Right Circulant Matrices with Complex Geometric Sequence

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***Abstract*—In this paper, we determine the determinant, eigenvalues, Euclidean norm, spectral norm and inverse of right circulant matrices with complex geometric sequence.**

***Index Terms*—complex geometric sequence, determinant, eigenvalues, Eucliden norm, matrix inverse, right circulant matrix, spectral norm**

***MSC 2010 Codes* – 15A15, 15A18, 15A60**

I. INTRODUCTION

IN [1], the determinant, eigenvalues, Euclidean norm, spectral norm and inverse of right circulant matrices with real geometric sequence $\left\{ar^k\right\}_{k=0}^{+\infty}$ were derived. The matrices take the form

$$\begin{aligned} R &= RCIRC_n(\vec{g}) \\ &= \begin{pmatrix} a & ar & ar^2 & \dots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & \dots & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & \dots & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & \dots & a & ar \\ ar & ar^2 & ar^3 & \dots & ar^{n-1} & a \end{pmatrix} \end{aligned}$$

where $\vec{g} = (a, ar, ..., ar^{n-1})$, $a \neq 0$ and $r \neq 0, 1$.

In this paper we will do the same but we will deal with the complex geometric sequence $\left\{sz^k\right\}_{k=0}^{+\infty}$ where $s = a + ib = |s|e^{i\alpha}$ and $z = c+id = |z|e^{i\beta}$. Hence we will be investigating the matrix

$$\begin{aligned} C &= RCIRC_n(\vec{v}) \\ &= \begin{pmatrix} s & sz & sz^2 & \dots & sz^{n-2} & sz^{n-1} \\ sz^{n-1} & s & sz & \dots & sz^{n-3} & sz^{n-2} \\ sz^{n-2} & sz^{n-1} & s & \dots & sz^{n-4} & sz^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ sz^2 & sz^3 & sz^4 & \dots & s & sz \\ sz & sz^2 & sz^3 & \dots & sz^{n-1} & s \end{pmatrix} \end{aligned}$$

where $\vec{v} = (s, sz, ..., sz^{n-1})$, $a \neq 0$ and $z \neq 0, 1$.

For the rest of the paper, for any matrix A we shall denote $|A|$, $||A||_E$ and $||A||_2$ for its determinant, Euclidean norm and spectral norm, respectively. We shall also use $e[i\theta] = e^{i\theta}$.

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II. PRELIMINARY RESULTS

We shall use the following lemmas to prove some our main results.

Lemma 2.1: Let

$$K = \begin{pmatrix} 1 & z & z^2 & \dots & z^{n-2} & z^{n-1} \\ z^{n-1} & 1 & z & \dots & z^{n-3} & z^{n-2} \\ z^{n-2} & z^{n-1} & 1 & \dots & z^{n-4} & z^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z^2 & z^3 & z^4 & \dots & 1 & z \\ z & z^2 & z^3 & \dots & z^{n-1} & 1 \end{pmatrix}$$

then

$$|K| = (1 - |z|^n e[in\beta])$$

Proof:
By applying the row operation $R_{k+1} - z^{n-k}R_1 \rightarrow R_{k+1}$ where k=1,2,...,n-1, K will be row equivalen to

$$\begin{pmatrix} 1 & z & z^2 & \dots & z^{n-2} & z^{n-1} \\ 0 & 1-z^n & r(1-z^n) & \dots & z^{n-3}(1-z^n) & z^{n-2}(1-r^n) \\ 0 & 0 & 1-z^n & \dots & z^{n-4}(1-z^n) & z^{n-3}(1-z^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1-z^n & z(1-z^n) \\ 0 & 0 & 0 & \dots & 0 & 1-z^n \end{pmatrix}$$

Since $z = |z|e[i\beta]$, the lemma immediately follows.

Lemma 2.2: In $j_k = \frac{1}{n}\sum_{m=0}^{n-1} \frac{1-t\omega^{-m}}{a(1-t^n)}\omega^{mk}$ where $\omega = e[2\pi i/n]$,

- $j_0 = \frac{1}{a(1-t^n)}$;
- $j_1 = \frac{-t}{a(1-t^n)}$; and
- $j_k = 0$ for $k = 2,\ 3,\ ...,\ n-1$

Proof:
For $k = 0$,

$$\begin{aligned} j_0 &= \frac{1}{n}\sum_{m=0}^{n-1} \frac{1-t\omega^{-m}}{a(1-t^n)} \\ &= \frac{1}{an(1-t^n)}\sum_{m=0}^{n-1}(1-t\omega^{-n}) \\ &= \frac{1}{an(1-t^n)}\left[\frac{t(1-\omega^{-n})}{1-\omega} - n\right] \\ &= \frac{1}{a(1-t^n)} \end{aligned}$$

For $k = 1$,

$$\begin{aligned} j_1 &= \frac{1}{n}\sum_{m=0}^{n-1}\frac{1-t\omega^{-m}}{a(1-t^n)}\omega^{-m} \\ &= \frac{1}{an(1-t^n)}\sum_{m=0}^{n-1}(\omega^{-m}-t) \\ &= \frac{1}{an(1-t^n)}\left[\frac{1-\omega^{-n}}{1-\omega}-tn\right] \\ &= \frac{-t}{a(1-t^n)} \end{aligned}$$

For $k = 2,\ 3,\ ...,\ n-1$,

$$\begin{aligned} j_k &= \frac{1}{n}\sum_{m=0}^{n-1}\frac{1-t\omega^{-m}}{1-a(t^n)}\omega^{-mk} \\ &= \frac{1}{an(1-t^n)}\sum_{m=0}^{n-1}\left(t\omega^{-m(k-1)}-\omega^{-mk}\right) \\ &= \frac{1}{an(1-t^n)}\left[\frac{t(1-\omega^{(1-k)n})}{1-\omega^{1-k}}-\frac{1-\omega^{-nk}}{1-\omega^{-k}}\right] \\ &= 0 \end{aligned}$$

III. Main Results

Theorem 3.1:

$$|C| = s^n(1-|z|^n e[in\beta])^{n-1}$$

Proof:

$$|C| = |sK| = s^n|K| = s^n(1-|z|^n e[in\beta])^{n-1}$$

Theorem 3.2: The eigenvalues of C are given by

$$\lambda_m = \frac{s(1-|z|^n e[in\beta])}{1-|z|e[i\xi]}$$

where $\xi = \frac{n\beta-2\pi m}{n}$ and $m = 0,\ 1,\ ...,\ n-1$.

Proof:

The eigenvalues of C are given by the DFT

$$\begin{aligned} \lambda_m &= \sum_{k=0}^{n-1} sz^k e[-2\pi imk/n] \\ &= s\sum_{k=0}^{n-1}|z|^k e[ik\beta]e[-2\pi imk/n] \\ &= s\sum_{k=0}^{n-1}|z|^k e\left[ik\left(\frac{n\beta-2\pi m}{n}\right)\right] \\ &= \frac{s(1-|z|^n e[in\beta])}{1-|z|e[i\frac{n\beta-2\pi m}{n}]} \end{aligned}$$

as desired.

Theorem 3.3:

$$||C||_E = |s|\sqrt{\frac{n(1-|z|^{2n}e[i2n\beta])}{1-|z|^2 e[i2\beta]}}$$

Proof:

$$\begin{aligned} ||C||_E &= \sqrt{\sum_{i,,j=1}^{m,n}|a_{ij}|^2} \\ &= \sqrt{\sum_{k=0}^{n-1}|sz^k|^2} \\ &= |s|\sqrt{\sum_{k=0}^{n-1}|z|^{2k}} \\ &= |s|\sqrt{\frac{1-|z|^{2n}}{1-|z|^2}} \\ &= s|\sqrt{\frac{n(1-|z|^{2n}e[i2n\beta])}{1-|z|^2 e[i2\beta]}} \end{aligned}$$

Theorem 3.4:

$$||C||_2 = max\left\{|s|\frac{\sqrt{1-2|z|^n\cos n\beta+|z|^{2n}}}{\sqrt{1-2|z|\cos\frac{n\beta-2\pi m}{n}+|z|^2}}\right\}$$

where $m = 0,\ 1,\ ...,\ n-1$.

Proof:

Computing for the modulus and finiding the maximum of the eigenvalues gives the result.

Theorem 3.5:

$$C^{-1} = RCIRC_n\left(\frac{1}{s(1-|z|^n e[in\beta])}, \frac{-|z|e[i\beta]}{s(1-|z|^n e[in\beta])}, 0, \ldots, 0\right)$$

Proof:

The entries of the inverse of C are given by the IDFT

$$\begin{aligned} w_k &= \tfrac{1}{n}\textstyle\sum_{m=0}^{n-1}\lambda_m^{-1}e[2\pi imk/n] \\ &= \tfrac{1}{n}\textstyle\sum_{m=0}^{n-1}\frac{1-|z|e[i\frac{n\beta-2\pi m}{n}]}{s(1-|z|^n e[in\beta])}e[2\pi imk/n] \end{aligned}$$

where $k = 0,\ 1,\ ...,\ n-1$.

Using Lemma 2.2 with $t = |z|e[i\beta]$ proves the theorem.

IV. Conclusion

In summary, we have extended the results of Bueno in [1] using the elements of complex geometric sequences as entries for the right circulant matrices.

References

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[2] R. Gray,” Toeplitz and Circulant Matrices: A Review,’ Technical Report, Stanford University, USA