Common Fixed Points Of ψ-Weak Generalized Geraghty Contractions In Partially Ordered Metric Spaces

G. V. R. Babu, K. K. M. Sarma and V. A. Kumari

***Abstract*—In this paper, we introduce ψ-weak generalized Geraghty contractions and prove the existence of common fixed points for a pair of weakly compatible maps in partially ordered complete metric spaces. Our result generalizes a result of Amini-Harindi, Emami(2010), Gordji, Ramezani, Cho and Pirbavafa(2012), Choudhury and Kundu (2013).**

***Index Terms*—Geraghty contraction, ψ-weak generalized Geraghty contraction, Compatible maps, Weakly Compatible maps, (CLRg)-property.**

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I. Introduction

The Banach contraction mapping principle is one of the fundamental results of nonlinear functional analysis to prove the existence of fixed points of contraction mappings in complete metric spaces. In 1973, Geraghty[1] introduced an extension of the contraction in which the contraction constant was replaced by a function having some specified properties. We use the following notation introduced by Geraghty, namely S = $\{\beta : [0,\ \infty) \to [0,1)/\ \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$.

II. Some Preliminary Results

Definition 1.1.[1] A selfmap $f : X \to X$ is said to be a *Geraghty contraction* if there exists $\beta \in S$ such that

$d(f(x), f(y)) \leq \beta(d(x,y))d(x,y)$ for all $x, y \in X$. (1.1.1)

Theorem 1.2[1] Let X be a complete metric space. Let $f : X \to X$ be a mapping such that there exists $\beta \in S$ such that

$d(f(x), f(y)) \leq \beta(d(x,y))d(x,y)$ for all $x, y \in X$. (1.2.1)

Then for any choice of initial point x_0, the iteration $x_n = f(x_{n-1})$, for n = 1, 2, 3,.... , converges to the unique fixed point z of f in X .

In 2010, Amimi-Harindi and Emami[2] extended Theorem 1.2 to partially ordered metric spaces.

Theorem 1.3.[2] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in S$ such that

$d(f(x), f(y)) \leq \beta(d(x,y))d(x,y)$ (1.3.1)

for all $x,\ y \in X$ with $x \succeq y$.

Assume that either

(i) f is continuous (or)

(ii) X is such that if an increasing sequence $\{x_n\} \to x$ in X, then $x_n \preceq x$ for all n.

Further if, for each $x, y \in X$, there exists $z \in X$ such that z is comparable to x and y. Then f has a fixed point in X.

In the following, we denote

$\Psi = \{\psi : [0,\infty) \to [0,\infty)/\psi$ is non-decreasing, continuous and $\psi(t) = 0 \Leftrightarrow t = 0\}$.

In 2012, Gordji, Ramezani, Cho and Pirbavafa[3] proved the following theorem by using an element $\psi \in \Psi$ along with the additional property namely sub-additivity.

i.e., $\psi(s + t) \leq \psi(s) + \psi(t)$ for all s, t ≥ 0.

Theorem 1.4.[3] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that

$\psi(d(f(x), f(y))) \leq \beta\Big(\psi(d(x,y))\Big)\psi(d(x,y))$ (1.4.1)

for all $x, y \in X$ with $x \succeq y$. Assume that either

(i) f is continuous (or)

(ii) X is such that if an increasing sequence $\{x_n\}$ converges to x then $x_n \preceq x$ for each n ≥ 1.

Then f has a fixed point. Further if, for each $x, y \in X$, there exists $z \in X$ which is comparable to x and y, then f has a unique fixed point in X.

When ψ is sub-additive and d is metric then $\psi\ o\ d$ is a metric and d is complete if and only if $\psi\ o\ d$ is complete so that the composition $\psi\ o\ d$ has the same properties as that of a metric. A detailed discussion is given in [4]. In 2014, Babu, Sarma and Krishna[4] relaxed the sub-additivity property of ψ of Theorem 1.4 and proved fixed point theorem by using more general Geraghty contraction than that of (1.4.1) which is as follows.

Theorem 1.5.[4] Let $(X, \preceq)$ be a partially ordered set and suppose that d is a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$. Assume that f is ψ-weak generalized Geraghty contraction (i.e., if there exist $\beta \in S$ and $L \geq 0$ such that

$\psi(d(f(x), f(y))) \leq \beta(\psi(M(x,y)))$

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$\psi(M(x,y)) + L.N(x,y)$ (1.5.1)
where $M(x,y) = \max\{d(x,y), \frac{d(x,f(x))+d(y,f(y))}{2}, \frac{d(x,f(y))+d(y,f(x))}{2}\}$ and
$N(x,y) = \min\{d(x,f(x)), d(x,f(y)), d(y,f(x))\}$
for all $x, y \in X$ with $x \succeq y$.
Furthermore, assume that either

(i) f is continuous (or)

(ii) X is such that if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 1$ and if for any $s > 0$, $\limsup_{t\to s} \beta(t) = \beta(s)$.

Then f has a fixed point in X.

Definition 1.6.[5] A selfmap $f : X \to X$, where (X,d) is a metric space is called a *Kannan type* mapping if there exists $0 < \lambda < 1$ such that

$$d(f(x), f(y)) \leq \frac{\lambda}{2}[d(x,f(x)) + d(y,f(y))] \quad (1.6.1)$$

for all $x, y \in X$.

In 2013, Choudhury and Kundu [6] extended Geraghty theorem to Kannan type mappings in partially ordered metric spaces.

Theorem 1.7.[6] Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in S$ such that

$$d(f(x), f(y)) \leq \beta(\tfrac{1}{2}(d(x,f(x)) + d(y,f(y)))) (\tfrac{1}{2}(d(x,f(x)) + d(y,f(y)))) \quad (1.7.1)$$

for all $x, y \in X$, x and y are comparable.
Also suppose that either

(i) f is continuous (or)

(ii) X has the property, if an increasing sequence $\{x_n\}$ converges to x then $x_n \preceq x$ for each n ≥ 0.

Then f has a fixed point in X.

In 1986, Jungck[7] introduced the concept of compatible maps and proved the existence of fixed points in metric spaces. In 1998, Jungck and Rhoades[8] introduced the concept namely weakly compatible maps as a generalization of compatible maps.

Definition 1.8.[7] Two self mappings f and g of a metric space (X,d) are said to be *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Definition 1.9.[8] Two self mappings f and g of a metric space (X,d) are said to be *weakly compatible* if they commute at their coincidence points, i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Definition 1.10.[9] Let (X,d) be a metric space and f, g be self maps of X. We say that f and g are *reciprocally continuous* if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$ whenever $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Lemma 1.11.[10] Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that
$d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$ and

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon$,

(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$,

(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$ and

(iv) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon$.

Definition 1.12.[11] Suppose $(X,\preceq)$ is a partially ordered set and $f, g : X \to X$ are mappings of X to itself. f is said to be g-non-decreasing if for $x, y \in X$,

$$gx \preceq gy \text{ implies } fx \preceq fy. \quad (1.12.1)$$

In this paper, we introduce ψ-weak generalized Geraghty contractions and prove the existence of common fixed points for a pair of weakly compatible maps in partially ordered complete metric spaces. Our result generalizes a result of Amini-Harindi, Emami[2], Gordji, Ramezani, Cho and Pirbavafa[3], Choudhury and Kundu [6].

In the following, we introduce generalized Geraghty contraction and ψ-weak generalized Geraghty contraction for a pair of selfmaps in partially ordered metric spaces.

Definition 1.13. Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a metric space. Let f and g be two self mappings on X. If there exists $\beta \in S$ such that

$$d(fx, fy) \leq \beta(M(x,y))M(x,y) \quad (1.13.1)$$

for all $x, y \in X$ with $gx \succeq gy$, where
$M(x,y) = \max\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{1}{2}(d(gx,fy)+d(gy,fx))\}$, then we say that (f,g) is a pair of *generalized Geraghty contraction maps*.

Example 1. Let $X = \{0,2,3\}$ with the usual metric. We define partial order $\preceq$ on X as follows,
$\preceq := \{(0,0),(2,2),(3,3),(2,0),(3,0),(3,2)\}$.
Let A = $\{((0,0),(2,2),(3,3),(2,0),(3,0),(3,2)\}$,
B = $\{(0,2),(0,3),(2,3)\}$. We define $f, g : X \to X$ by $f0 = f2 = 2$, $f3 = 3$; $g0 = 2$, $g2 = g3 = 0$, and $\beta : [0,\infty) \to [0,1)$ by $\beta(t) = \begin{cases} \frac{2}{3} & if\ t \geq 1 \\ \frac{1}{1+t} & if\ t \in (0,1) \\ 0 & if\ t = 0. \end{cases}$

The following three cases arise to verify the inequality (1.13.1).

<u>*Case*</u> (i): $(x,y) = (0,2)$.
In this case, the inequality (1.13.1) trivially holds.

<u>*Case*</u> (ii): $(x,y) = (0,3)$.
In this case,
$d(f0, f3) = 1$, $M(0,3) = 3$ and $\beta(3) = \frac{2}{3}$. Now

$$\begin{aligned} d(fx,fy) &= d(f0,f3) = 1 \\ &\leq \beta(3).3 \\ &= \beta(M(0,3))M(0,3) = \beta(M(x,y))M(x,y). \end{aligned}$$

<u>*Case*</u> (iii): $(x,y) = (2,3)$.
In this case,
$d(f2, f3) = 1$, $M(2,3) = 3$, $\beta(3) = \frac{2}{3}$. Now

$$\begin{aligned} d(fx,fy) &= d(f2,f3) = 1 \\ &\leq \beta(3).3 \\ &= \beta(M(2,3))M(2,3) = \beta(M(x,y))M(x,y). \end{aligned}$$

Therefore (f,g) is a pair of generalized Geraghty contraction maps. Here we observe that f and g are not Geraghty

contractions.
For, when $(x, y) = (2, 3)$ we have
$d(f2, f3) = 1$ and $d(2, 3) = 1$, we observe that
for any $\beta \in S$,

$$d(fx, fy) = d(f2, f3) = 1 \not\leq \beta(1).1 = \beta(d(2,3))d(2,3) = \beta(d(x,y))d(x,y),$$

so that the inequality (1.1.1) fails to hold, and f is not a Geraghty contraction.
Further, when $(x, y) = (0, 2)$ we have

$$d(gx, gy) = d(g0, g2) = 2 \not\leq \beta(2).2 = \beta(d(0,2))d(0,2) = \beta(d(x,y))d(x,y)$$

for any $\beta \in S$.
Therefore the inequality (1.1.1) fails to hold, and g is not a Geraghty contraction.

Definition 1.14. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a metric space. Let f and g be two self mappings on X. If there exists $\psi \in \Phi$, $\beta \in S$ and $L \geq 0$ such that

$$\psi(d(fx, fy)) \leq \beta(\psi(M(x,y)))\psi(M(x,y)) + L.N(x,y) \qquad (1.14.1)$$

for all $x, y \in X$ with $gx \succeq gy$, where

$$M(x,y) = \max\{d(gx,gy), d(gx,fx), d(gy,fy), \tfrac{1}{2}(d(gx,fy) + d(gy,fx))\}$$

and $N(x,y) = \min\{d(gx,fx), d(gx,fy), d(gy,fx)\}$, then we say that (f, g) is a pair of *ψ-weak generalized Geraghty contraction maps*.

If $g = I_X$, the identity map of X in (1.14.1), then we call f is a ψ-weak generalized Geraghty contraction map. Further, if ψ is also the identity map then we call f is a weak generalized Geraghty contraction map.
It is clear that every pair of generalized Geraghty contraction maps is a pair of weak generalized Geraghty contraction maps. But its converse is not true due to the following example.

Example 2. Let $X = [0, 2]$ with the usual metric. We define partial order $\preceq$ on X as follows
$\preceq := \{(x, x)/x \in X\} \cup \{(0, 1), (0, 2), (1, 2)\}$. We define $f, g : X \to X$ by $f(x) = \begin{cases} \frac{x^2}{2} & if\ x \in [0,1] \\ 1 + \frac{x}{2} & if\ x \in (1,2] \end{cases}$ and

$$g(x) = \begin{cases} \frac{x}{2} & if\ x \in [0,1] \\ \frac{3}{2} & if\ x \in (1,2] \end{cases}$$

We define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = 2t$ and
$\beta : [0, \infty) \to [0, 1)$ by $\beta(t) = \begin{cases} \frac{3}{4} & if\ t \geq 1 \\ \frac{1}{1+t} & if\ t \in (0,1) \\ 0 & if\ t = 0. \end{cases}$

The following two cases arise to verify the inequality (1.14.1).
Case (i): $(x, y) = (2, 1)$.
In this case,
$d(f2, f1) = \frac{3}{2}$, $M(2, 1) = \frac{5}{4}$, $N(2, 1) = \frac{1}{2}$, $\psi(\frac{5}{4}) = \frac{5}{2}$ and $\beta(\frac{5}{2}) = \frac{3}{4}$. Now

$$\begin{aligned}\psi(d(fx, fy)) = \psi(d(f2, f1)) &= 3 \\ &\leq \beta(\tfrac{5}{2}).\tfrac{5}{2} + L.\tfrac{1}{2} \\ &= \beta(\psi(M(2,1)))\psi(M(2,1)) + L.N(2,1) \\ &= \beta(\psi(M(x,y)))\psi(M(x,y)) + L.N(x,y)\end{aligned}$$

holds with L = 3.
Case (ii): $(x, y) = (2, 0)$.
In this case,
$d(f2, f0) = 2$, $M(2, 0) = \frac{7}{4}$, $N(2, 0) = \frac{1}{2}$, $\psi(\frac{7}{4}) = \frac{7}{2}$ and $\beta(\frac{7}{2}) = \frac{3}{4}$. Now

$$\begin{aligned}\psi(d(fx, fy)) = \psi(d(f2, f1)) &= 4 \\ &\leq \beta(\tfrac{7}{2}).\tfrac{7}{2} + L.\tfrac{1}{2} \\ &= \beta(\psi(M(2,0)))\psi(M(2,0)) + L.N(2,0) \\ &= \beta(\psi(M(x,y)))\psi(M(x,y)) + L.N(x,y)\end{aligned}$$

holds with L = 3.
Therefore (f, g) is a pair of ψ-weak generalized Geraghty contraction maps.
In the following, we mention the importance of L and ψ in the inequality (1.14.1). If $L = 0$ then the inequality (1.14.1) fails to hold.
For, by choosing $x = 2$ and $y = 1$ we have
$\psi(d(f2, f1)) = 3 \not\leq \beta(\frac{5}{2}).\frac{5}{2} = \beta(\psi(M(2,1))).\psi(M(2,1))$ for any $\beta \in S$.
If $L = 0$ and ψ is the identity mapping then the inequality (1.14.1) fails to hold.
For, by choosing $x = 2$ and $y = 1$ we have
$d(f2, f1) = \frac{3}{2} \not\leq \beta(\frac{5}{4}).\frac{5}{4} = \beta(M(2,1)).M(2,1)$
for any $\beta \in S$. *i.e.*, (f, g) is not a pair of generalized Geraghty contraction maps.

III. Main Results

Theorem 2.1. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let f and g be two self mappings on X, f is g-non-decreasing. Suppose that (f, g) is a pair of *ψ-weak generalized Geraghty contraction maps*. Assume that

(i) $fX \subseteq gX$
(ii) there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$
(iii) $g(X)$ is a complete subset of X.
(iv) if any nondecreasing sequence $\{x_n\}$ in X converges to u, then that $x_n \preceq u$ for all $n \geq 0$.

Then f and g have a coincidence point in X.

Proof: By (ii), let $x_0 \in X$ be such that $gx_0 \preceq fx_0$. Since $fX \subseteq gX$, we choose $x_1 \in X$ such that $gx_1 = fx_0$. Since $gx_0 \preceq fx_0 = gx_1$, and f is g-non-decreasing, we have $fx_0 \preceq fx_1$, so that $gx_1 \preceq gx_2$. By using similar technique there exists a sequence $\{x_n\}$ in X with

$$fx_n = gx_{n+1} \text{ for } n = 1, 2, \ldots . \qquad (2.1.1)$$

Further,since $gx_1 \preceq gx_2$ and f is g-non-decreasing, we have $fx_1 \preceq fx_2$ so that $gx_2 \preceq gx_3$. Inductively, it follows that $gx_n \preceq gx_{n+1}$ for all $n = 0, 1, 2, \ldots$.
If $gx_{n+1} = gx_{n+2}$, for some n, then $gx_{n+1} = fx_{n+1}$ so that x_{n+1} is a coincidence point of f and g .
If $gx_{n+1} \neq gx_{n+2}$, for all n then, we have $d(gx_{n+2}, gx_{n+1}) > 0$.
Now from (1.14.1), we have

$$\begin{aligned}\psi(d(gx_{n+2}, gx_{n+1})) &= \psi(d(fx_{n+1}, fx_n)) \\ &\leq \beta(\psi(M(x_{n+1}, x_n)))\psi(M(x_{n+1}, x_n)) \\ &\quad + L.N(x_{n+1}, x_n), \qquad (2.1.2)\end{aligned}$$

where
$M(x_{n+1}, x_n) = \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, fx_{n+1}),$

$$\begin{aligned} & d(gx_n, fx_n), \tfrac{1}{2}(d(gx_{n+1}, fx_n) + d(gx_n, fx_{n+1}))\} \\ &= \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1}), \tfrac{1}{2}(d(gx_{n+1}, gx_{n+1}) + d(gx_n, gx_{n+2}))\} \\ &= \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}), \tfrac{1}{2}(d(gx_n, gx_{n+2}))\} \\ &\le \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}), \tfrac{1}{2}(d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}))\} \\ &= \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2})\} \\ &\le M(x_{n+1}, x_n). \end{aligned}$$

Hence $M(x_{n+1}, x_n) = \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2})\}$.

Now

$$\begin{aligned} N(x_{n+1}, x_n) &= \min\{d(gx_{n+1}, fx_{n+1}), d(gx_{n+1}, fx_n), d(gx_n, fx_{n+1})\} \\ &= \min\{d(gx_{n+1}, gx_{n+2}), d(gx_{n+1}, gx_{n+1}), d(gx_n, gx_{n+2})\} = 0. \end{aligned}$$

If $M(x_{n+1}, x_n) = d(gx_{n+1}, gx_{n+2})$ then form (2.1.2) we have

$$\begin{aligned} \psi(d(gx_{n+2}, gx_{n+1})) &\le \beta(\psi(d(gx_{n+2}, gx_{n+1}))) \psi(d(gx_{n+2}, gx_{n+1})) \\ &< \psi(d(gx_{n+2}, gx_{n+1})), \end{aligned}$$

a contradiction.

Hence $M(x_{n+1}, x_n) = d(gx_{n+1}, gx_n)$ so that from (2.1.2) we have

$$\psi(d(gx_{n+2}, gx_{n+1})) \le \beta(\psi(d(gx_{n+1}, gx_n))) \psi(d(gx_{n+1}, gx_n)) \quad (2.1.3)$$

which implies that $\psi(d(gx_{n+2}, gx_{n+1})) < \psi(d(gx_{n+1}, gx_n))$ for all n.

Thus it follows that $\{\psi(d(gx_{n+2}, gx_{n+1})\}$ is a decreasing sequence and $\lim_{n\to\infty} d(gx_{n+2}, gx_{n+1})$ exists and it is r (say). $i.e.$, $\lim_{n\to\infty} d(gx_{n+2}, gx_{n+1}) = r \ge 0$.

We now show that r = 0.

Suppose that $r > 0$. Then from (2.1.3), we have

$$\psi(d(gx_{n+2}, gx_{n+1})) \le \beta(\psi(d(gx_{n+1}, gx_n)))\psi(d(gx_{n+1}, gx_n))$$

On letting $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{\psi(d(gx_{n+2}, gx_{n+1}))}{\psi(d(gx_{n+1}, gx_n))} \le \lim_{n\to\infty} \beta(\psi(d(gx_{n+1}, gx_n))) \le 1$$

$$1 \le \lim_{n\to\infty} \beta(\psi(d(gx_{n+1}, gx_n))) \le 1$$

so that $\beta(\psi(d(gx_{n+1}, gx_n))) \to 1$, as $n \to \infty$.

Since $\beta \in S$, it is follows that

$\lim_{n\to\infty} \psi(d(gx_{n+1}, gx_n))$ = 0.

$r = \lim_{n\to\infty} d(gx_{n+1}, gx_n) = 0 < r$,

a contradiction.

Hence $\lim_{n\to\infty} d(gx_{n+2}, gx_{n+1})) = 0$. $i.e.$, r = 0.

We shall divide the remaining proof into two steps.

Step (i): $\{gx_n\}$ is a Cauchy sequence in X.

Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then, there exist $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$d(gx_{m(k)}, gx_{n(k)}) \ge \epsilon. \quad (2.1.4)$$

We choose $m(k)$, the least positive integer satisfying (2.1.4). Then, we have $m(k) > n(k) > k$ with

$$d(gx_{m(k)}, gx_{n(k)}) \ge \epsilon, \qquad d(gx_{m(k)-1}, gx_{n(k)}) < \epsilon$$

and by Lemma 1.11, it follows that $\lim_{k\to\infty} d(gx_{m(k)+1}, gx_{n(k)+1}) = \epsilon$.

Now from (1.14.1), we have

$$\begin{aligned} \psi(d(gx_{m(k)+1}, gx_{n(k)+1})) &= \psi(d(fx_{m(k)}, fx_{n(k)})) \\ &\le \beta(\psi(M(x_{m(k)}, x_{n(k)})))\psi(M(x_{m(k)}, x_{n(k)})) + L.N(x_{m(k)}, x_{n(k)}), \end{aligned}$$

where

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}) &= \max\{d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)}, fx_{m(k)}), d(gx_{n(k)}, fx_{n(k)}), \tfrac{1}{2}(d(gx_{m(k)}, fx_{n(k)}) + d(gx_{n(k)}, fx_{m(k)}))\} \\ &= \max\{d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)}, gx_{m(k)+1}), d(gx_{n(k)}, gx_{n(k)+1}), \tfrac{1}{2}(d(gx_{m(k)}, gx_{n(k)+1}) + d(gx_{n(k)}, gx_{m(k)+1}))\} \end{aligned}$$

and

$$\begin{aligned} N(x_{m(k)}, x_{n(k)}) &= \min\{d(gx_{m(k)}, fx_{m(k)}), d(gx_{m(k)}, fx_{n(k)}), d(gx_{n(k)}, fx_{m(k)})\} \\ &= \min\{d(gx_{m(k)}, gx_{m(k)+1}), d(gx_{m(k)}, gx_{n(k)+1}), d(gx_{n(k)}, gx_{m(k)+1})\}. \end{aligned}$$

On letting $k \to \infty$, and using the conclusion of Lemma 1.11, it follows that $\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon+\epsilon)\} = \epsilon$, $\lim_{k\to\infty} N(x_{m(k)}, x_{n(k)}) = 0$ and

$\psi(\epsilon) \le \lim_{k\to\infty} \beta(\psi(M(x_{m(k)}, x_{n(k)}))\psi(\epsilon) + L.0$.

Hence $1 = \frac{\psi(\epsilon)}{\psi(\epsilon)} \le \lim_{k\to\infty} \beta(\psi(M(x_{m(k)}, x_{n(k)}))) \le 1$ so that $\beta(\psi(M(x_{m(k)}, x_{n(k)}))) \to 1$ as $k \to \infty$.

Since $\beta \in S$, we have $M(x_{m(k)}, x_{n(k)}) \to 0$ as k $\to \infty$. $i.e.$, ϵ = 0,

a contradiction.

Therefore $\{gx_n\}$ is a Cauchy sequence in (X, d). Since $g(X)$ is complete, there exists $x \in g(X)$ such that

$\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} fx_n = gy = x$ for some $y \in X$.

Step (ii): $fy = gy$.

Suppose that $fy \neq gy$, $i.e.$, $d(fy, gy) > 0$.

Since $\{gx_n\}$ is a non-decreasing sequence and $\{gx_n\}$ converges to gy for some y in X, by (iv), we have $gx_n \preceq gy$ for all $n \ge 0$.

Now from (1.14.1), we have

$$\psi(d(fx_n, fy)) \le \beta(\psi(M(x_n, y)))\psi(M(x_n, y)) + L.N(x_n, y), \quad (2.1.5)$$

where

$$\begin{aligned} M(x_n, y) &= \max\{d(gx_n, gy), d(gx_n, fx_n), d(gy, fy), \tfrac{1}{2}(d(gx_n, fy) + d(gy, fx_n))\} \\ &= \max\{d(gx_n, gy), d(gx_n, gx_{n+1}), d(gy, fy), \tfrac{1}{2}(d(gx_n, fy) + d(gy, gx_{n+1}))\} \text{ and} \end{aligned}$$

$$\begin{aligned} N(x_n, y) &= \min\{d(gx_n, fx_n), d(gx_n, fy), d(gy, fx_n)\} \\ &= \min\{d(gx_n, gx_{n+1}), d(gx_n, fy), d(gy, gx_{n+1})\}. \end{aligned}$$

On letting $n \to \infty$, we get

$\lim_{n\to\infty} M(x_n, y) = d(gy, fy)$ and $\lim_{n\to\infty} N(x_n, y) = 0$.

Now from (2.1.5), we have

$$\psi(d(gy, fy)) \le \lim_{n\to\infty} \beta(\psi(M(x_n, y)))\psi(d(gy, fy)) + L.0$$

$1 = \frac{\psi(d(gy,fy))}{\psi(d(gy,fy))} \le \lim_{n\to\infty} \beta(\psi(M(x_n, y))) \le 1$ so that $\lim_{n\to\infty} \beta(\psi(M(x_n, y)) = 1$ and also $\lim_{n\to\infty} (\psi(M(x_n, y)) = 0$, $i.e.$, $d(gy, fy) = 0$,

a contradiction .

Therefore $gy = fy$.

Therefore y is a coincidence point of f and g.

Theorem 2.2. In addition to the hypotheses of Theorem 2.1, if $gy \preceq ggy$ where y is as in (iv) and f and g are weakly compatible then f and g have a common fixed point in X.

Proof: From the proof of Theorem 2.1 we have $\{gx_n\}$ is non-decreasing sequence that converging to gy and $fy = gy$. Since f and g are weakly compatible, we have $fgy = gfy$.
Now $fy = gy = u$ (say) (2.2.1)
Also, $fu = fgy = gfy = gu$. (2.2.2)
If $y = u$ then u is a common fixed point of f and g.
If $y \neq u$, *i.e.*, $d(y, u) > 0$.
Now from inequality (1.14.1), we have

$$\psi(d(gy, gu)) = \psi(d(fy, fu)) \leq \beta(\psi(M(y, u)))\psi(M(y, u)) + L.N(y, u), \quad (2.2.3)$$

where

$$M(y, u) = \max\{d(gy, gu), d(gy, fy), d(gu, fu), \tfrac{1}{2}(d(gz, fu) + d(gu, fz))\} = \max\{d(gy, gu), 0, 0, d(gy, gu)\} = d(gy, gu)$$

and

$$N(y, u) = \min\{d(gy, fy), d(gy, fu), d(gu, fy)\} = \min\{0, d(gy, gu)\} = 0.$$

Now from (2.2.3), we have

$$\psi(d(gy, gu)) \leq \beta(\psi(d(gy, gu)))\psi(d(gy, gu)) + L.0$$

$$1 = \frac{\psi(d(gy,gu))}{\psi(d(gy,gu))} \leq \beta(\psi(d(gy, gu))) < 1.$$

Hence $gu = gy$. Then by (2.2.1) and (2.2.2), we have $u = gu = fu$.
Therefore u is a common fixed point of f and g.

In the following we prove Theorem 2.1 by replacing the condition (iii) and (iv) by 'reciprocal continuity' and 'compatibility'.

Theorem 2.3. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let f and g be two self mappings on X, f is g-non-decreasing. Suppose that (f, g) is a pair of ψ*-weak generalized Geraghty contraction maps*. Assume that

(i) $fX \subseteq gX$
(ii) f and g are compatible
(iii) there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$
(iv) f and g are reciprocally continuous.

Then f and g have a coincidence point.

Proof: From the proof of Theorem 2.1 we have $\{gx_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $z \in X$ such that
$\lim_{n\to\infty} gx_n = z$, $z \in X$.
Hence $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z$.
Since f and g are reciprocally continuous, we have $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$.
Since f and g are compatible, we have $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ so that $d(fz, gz) = 0$.
Hence $fz = gz$ so that z is a coincidence point of f and g.

IV. Corollaries and Examples

By choosing $g = I_X$ in Theorem 2.1, we have the following corollary.

Corollary 3.1. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space and let $f : X \to X$ be a ψ-weak generalized Geraghty contraction map. If there exists x_0 in X such that $x_0 \preceq fx_0$ and f is non-decreasing, if any non-decreasing sequence $\{x_n\}$ in X converges to u, then we assume that $x_n \preceq u$ for all $n \geq 0$. Then f has a fixed point.

By choosing $g = I_X$ and $\psi(t) = t$ in Theorem 2.1, we have the following corollary.

Corollary 3.2. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space and let $f : X \to X$ be a weak generalized Geraghty contraction map on X. If there exists x_0 in X such that $x_0 \preceq fx_0$ and f is non-decreasing, if any non-decreasing sequence $\{x_n\}$ in X converges to u, then we assume that $x_n \preceq u$ for all $n \geq 0$. Then f has a fixed point.

By choosing $g = I_X$ in Theorem 2.3, we have the following corollary.

Corollary 3.3. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space and let $f : X \to X$ be a ψ-weak generalized Geraghty contraction map. If there exists x_0 in X such that $x_0 \preceq fx_0$, if f is non-decreasing and continuous. Then f has a fixed point.

By choosing $g = I_X$ and $\psi(t) = t$ in Theorem 2.3, we have the following corollary.

Corollary 3.4. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space and let $f : X \to X$ be a weak generalized Geraghty contraction map on X. If there exists x_0 in X such that $x_0 \preceq fx_0$, if f is non-decreasing and continuous. Then f has a fixed point.

Remark 3.3. (i) Theorem 1.3 follows as a corollary to Theorem 2.2 by choosing $\psi(t) = t$, $L = 0$ and $g = I_X$, the identity map of X in the inequality (1.14.1) of Theorem 2.2 is a partial generalization of Theorem 1.3, provided β is non-decreasing.
(ii) Theorem 1.4 follows as a corollary to Theorem 2.3, by choosing $L = 0$ and $g = I_X$, the identity map of X in the inequality (1.14.1) of Theorem 2.3 is a partial generalization of Theorem 1.4, provided β is non-decreasing.

The following are the examples in support of our main results of Section 2.

Example 3. Let $X = \{0, 1, 2, 8\}$ with the usual metric. We define partial order $\preceq$ on X as follows,
$\preceq := \{(0, 0), (1, 1), (2, 2), (8, 8), (0, 1), (0, 2), (0, 8), (1, 2), (1, 8)\}$.
We define $f, g : X \to X$ by $f0 = f1 = 1,\ f2 = 2, f8 = 8$ and also $g0 = 0,\ g1 = 1,\ g2 = 8,\ g8 = 2$.
We define $\psi : [0, \infty) \to [0, 1)$ by $\psi(t) = \frac{8t}{7}$ and

$$\beta(t) = \begin{cases} \frac{7}{8} & if\ t > 1 \\ \frac{1}{1+t} & if\ t \in (0,1) \\ 0 & if\ t = 0. \end{cases}$$

Clearly $fX \subseteq gX$. We choose $x_0 = 0$ then $gx_0 \preceq fx_0$. Also f and g are weakly compatible and f is g-non-decreasing. Further, f and g, ψ and β satisfy the inequality (1.14.1). Hence f and g satisfy all the hypotheses of Theorem 2.2 and 1 is the unique common fixed point of f and g.

In the following, we mention the importance of L and ψ in the inequality (1.14.1) of Theorem 2.2.

If $L = 0$ and ψ is the identity mapping then the inequality (1.14.1) fails to hold.

For, by choosing $x = 8$ and $y = 1$ we have

$d(f8, f1) = 7 \nleq \beta(6).6 = \beta(M(8,1)).M(8,1)$

for any $\beta \in S$.

Further, if we choose $x = 8$ and $y = 1$ then the inequality (1.3.1) fails to hold;

for, $d(f8, f1) = 7 \nleq \beta(7).7 = \beta(d(8,1)).d(8,1)$

for any $\beta \in S$, and hence Theorem 1.3 is not applicable.

If we choose $x = 8$ and $y = 1$ then the inequality (1.4.1) fails to hold;

for, $\psi(d(f8, f1)) = 8 \nleq \beta(8).8 = \beta(\psi(d(8,1))).\psi(d(8,1))$

for any $\beta \in S$, and hence Theorem 1.4 is not applicable.

If we choose $x = 8$ and $y = 1$ then the inequality (1.7.1) fails to hold;

for, $d(f8, f1) = 7 \nleq \beta(0).0 = \beta(\frac{1}{2}(d(8, f8) + d(1, f1)))$
$(\frac{1}{2}(d(8, f8) + d(1, f1)))$

for any $\beta \in S$, and hence Theorem 1.7 is not applicable.

Remarks 3.3 $(i), (ii)$ and Example 3 suggest that Theorem 2.2 is a generalization of Theorem 1.3 and Theorem 1.4, provided β is non-decreasing

The following is an example in support of Theorems 2.3.

Example 4. Let $X = \{1, 2, 4, 6\}$ with the usual metric. We define partial order $\preceq$ on X as follows,

$\preceq := \{(1,1),(2,2),(4,4),(6,6),(1,2),(1,4),(1,6),(2,4),(2,6)\}$.

We define $f, g : X \to X$ by $f1 = f2 = 2,\ f4 = 4, f6 = 6$ and also $g1 = 1,\ g2 = 2,\ g4 = 6,\ g6 = 4$.

We define $\psi : [0,\infty) \to [0,1)$ by $\psi(t) = \frac{t}{2}$ and

$$\beta(t) = \begin{cases} \frac{2}{3} & if\ t > 1 \\ \frac{1}{1+t} & if\ t \in (0,1) \\ 0 & if\ t = 0. \end{cases}$$

Clearly $fX \subseteq gX$. We choose $x_0 = 1$ then $gx_0 \preceq fx_0$. Also f and g are compatible and f and g are reciprocally continuous and f is g-non-decreasing. Further, f and g, ψ and β satisfy the inequality (1.14.1). Hence f and g satisfy all the hypotheses of Theorem 2.3 and 2 is the unique common fixed point of f and g.

In the following, we mention the importance of L and ψ in the inequality (1.14.1) of Theorem 2.3.

If $L = 0$ and ψ is the identity mapping then the inequality (1.14.1) fails to hold.

For, by choosing $x = 6$ and $y = 1$ we have

$d(f6, f1) = 4 \nleq \beta(\frac{7}{2}).\frac{7}{2} = \beta(M(6,1)).M(6,1)$

for any $\beta \in S$.

Further, if we choose $x = 6$ and $y = 2$ then the inequality (1.3.1) fails to hold;

for, $d(f6, f2) = 4 \nleq \beta(4).4 = \beta(d(6,2)).d(6,2)$

for any $\beta \in S$, and hence Theorem 1.3 is not applicable.

If we choose $x = 6$ and $y = 2$ then the inequality (1.4.1) fails to hold;

for, $\psi(d(f6, f2)) = 2 \nleq \beta(2).2 = \beta(\psi(d(6,2))).\psi(d(6,2))$

for any $\beta \in S$, and hence Theorem 1.4 is not applicable.

If we choose $x = 6$ and $y = 2$ then the inequality (1.7.1) fails to hold;

for, $d(f6, f2) = 4 \nleq \beta(0).0 = \beta(\frac{1}{2}(d(6, f6) + d(2, f2)))$
$(\frac{1}{2}(d(6, f6) + d(2, f2)))$

for any $\beta \in S$, and hence Theorem 1.7 is not applicable.

V. Conclusion

In this paper, we introduced ψ-weak Geraghty contractions and proved the existence and uniqueness of common fixed points for a pair of weakly compatible maps in partially ordered complete metric spaces. Remarks 3.3 $(i), (ii)$ and Examples 3 and 4 suggest that Theorem 2.2 and Theorem 2.3 generalize Theorem 1.3 and partially generalize Theorem 1.4, provided β is non-decreasing. Our results generalize the results of Amini-Harindi, Emami[2], Gordji, Ramezani, Cho and Pirbavafa[3], Choudhury and Kundu [6].

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