} = F(x) \cup G(x) \cup H(x)$$
$$= (F \cup G \cup H)(x).$$

If we cannot write x as $x = abc$ for some $a, b, c \in S$. Then

$$(F \circ G \circ H)(x) = U \supseteq (F \cup G \cup H)(x).$$

Hence $(F \circ G \circ H, S) \supseteq (F \cup G \cup H, S)$.

Theorem 3.15. For a regular ternary semigroup S, if (G, S) is a uni-soft lateral ideal over U then $(F \cup G \cup H, S) \supseteq (F \circ G \circ H, S)$ for every soft set (F, S) and (H, S) over U.

Proof: Suppose S a regular ternary semigroup and $w \in S$. Also let (F, S), (G, S) and (H, S) are soft sets over U such that (G, S) is a uni-soft lateral ideal over U. Now if $w = xyz$, for some $x, y, z \in S$, then

$$(F \circ G \circ H)(w) = \bigcap_{w=xyz} \{F(x) \cup G(y) \cup H(z)\}.$$

Since S is a regular ternary semigroup, therefore $w = waw$ for some $a \in S$. Thus

$$\begin{aligned}(F \cup G \cup H)(w) &= F(w) \cup G(w) \cup H(w) \\ &= F(w) \cup G(waw) \cup H(w) \\ &\supseteq F(w) \cup G(a) \cup H(w).\end{aligned}$$

Further,

$$\begin{aligned}&(F \cup G \cup H)(w) \supseteq F(w) \cup G(a) \cup H(w) \\ &\supseteq \bigcap_{w=xyz} \{F(x) \cup G(y) \circ H(z)\} = (F \circ G \circ H)(w) \\ &\Rightarrow (F \cup G \cup H, S) \supseteq (F \circ G \circ H, S).\end{aligned}$$

Hence $(F \cup G \cup H, S) \supseteq (F \circ G \circ H, S)$.

Theorem 3.16. If S is a regular ternary semigroup and $(F, S), (G, S)$ and (H, S) are uni-soft left, uni-soft lateral and uni soft right ideals over U, respectively. Then

$$(F \circ G \circ H, S) = (F \cup G \cup H, S).$$

Proof: Suppose S is a regular ternary semigroup and $(F, S), (G, S)$ and (H, S) are uni-soft left, uni-soft lateral and uni-soft right ideals over U, respectively. Then by Theorem (3.14) we have:

$$(F \circ G \circ H, S) \supseteq (F \cup G \cup H, S).$$

Also by Theorem (3.15) we have:

$$(F \cup G \cup H, S) \supseteq (F \circ G \circ H, S).$$

Hence $(F \circ G \circ H, S) = (F \cup G \cup H, S)$.

Definition 3.7 A soft set (F, S) over U is called a uni-soft quasi ideal over U if

$$(F \circ \chi_S \circ \chi_S, S) \cup (\chi_S \circ F \circ \chi_S, S) \cup (\chi_S \circ \chi_S \circ F, S) \supseteq (F, S).$$

Example 3.3. Let $S = \{0, a, b, c\}$ be a semigroup under the operation ( ), given below:

( )	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Define ternary operator [ ], as $[xyz] = (xy)z = x(yz)$. Then clearly (S,[ ]) is a ternary semigroup. Now let (F, S) be a soft set over U defined by:

$$F : S \to P(U),\ x \mapsto \begin{cases} \eta & \text{if } x = \{b, c\}, \\ \phi & \text{if } x = \{0, a\}. \end{cases}$$

where η is any nonempty subset of U.
Then (F, S) is a uni-soft quasi ideal over U. Further we can also check that (F, S) is a uni-soft left, uni-soft right and uni-soft lateral ideal over U.

From Example (3.3) we see that every uni-soft left(right, lateral) ideal over U is a uni-soft quasi ideal over U but the converse does not holds in general as we see in the following example.

Example 3.4. Let $S = \{0, a, b, c\}$ be a semigroup under the operation ( ), given below:

( )	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	0	0
c	0	c	0	0

Define ternary operator [ ], as $[xyz] = (xy)z = x(yz)$. Then (S,[ ]) is a ternary semigroup. Now let (F, S) be a soft set over U defined by:

$$F : S \to P(U),\ x \mapsto \begin{cases} \phi & \text{if } x = 0, \\ \eta_1 & \text{if } x = a, \\ \eta_2 & \text{if } x = b, \\ \eta_3 & \text{if } x = c. \end{cases}$$

where $\eta_1, \eta_2, \eta_3 \in P(U)$ with $\eta_1 \subset \eta_2 \subset \eta_3$.
Then (F, S) is a uni-soft quasi ideal over U. But (F, S) is not a uni-soft left ideal over U as:

$$\eta_3 = F(c) = F(caa) \nsubseteq F(a) = \eta_1.$$

Also, (F, S) is not a uni-soft right(lateral) ideal over U as:

$$\eta_2 = F(b) = F(aab) \nsubseteq F(a) = \eta_1.$$

Theorem 3.17. For a nonempty subset K of S the following are equivalent:

(1) K is a quasi ideal of S.

(2) The uni-characteristic soft set (χ_K, S) is a uni-soft quasi ideal over U.

Proof: Let K be a quasi ideal of S and let a be any element of S. Now if $a \in K$, then

$$((\chi_K \circ \chi_S \circ \chi_S) \cup (\chi_S \circ \chi_K \circ \chi_S) \cup (\chi_S \circ \chi_S \circ \chi_K))(a) \supseteq \phi = \chi_K(a).$$

Also if $a \notin K$ then $\chi_K(a) = U$. Take

$$((\chi_K \circ \chi_S \circ \chi_S) \cup (\chi_S \circ \chi_K \circ \chi_S) \cup (\chi_S \circ \chi_S \circ \chi_K))(a) = \phi.$$

Then,

$$\phi = ((\chi_K \circ \chi_S \circ \chi_S))(a) = \bigcap_{a=xyz} \{\chi_K(x) \cup \chi_S(y) \cup \chi_S(z)\},$$

$$\phi = ((\chi_S \circ \chi_K \circ \chi_S))(a) = \bigcap_{a=xyz} \{\chi_S(x) \cup \chi_K(y) \cup \chi_S(z)\}$$

and

$$\phi = ((\chi_S \circ \chi_S \circ \chi_K))(a) = \bigcap_{a=xyz} \{\chi_S(x) \cup \chi_S(y) \cup \chi_K(z)\}.$$

Which shows that there exist elements b, c, d, e, f, g, h, i and $j \in S$ with $a = bcd = efg = hij$ such that $\chi_K(b) = \chi_K(f) = \chi_K(j) = \phi$. This shows that $b, f, j \in K$ and $a = bcd = efg = hij \in KSS \cap SKS \cap SSK \subseteq K$, which is a contradiction to the fact that $a \notin K$. Therefore we have $(\chi_K \circ \chi_S \circ \chi_S, S) \cup (\chi_S \circ \chi_K \circ \chi_S, S) \cup (\chi_S \circ \chi_S \circ \chi_K, S) \supseteq (\chi_K, S)$. Hence (χ_K, S) is a uni-soft quasi ideal over U.

Conversely, assume that (χ_K, S) is a uni-soft quasi ideal over U and let $a \in KSS \cap SKS \cap SSK$. Then we can write $a = buv = wcx = yzd$, for some $b, c, d \in K$ and $u, v, w, x, y, z \in S$. Since (χ_K, S) is a uni-soft quasi ideal over U, so by Definition (3.7) we can write

$$\chi_K(a) \subseteq ((\chi_K \circ \chi_S \circ \chi_S) \cup (\chi_S \circ \chi_K \circ \chi_S) \cup (\chi_S \circ \chi_S \circ \chi_K))(a)$$

$$= (\chi_K \circ \chi_S \circ \chi_S)(a) \cup (\chi_S \circ \chi_K \circ \chi_S)(a) \cup (\chi_S \circ \chi_S \circ \chi_K)(a)$$

$$= (\bigcap_{a=pqr} \{\chi_K(p) \cup \chi_S(q) \cup \chi_S(r)\}) \cup (\bigcap_{a=pqr} \{\chi_S(p) \cup \chi_K(q) \cup$$

$$\chi_S(r)\}) \cup (\bigcap_{a=pqr} \{\chi_S(p) \cup \chi_S(q) \cup \chi_K(r)\})$$

$$= (\bigcap_{a=pqr} \{\chi_K(p)\}) \cup (\bigcap_{a=pqr} \{\chi_K(q)\}) \cup (\bigcap_{a=pqr} \{\chi_K(r)\}) = \phi,$$

which shows that $a \in K$ and thus $KSS \cap SKS \cap SSK \subseteq K$.

Again, let $a \in KSS \cap SSKSS \cap SSK$. Since (χ_K, S) is a uni-soft quasi ideal over U, so we can write

$$\chi_K(a) \subseteq ((\chi_K \circ \chi_S \circ \chi_S) \cup (\chi_S \circ \chi_K \circ \chi_S) \cup (\chi_S \circ \chi_S \circ \chi_K))(a)$$

$$= (\chi_K \circ \chi_S \circ \chi_S)(a) \cup (\chi_S \circ \chi_K \circ \chi_S)(a) \cup (\chi_S \circ \chi_S \circ \chi_K)(a).$$

But $(\chi_S \circ \chi_K \circ \chi_S, S) \subseteq (\chi_S \circ \chi_S \circ \chi_K \circ \chi_S \circ \chi_S, S)$ implies that

$$\chi_K(a) \subseteq (\chi_K \circ \chi_S \circ \chi_S)(a) \cup (\chi_S \circ \chi_S \circ \chi_K \circ \chi_S \circ \chi_S)(a)$$

$$\cup (\chi_S \circ \chi_S \circ \chi_K)(a) = (\bigcap_{a=pqr} \{\chi_K(p) \cup \chi_S(q) \cup \chi_S(r)\})$$

$$\cup (\bigcap_{a=pqr} \{\chi_S(p) \cup (\chi_S \circ \chi_K \circ \chi_S)(q) \cup \chi_S(r)\})$$

$$\cup (\bigcap_{a=pqr} \{\chi_S(p) \cup \chi_S(q) \cup \chi_K(r)\})$$

$$= (\bigcap_{a=pqr} \{\chi_K(p)\}) \cup (\bigcap_{a=pqr} \{\bigcap_{q=mno} \{\chi_S(m) \cup \chi_K(n) \cup \chi_S(o)\}\})$$

$$\cup (\bigcap_{a=pqr} \{\chi_K(r)\}) = \phi \cup (\bigcap_{a=pqr} \{\bigcap_{q=mno} \{\chi_K(n)\}\}) \cup \phi = \phi$$

$$\Rightarrow \chi_K(a) = \phi.$$

Therefore $a \in K$ and thus $KSS \cap SSKSS \cap SSK \subseteq K$. Hence K is a quasi ideal of S.

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