A Common Fixed Point Theorem for Four Self Mappings in Fuzzy Metric Spaces

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***Abstract*—In this Paper we prove a fixed point theorem for four self mappings in Fuzzy Metric Spaces using the concept of R - weakly commutativity.**

***Index Terms*—Common fixed point, R - weakly commuting maps, Fuzzy metric spaces.**

***MSC 2000 Codes* – 47H10, 54H25**

I. Introduction

ZADEH ([8]) introduced the concept of fuzzy set in 1965. Kramosil and Michalek ([4]) introduced Fuzzy metric space in 1975 by generalizing the concept of probabilistic metric space. George and Veermani ([1]) introduced Fuzzy metric spaces with the help of continuous t - norms. Vasuki ([6]) introduced the concept of R - weakly commutativity in Fuzzy metric space. A. k. Sharma, V. H. Badashah, V. K. Gupta and Ajay Sharma ([9]) generalized the concept of R - weakly commutativity from two mappings to three mappings.

In this paper we generalize this result to four mappings.

II. Some Preliminary Results

Definition 2.1([8]) A Fuzzy set A in X is a function with domain X and values in $[0, 1]$.□

Definition 2.2([5]) A binary operation $* : [0, 1]\times[0, 1] \to [0, 1]$ is a continuous t-norm if for each a, b, c, d in $[0, 1]$, $*$ satisfies the following conditions:

1) $*$ is associative and commutative;
2) $*$ is continuous;
3) $a * 1 = a$ for all $a \in [0, 1]$;
4) $a * b \le c * d$ when $a \le c$ and $b \le d$.

□

Example 2.1. ([1]) t - norms are $a * b = ab$, $a * b = min\{a, b\}$

Definition 2.2([4]) The triplet $(X, M, *)$ is a Fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ if and only if $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.
6) $lim_{t\to\infty}M(x, y, t) = 1$ for all $x, y \in X$.

□

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$.

The following example shows every metric space induces a Fuzzy metric spaces.

Example 3.1.([2]) Let (X, d) be a metric space and define $a * b = min\{a, b\}$. Also define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space. It is called standard fuzzy metric space induced by (X, d).

Lemma 2.1([3]) For all $x, y \in X$, $M(x, y, .)$ is non - decreasing function.

Definition 2.3([3]) Let $(X, M, *)$ be a fuzzy metric space, then

1 a sequence $\{x_n\}$ in X is said to be convergent to x if
$$lim_{n\to\infty}M(x_n, x, t) = 1,$$
for all $t > 0$;

2 a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that
$$M(x_n, x_m, t) > 1 - \epsilon,$$
for all $t > 0$ and $n, m \ge n_0$;

3 a fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

□

Definition 2.4([5]) The mappings f and g of a Fuzzy metric space $(X, M, *)$ into itself said to be weakly commuting if
$$M(fgx, gfx, t) \ge M(fx, gx, t) \quad \forall x \in X.$$
□

Definition 2.5([5]) The mappings f and g of a Fuzzy metric space $(X, M, *)$ into itself said to be R - weakly commuting if
$$M(fgx, gfx, t) \ge M(fx, gx, \tfrac{t}{R}) \text{ where } R > 1, \quad \forall x \in X.$$
□

Remark 2.1 R - weakly commuting mapping implies weakly commuting only when $R \le 1$.

Example 2.2 Let $X = \mathbb{R}$ the set of real number.

Define $a * b = ab$ and $M(x, y, t) = (exp(\frac{|x-y|}{t}))^{-1}$ for all $x, y \in X, t > 0$.

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$M(x,y,t) = 0$ when $t = 0$. Then $(X, M, *)$ is a Fuzzy metric spaces.

Define $f(x) = 2x - 1$ and $g(x) = x^2$. Then

$$M(fgx, gfx, t) = (exp(\tfrac{2|x-1|^2}{t}))^{-1}$$

$$M(fgx, gfx, \tfrac{t}{2}) = (exp(\tfrac{2|x-1|^2}{t}))^{-1}$$

Therefore for $R = 2$, f and g are R - weakly commuting.

Example 2.3 Every metric space induces a Fuzzy metric space. Let (X, d) be a metric space.

Define $a * b = ab$ and $M(x,y,t) = \frac{kt^n}{kt^n + md(x,y)}$ $k, m, n \in \mathbb{R}^+$

Then $(X, M, *)$ is a Fuzzy metric spaces.

If we put $k = m = n = 1$, then

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

This fuzzy metric space induced by a metric d, the standard metric.

The following theorem was proved by Vasuki ([7]).

Theorem 2.1 Let $(X, M, *)$ be complete Fuzzy metric space and let f and g be R - weakly commuting mapping of X satisfying the condition

$$M(fx, fy, t) \geq r\{M(gx, gy, \tfrac{t}{R})\} \text{ for all } x, y \in X.$$

where $r : [0,1] \to [0,1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. The sequence $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x, y_n \to y, t > 0$ implies that $M(x_n, y_n, t) \to M(x, y, t)$ as $n \to \infty$ If the range of g contain the range of f if either f or g is continuous, then f and g have a unique common fixed point in X. □

The following theorem was proved by A. K. Sharma, N. Badasha, et. al. ([9])

Theorem 2.2 Let f, g and h be three self mappings on a Fuzzy metric space $(X, M, *)$ satisfying the conditions.

$$f(X) \cap g(X) \subset h(X) \text{ and}$$

$$M(fx, gy, t) \geq r\{M(hx, hy, t)\} \text{ for all } x, y \in X.$$

Where $r : [0,1] \to [0,1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. Suppose that h is continuous and pairs (f, h) and (g, h) are R - weakly commuting on X. Then f, g and h have a unique common fixed point in X.

III. Main Result

To prove the main result we need the following lemma.

Lemma 3.1 Let $(X, M, *)$ be a complete fuzzy metric space and let f, g, h and I be mapping of X into X satisfying the conditions

$$f(X) \cap g(X) \subset h(X) \tag{1}$$

. and

$$M(fx, gy, t) \geq r\{M(hx, Iy, t)\} \forall \quad x, y \in X. \tag{2}$$

Where $r : [0,1] \to [0,1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$.

Then the sequence $\{y_n\}$ is a cauchy sequence in X.

Proof: For $x_0 \in X$, we can choose $fx_0 = hx_1$ and for this $x_1 \in X$, then $\exists \quad x_2 \in X$ such that $gx_1 = Ix_2$ and so on. Continue in this manner we can choose a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = hx_{2n+1}, y_{2n+1} = gx_{2n+1} = Ix_{2n+2} \tag{3}$$

For $t > 0$,

$$\begin{aligned} M(y_{2n}, y_{2n+1}, t) &= M(fx_{2n}, gx_{2n+1}, t) \geq r\{M(hx_{2n}, Ix_{2n+1}, t)\} \\ &= r\{M(y_{2n-1}, y_{2n}, t)\} > M(y_{2n-1}, y_{2n}, t) \quad (4) \end{aligned}$$

i.e., $M(y_{2n}, y_{2n+1}, t) > M(y_{2n-1}, y_{2n}, t)$ for all $n \geq 0$.

Thus $\{M(y_{2n}, y_{2n+1}, t) : n \geq 0\}$ is a increasing sequence of positive number in $[0,1]$ and therefore to a limit $l(\leq 1)$. We claim that $l = 1$.For if $l < 1$, on letting $n \to \infty$ in (4), we have

$$l \geq r(l) > l,$$

which is a contradiction. Hence $l = 1$.

Similarly $M(y_{2n+2}, y_{2n+1}, t) > M(y_{2n}, y_{2n+1}, t) \quad \forall n \geq 0$.

Therefore $\{M(y_{2n+1}, y_{2n+2}, t) : n \geq 0\}$ is a increasing sequence of positive numbers in $[0,1]$ and therefore to a limit $l = 1$.

Therefore for every $n \in \mathbb{N}$, $M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t)$ and $lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$ for $t > 0$.

Now for any positive integer p,

$$\begin{aligned} M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, \frac{t}{p}) * \cdots * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\geq M(y_n, y_{n+1}, \frac{t}{p}) * \cdots * M(y_n, y_{n+1}, \frac{t}{p}) \\ &= 1 * 1 * 1 \cdots * 1 \end{aligned}$$

i. e., $M(y_n, y_{n+1}, t) \geq 1$

Thus $\{y_n\}$ is a cauchy sequence in X.

Now we prove a theorem with four mappings.

Theorem 3.1 Let $(X, M, *)$ be complete Fuzzy metric space and let (f, g) and (h, I) be R - weakly commuting mappings satisfying the conditions(1) and (2).Suppose that h and I are continuous and pairs (f, h) and (g, I) are R - weakly commuting on X. Then f, g, h and I have a unique common fixed point.

Proof: By Lemma (III) we have sequence $\{y_n\}$ is a cauchy sequence in X. But X is complete and so by completeness of X, $\{y_n\}$ converges to $u \in X$.

Consequently, the subsequences $\{fx_{2n}\}, \{hx_{2n+1}\}, \{gx_{2n+1}\}, \{Ix_{2n+2}\}$ of y_n also converges to u in X.

Since h is continuous. It follows that $hfx_n \to hu$.

Since (f, h) is R - weakly commuting, therefore

$$M(fhx_n, hfx_n, t) \geq M(fx_n, hx_n, \tfrac{t}{R}) \quad \forall \quad x \in X.$$

On letting as $n \to \infty$,

$$M(fhx_n, hu, t) \geq M(u, u, \frac{t}{R}) = 1.$$

Hence $fhx_n \to hu$.

As I is continuous. It follows that $Igx_n \to Iu$.

Since (g, I) is R - weakly commutes, therefore

$$M(gIx_n, Iu, t) \geq M(gx_n, Ix_n, \frac{t}{R})$$

On letting $n \to \infty$, we get

$$M(gIx_n, Iu, t) \geq M(u, u, \frac{t}{R}) = 1.$$

Hence $gIx_n \to Iu$.

Also by(1)

$$M(fhx_{2n}, gx_{2n+1}, t) \geq r\{M(hhx_{2n}, Ix_{2n+1}, t)\}$$

On letting as $n \to \infty$, we get

$$M(hu, u, t) \geq r(M(hu, u, t)) > M(hu, u, t)$$

which is a contraction. Hence $hu = u$.

Again,

$$M(fu, gx_{2n+1}, t) \geq r\{M(hu, Ix_{2n+1}, t)\}$$

On letting as $n \to \infty$, we get

$$M(fu, u, t) \geq r(M(hu, u, t)) = r(M(u, u, t)) = r(1) = 1.$$

Hence $fu = u$.

Now consider(1)

$$M(fu, gIx_{2n+1}, t) \geq r\{M(hu, Ix_{2n+1}, t)\}$$

Since $hu = u, fu = u$.

$$M(u, Iu, t) \geq r(M(u, Iu, t)) > M(u, Iu, t)$$

Therefore, $u = Iu$

Also

$$M(fu, gx_{2n+1}, t) \geq r\{M(hu, Ix_{2n+1}, t)\}$$

$\Rightarrow M(u, gu, t) \geq r(M(u, u, t)) = r(1) = 1$
therefore $u = g(u)$ Hence f, g, h and I have a common fixed point.

Uniqueness of fixed point: Suppose $u \neq v$, where u, v is a fixed point of f, g, h and I. Then there exist $t > 0$ such that $M(u, v, t) < 1$

$$M(u, v, t) = M(fu, gv, t) \geq r(M(hu, Iv, t)) > M(u, v, t).$$

which is a contradiction. Hence $u = v$. therefore u is a unique common fixed point of f, g, h and I.

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References

[1] A. George and P. Veeramani, "On some results in fuzzy metric spaces", *Fuzzy Sets and Systems*, vol. 64, pp. 395-399, 1994.
[2] A. George and P. Veeramani, "On some results of analysis for fuzzy metric spaces", *Fuzzy Sets and Systems*, vol. 90, pp. 365-368, 1997.
[3] M. Grabiec, "Fixed points in fuzzy metric space", *Fuzzy Sets and Systems*, vol. 27, pp. 385-389, 1988.
[4] I. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces", *Kybernetica*, vol. 15, pp. 326 - 334, 1975.
[5] B. Schweizer, A. Sklar, "Statistical metric spaces", *Pac. J. Math.*, vol. 10, pp. 314 - 334, 1960.
[6] R. Vasuki, "Common fixed points for R-weakly commuting maps in fuzzy metric spaces", *Indian J. Pure Appl. Math.*, vol. 30, pp. 419 - 423,(1999).
[7] R. P. Pant," Common fixed points of noncommuting mappings", *J. Math. Anal. Appl.*, vol. 188, pp. 436 - 440, 1994.
[8] L. A. Zadeh, "Fuzy sets", *Information and Control*, vol. 8, pp. 338 - 353, 1965.
[9] A. K. Sharma, V. H. Badshah, V. K. Gupta and A. Sharma, "A common fixed point theorem for three self mappings in a Fuzzy metric spaces", *Int. J. Contemp. Math. Sciences, vol. 7, no. 31, pp. 1509 - 1518, 2012.*