Some Fixed Point Results in Fuzzy Inner Product Spaces

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***Abstract*—In this paper, some properties of fuzzy inner product spaces are given and some fixed point results are established.**

***Index Terms*—Fuzzy inner product space, decomposition theorem, fixed point theorems.**

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I. Introduction

THE idea of real probabilistic inner product space introduced by Sklar [1] and after that some authors established this definition in different approach (for references please see [5], [6], [8], [10], [11], [15]). Mukherjee & Bag [12] redefine fuzzy real inner product space introduced by Goudarzi & Vaezpour [7] in order to establish a decomposition theorem from a fuzzy real inner product into a family of crisp inner products. Some recent research on fuzzy inner product spaces is carried out in [16]-[25]. In [14], the present authors try to give a concept of fuzzy inner product space in complex field and study some properties. In this paper some observations and some fixed point theorems namely Browder-Petryshiyn, D. de Figueiredo etc. are studied in fuzzy setting.

We have organized the paper in the following way: Preliminary results are given in Section II. In Section III, some basic results space are studied. In Section IV, various type of fixed point results in fuzzy inner product space are established.

II. Some Preliminaries

Definition 2.1 [2] Let X be a linear space over a field F(field of real / complex numbers). A fuzzy subset N of $X \times R$ (R is the set of real numbers) is called a fuzzy norm on X if $\forall x, u \in X$ and $c \in F$, following conditions are satisfied:
(N1) $\forall t \in R$ with $t \leq 0,\ N(x\ ,\ t) = 0$;
(N2) ($\forall t \in R,\ t > 0,\ N(x\ ,\ t) = 1$) iff $x = \underline{0}$;
(N3) $\forall t \in R,\ t > 0,\ N(cx\ ,\ t) = N(x\ ,\ \frac{t}{|c|})$ if $c \neq 0$;
(N4) $\forall s, t \in R,\ x, u \in X$;
$N(x+u\ ,\ s+t) \geq min\{N(x\ ,\ s)\ ,\ N(u\ ,\ t)\}$
(N5) $N(x\ ,\ .)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x\ ,\ t) = 1$.
The pair $(X\ ,\ N)$ will be referred to as a fuzzy normed linear space.

Definition 2.2 [2] Let $(X\ ,\ N)$ be a fuzzy normed linear space. A subset A of X is said to be bounded iff $\exists\ t > 0$ and $0 < r < 1$ such that $N(x,\ t) > 1 - r\ \forall\ x \in A$.

Definition 2.3 [14] Let V be a linear space over $F(R$ or $C)$. Define $\mu : V \times V \times F \to [0,\ 1]$ such that $\forall x,\ y,\ z \in V,\ t \in F$, the following conditions hold:
(CFI-1) $\mu(x,\ x,\ t) = 0\ \ \forall\ t \in F - (R^+ \cup \{0\})$.
(CFI-2) $(\mu(x,\ x,\ t) = 1\ \ \forall\ t \in R^+$ iff $x = \theta$.
(CFI-3) $\mu(x,\ y,\ t) = \mu(y,\ x,\ \bar{t})$.
(CFI-4) For any scalar k,

$$\mu(kx,\ y,\ t) = \begin{cases} 1 - \mu(x,\ y,\ \frac{t}{k}) & \text{if } k \in R^- \\ H(t) & \text{if } k = 0 \\ F(x,\ y,\ \frac{t}{k}) & \text{otherwise} \end{cases}$$

Where

$$H(t) = \begin{cases} 1 & \text{if } \ t \in R^+ \\ 0 & \text{otherwise.} \end{cases}$$

(CFI-5) (a) $\mu(x+y,\ z,\ Rl\ t + Rl\ s) \geq \mu(x,\ z,\ Rl\ t) \wedge \mu(y,\ z,\ Rl\ s)$.
(b) $\mu(x+y,\ z,\ Im\,t{+}Ims) \geq \mu(x,\ z,\ Im\,t) \wedge \mu(y,\ z,\ Ims)$.
where $Rl\ t$ is the real part of t and $Im\ t$ is the imaginary part of t.
(CFI-6) $\lim_{Rl\ t\to\infty} \mu(x,\ y, Rl\ t) = 1$.
Then μ is said to be a fuzzy inner product and $(V,\ \mu)$ is a fuzzy inner product space.

Definition 2.4 [4] Let $(X,\ N)$ be a fuzzy normed linear space.
A mapping $T : (X,\ N) \to (X,\ N)$ is said to be fuzzy non-expansive if
$N(T(x) - T(y),\ t) \geq N(x - y,\ t)\ \forall\ x,\ y \in X\ \forall\ t \in R$.

Proposition 2.1 [4] Let $(X,\ N)$ be a fuzzy normed linear space satisfying (N6) and $T : X \to X$ be a fuzzy non-expansive mapping. Then T is a non-expansive mapping w.r.t. each α-norm of N, where $\alpha \in (0,\ 1)$.

Remark 2.1 [4] If $N(x,\ .)$ is upper semicontinuous, then the converse of the Proposition 2.1 is also true.

Definition 2.5 [3] Let $(X,\ N)$ be a fuzzy normed linear space and $\alpha \in (0,\ 1)$. A sequence $\{x_n\}$ in X is said to be α-convergent in X if $\exists\ x \in X$ such that
$lim_{n\to\infty} N(x_n - x, t)\ > \alpha\ \forall\ t > 0$ and x is called the limit of $\{x_n\}$.

Proposition 2.2 [3] Let $(X\ ,\ N)$ be a fuzzy normed linear space satisfying (N6). If $\{x_n\}$ be an α-convergent sequence in $(X\ ,\ N)$. Then $||x_n - x||_\alpha \to 0$ as $n \to \infty$ ($||\ ||_\alpha$ denotes the α-norm of N).

Definition 2.6 [4] Let $(X\ ,\ N)$ be a fuzzy normed linear space. A subset F of X is said to be l-fuzzy closed if for

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each $\alpha \in (0 , 1)$ and for any sequence $\{x_n\}$ in F and $x \in X, (\underline{lim}_{n\to\infty} N(x_n - x , t) \geq \alpha \; \forall t > 0) \Rightarrow x \in F$.

Proposition 2.3 [4] Let (X , N) be a fuzzy normed linear space satisfying (N6) and $F \subset X$. Then F is l-fuzzy closed iff F is closed w.r.t. $|| \; ||_\alpha$ $(\alpha - norm \; of \; N)$ for each $\alpha \in (0 , 1)$.

Definition 2.7 [4] Let (X , N) be a fuzzy normed linear space. X is said to be l-fuzzy bounded if for any $\alpha \in (0, 1)$, $\exists \; t(\alpha)$ such that $N(x, t(\alpha)) \geq \alpha \;\; \forall \; x \in X$.

Proposition 2.4 [4] Let (X , N) be a fuzzy normed linear space satisfying (N6). Then X is l-fuzzy bounded iff X is bounded w.r.t. $|| \; ||_\alpha$, for all $\alpha \in (0 , 1)$, where $|| \; ||_\alpha$ denotes the α-norm of N

Definition 2.8 [3] Let (X, N) be a fuzzy normed linear space and $\alpha \in (0, 1)$. A sequence $\{x_n\}$ in X is said to be α-Cauchy if

$\underline{lim}_{n\to\infty} N(x_n - x_{n+p}, t) \geq \alpha \; \forall \; t > 0, \; p = 1, 2, ...$.

Definition 2.9 [3] Let (X, N) be a fuzzy normed linear space and $\alpha \in (0, 1)$. It is said to be α-complete if any α-Cauchy sequence in X, α-converges to a point in X.

Definition 2.10 [3] Let (X, N) be a fuzzy normed linear space. It is said to be l-fuzzy complete if it is α-complete for all $\alpha \in (0, 1)$.

Theorem 2.1 [9](Browder-Petryshyin). Let H be a Hilbert space and C be a closed, convex and bounded subset of H. If $f : C \to C$ is a nonexpansive mapping on C, then f has a fixed point in C.

Theorem 2.2 [9] (D. de Figueiredo) Let H be a Hilbert space and C be a closed, convex and bounded set in H containing θ. If $T : C \to C$ is any nonexpansive mapping, then for any x_0 in C the sequence $\{x_n\}$, with

$x_n = T_n^{n^2} x_{n-1}, \;\; n = 1, 2, 3,....$

and $T_n x = \frac{n}{n+1} T x$ converges strongly to a fixed point of T.

Theorem 2.3 [9] Let $f : C \to C$ be nonexpansive and demicompact on the closed, convex and bounded set C is a Hilbert space or strictly convex fuzzy normed linear space. Then the set of fixed points of f, $F(f)$ is nonempty and convex set. Also each $s \in (0, 1)$, the sequence $\{x_n\}$ where $x_n = sf(x_{n-1}) + (1 - s)x_{n-1}$ $n = 1, 2, 3, ...$ converges strongly to a fixed point of f.

Definition 2.11 [9] The space X is said to have the 'fixed point property' if for any continuous function $f : X \to X$, there exists $x \in X$ such that $f(x_0) = x_0$.

Theorem 2.4 [9] Let K is a closed and convex set in a real Hilbert space H. Then K has the fixed point property for nonexpansive mappings iff K is bounded.

Definition 2.12 [4] Let X and Y be two linear spaces over the same field of scalars. Let N_1 and N_2 be two fuzzy norms on X and Y respectively. A mapping $T : (X, N_1) \to (Y, N_2)$ is said to be sectional fuzzy continuous at $x_0 \in U$, if $\exists \; \alpha \in (0, 1)$ such that for each $\epsilon > 0, \exists \; \delta > 0$ such that

$N_1(x - x_0, \delta) \geq \alpha \Rightarrow N_2(T(x) - T(x_0), \epsilon) \geq \alpha \;\; \forall \; x \in X$.

If T is sectional fuzzy continuous at each point of X, then T is said to be sectional fuzzy continuous on X.

Theorem 2.5 [4] Let (X, N_1) and (Y, N_2) are fuzzy normed linear spaces satisfying (N6). Then a mapping $T : (X, N_1) \to (Y, N_2)$ is sectional fuzzy continuous iff $T : (X, || \; ||_\alpha^1) \to (Y, || \; ||_\alpha^2)$ is continuous for some $\alpha \in (0, 1)$.

Theorem 2.6 [14] Let (V, μ) be a fuzzy inner product space. Further assume that for $x, y \in V$

(CFI-7) $\mu(x, y, st) \geq \mu(x, x, s^2) \wedge \mu(y, y, t^2), \;\; \forall \, s, t \in R$ and $\forall \; x, y \in X$.

Define a function $N : X \times R \to [0, 1]$ by

$$N(cx, t) = \begin{cases} \mu(|c|x, |c|x, t^2) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then N is a fuzzy norm on X. We call this norm as induced norm of μ.

Theorem 2.7 [14] Let (V, μ) be a fuzzy inner product space. Further assume that for $x, y \in V$

(CFI-8) $-\infty < \{\wedge Rl \; t, \; \vee Rl \; t, \; \wedge Im \; t, \; \vee Im \; t : \mu(x, y, Rl \; t) \geq \alpha, \; \mu(x, y, Im \; t) \geq \alpha\} < +\infty$ for each $\alpha \in (0, 1)$ and

$\{\forall \; t > 0 : \mu(x, x, t) > 0\} \Rightarrow x = \theta\}$.

Also assume that

(CFI-9) $\mu(ix, y, Rl \; t) = \mu(x, y, Im\frac{t}{i}), \; \mu(ix, y, Im \; t) = \mu(x, y, Rl\frac{t}{i})$.

Define for $\alpha \in (0, 1)$,

$\langle x, y\rangle_\alpha = \{\wedge t : \mu(x, y, t) \geq \alpha\}$if $t \in R$

$\langle x, y\rangle_\alpha = \{\wedge Rl \; t + \vee Rl \; t + i \wedge Im \; t + i \vee Im \; t : \mu(x, y, Rl \; t) \geq \alpha, \; \mu(x, y, Im \; t) \geq \alpha\}$ otherwise.

Then $\{\langle \; , \; \rangle_\alpha : \alpha \in (0, 1)\}$ is a family of inner product in V. We call these inner products as α-inner products corresponding to the fuzzy inner product μ.

III. Some Observations on Induced Fuzzy Norm

In this section, we show that α-norms derived from induced fuzzy norm and from α-inner product are same.

Observation 3.1 Let (V, μ) be a fuzzy inner product space satisfying **(CFI-7)**, **(CFI-8)**, and **(CFI-9)** and N be its induced fuzzy norm. The α-norms derived from induced fuzzy norm and from α-inner products, $\alpha \in (0, 1)$ are same.

Proof. Since (V, μ) satisfy **(CFI-7)**, the induced fuzzy norm N of μ given by:

$$N(cx, t) = \begin{cases} \mu(|c|x, |c|x, t^2) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now **(CFI-8)** gives $\mu(x, x, t) > 0 \;\; \forall \; t > 0 \Rightarrow x = \underline{0}$

$\Rightarrow N(x, t) = \mu(x, x, t^2) > 0 \;\; \forall \; t > 0 \Rightarrow x = \underline{0}$.

i.e N satisfy (N6).

Therefore for $\alpha \in (0, 1)$

$||x||_\alpha = \wedge\{t > 0 : N(x, t) \geq \alpha\}$

$= \wedge\{t > 0 : \mu(x, x, t^2) \geq \alpha\}$

So $\{||x||_\alpha\}^2 = \wedge\{t^2 > 0 : \mu(x, x, t^2) \geq \alpha\}$

$= \wedge\{s > 0 : \mu(x, x, s) \geq \alpha\}$ (3.1.1).

On the other hand since μ satisfies **(CFI-9)**, we have

$\langle x, x\rangle_\alpha = \wedge\{t \in R : \mu(x, x, t) \geq \alpha\}$.

Now $\mu(x, x, t) = 0 \;\; \forall \; t \leq 0$

$\Rightarrow \langle x, x\rangle_\alpha = \wedge\{t > 0 : \mu(x, x, t) \geq \alpha\}$

$\Rightarrow \{||x||'_\alpha\}^2 = \wedge\{t > 0 : \mu(x, x, t) \geq \alpha\}$ (3.1.2).

From (3.1.1) and (3.1.2) we get

$\{||x||_\alpha\}^2 = \{||x||'_\alpha\}^2$

$\Rightarrow ||x||_\alpha = ||x||'_\alpha \qquad \forall \; \alpha \in (0, 1)$.

Observation 3.2. (CFI-7) is necessary but not sufficient.

Proof. Here we justify it by an example of a fuzzy inner product to show that it induces a fuzzy norm N without satisfying **(CFI-7)**.

Example 3.1. Consider the inner product space $(R^2, \langle\ ,\ \rangle)$ where $\langle\ ,\ \rangle$ is given by
$\langle x,\ y\rangle = a_1a_2 + b_1b_2$ where $x = (a_1,\ b_1)$, $y = (a_2,\ b_2)$, $a_1,\ b_1,\ a_2,\ b_2 \in R$
Consider the fuzzy inner product as
For $c = 0$, $\mu(cx,\ y,\ t) = H(t)$.
For $c \in R^-$,
$\mu(cx,\ y,\ t)$

$$= \begin{cases} 1 & \text{if } Rl\ t > Rl\langle cx, y\rangle\ or\ Im\ t \neq Im\langle cx, y\rangle \\ \frac{1}{2} & \text{if } Rl\ t = Rl\langle cx,\ y\rangle\ and\ Im\ t = Im\langle cx,\ y\rangle \\ 0 & \text{if } Rl\ t < Rl\langle cx,\ y\rangle\ and\ Im\ t = Im\langle cx,\ y\rangle. \end{cases}$$

For $c \in F - \{R^- \cup \{0\}\}$,
$\mu(cx,\ y,\ t)$

$$= \begin{cases} 1 & \text{if } Rl\frac{t}{c} > Rl\langle x,\ y\rangle\ and\ Im\frac{t}{c} = Im\langle x,\ y\rangle \\ \frac{1}{2} & \text{if } Rl\frac{t}{c} = Rl\langle x,\ y\rangle\ and\ Im\frac{t}{c} = Im\langle x,\ y\rangle \\ 0 & \text{if } Rl\frac{t}{c} < Rl\langle x,\ y\rangle\ or\ Im\frac{t}{c} \neq Im\langle x,\ y\rangle. \end{cases}$$

We have shown that $(R^2,\ \mu)$ is a fuzzy inner product space (Example 3.3[14]).
Now we show that μ does not satisfy **(CFI-7)**.

Proof. Take $s = -1$ and $t = 1$.
Then for $x = (1,\ 0)$, $y = (0,\ 1)$ we have
$Rl\ st = -1 < 0 = Rl\langle x,\ y\rangle$.
Therefore $\mu(x,\ y,\ st) = 0$.
Again $\langle x,\ x\rangle = 1$ and $\langle y,\ y\rangle = 1$.
So $s^2 = 1 = \langle x,\ x\rangle$ and $t^2 = 1 = \langle y,\ y\rangle$
$\Rightarrow \mu(x,\ x,\ s^2) = \frac{1}{2}$ and $\mu(y,\ y,\ t^2) = \frac{1}{2}$
$\Rightarrow min\{\mu(x,\ x,\ s^2),\ \mu(y,\ y,\ t^2) = \frac{1}{2}$.
Therefore $\mu(x,\ y,\ st) < min\{\mu(x,\ x,\ s^2),\ \mu(y,\ y,\ t^2)$.
Thus μ does not satisfy **(CFI-7)**.
Now define $N : R^2 \times R \to [0,\ 1]$ by

$$N(cx,\ t) = \begin{cases} \mu(|c|x,\ |c|x,\ t^2) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

We shall show that N is a fuzzy norm in R^2.
We have for $t > 0$

$$N(cx,\ t) = \begin{cases} 1 & \text{if } Rl\ t^2 > Rl|c|^2||x||^2 \\ \frac{1}{2} & \text{if } Rl\ t^2 = Rl|c|^2||x||^2 \\ 0 & \text{if } Rl\ t^2 < Rl|c|^2||x||^2 \end{cases}$$

and for $t \leq 0$, $N(cx,\ t) = 0$.
i.e. for $t > 0$

$$N(cx,\ t) = \begin{cases} 1 & \text{if } Rl\ t > Rl|c|||x|| \\ \frac{1}{2} & \text{if } Rl\ t = Rl|c|||x|| \\ 0 & \text{if } Rl\ t < Rl|c|||x|| \end{cases}$$

and for $t \leq 0$, $N(cx,\ t) = 0$.
(N1) By definition for $t \leq 0$, $N(x,\ t) = 0$.
(N2) $N(x,\ t) = 1\ \ \forall\ t > 0$
$\Leftrightarrow t > ||x||\ \ \forall\ t > 0$
$\Leftrightarrow ||x|| = 0$
$\Leftrightarrow x = \theta$.
(N3) For $t \leq 0$, $N(cx,\ t) = 0 = N(x,\ \frac{t}{|c|})$ and for $t > 0$,
$N(cx,\ t) = 1$
$\Rightarrow Rl\ t > Rl|c|||x||$
$\Rightarrow Rl\frac{t}{|c|} > Rl||x||$
$\Rightarrow N(x,\ \frac{t}{|c|}) = 1$.
Similarly if $N(cx,\ t) = \frac{1}{2}$ then $N(x,\ \frac{t}{|c|}) = \frac{1}{2}$ and $N(cx,\ t) = 0$ then $N(x,\ \frac{t}{|c|}) = 0$.
Thus $N(cx,\ t) = N(x,\ \frac{t}{|c|})$.
(N4) For $s + t \leq 0$ there is nothing to prove.
Let $s + t > 0$.
$N(x + y,\ s + t) = 1$ implies $N(x + y,\ s + t) \geq N(x,\ s) \wedge N(y,\ t)$.
$N(x + y,\ s + t) = \frac{1}{2}$
$\Rightarrow s + t = ||x + y|| \leq ||x|| + ||y||$.
Then $N(x,\ s) \wedge N(y,\ t) = 0$ or $\frac{1}{2}$.
$\Rightarrow N(x + y,\ s + t) \geq N(x,\ s) \wedge N(y,\ t)$.
Similarly for $N(x + y,\ s + t) = 0$ we get
$N(x + y,\ s + t) \geq N(x,\ s) \wedge N(y,\ t)$.
(N5) This follows from $N(x,\ t) = 1$ for $t > ||x||$.

IV. Some Fixed Point Theorems

In this Section Browder-Petryshyin and D.de Figueiredo theorem have been studied in fuzzy setting. Some other fixed point theorems also establish in this Section.

Definition 4.1. Let $(X,\ \mu)$ be a fuzzy inner product space satisfying **(CFI-7)** and N be its induced fuzzy norm.
Then (i) $(H,\ \mu)$ is said to be fuzzy Hilbert space if $(H,\ N)$ is complete fuzzy normed linear space.
(ii) $(H,\ \mu)$ is said to be α-complete fuzzy inner product space if $(H,\ N)$ is α-complete fuzzy normed linear space.
(iii) $C \subset X$ is said to be l-fuzzy closed if C is l-fuzzy closed w.r.t. N norm.
(iv) $C \subset X$ is said to be bounded if C is bounded w.r.t. N norm.
(v) $C \subset X$ is said to be l-fuzzy bounded if C is l-fuzzy bounded w.r.t. N norm.
(vi) $(H,\ \mu)$ is said to be l-fuzzy complete inner product space if $(H,\ N)$ is l-fuzzy complete fuzzy normed linear space.

Remark 4.1. If $(X,\ \mu)$ is a fuzzy inner product space satisfying **(CFI-7)** and **(CFI-8)**, then induced fuzzy norm N of μ satisfy **(N6)**.

Proof. Since $(X,\ \mu)$ is a fuzzy inner product space satisfying **(CFI-7)**, then we get induced norm N as

$$N(cx,\ t) = \begin{cases} \mu(|c|x,\ |c|x,\ t^2) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

From above it follows that $\forall\ t > 0,\ \mu(x,\ x,\ t) > 0 \Leftrightarrow N(x,\ t) > 0$.
Now by **(CFI-8)**,
$\mu(x,\ x,\ t) > 0\ \ \forall\ t > 0\ \ \Rightarrow x = \theta$
i.e. $N(x,\ t) > 0\ \ \forall\ t > 0\ \ \Rightarrow x = \theta$
i.e. N satisfy **(N6)**.

Theorem 4.1. (Browder-Petryshyin). Let $(H,\ \mu)$ be an l-fuzzy complete inner product space satisfying **(CFI-7), (CFI-8), (CFI-9)** and C be an l-fuzzy closed, convex and bounded set in H. If $f : C \to C$ is a fuzzy nonexpansive mapping on C, then f has a fixed point in C.

Proof. Since $(H,\ \mu)$ is l-fuzzy complete inner product space satisfying **(CFI-7), (CFI-8), (CFI-9)** then for each $\alpha \in (0,\ 1)$, $(H,\ \langle\ ,\ \rangle_\alpha)$ are Hilbert space. Since C is l-fuzzy closed then it is closed in $(H,\ \langle\ ,\ \rangle_\alpha)$, $\alpha \in (0,\ 1)$ and f is fuzzy nonexpansive implies f is nonexpansive in $(H,\ \langle\ ,\ \rangle_\alpha)$, $\alpha \in (0,\ 1)$.

Now C is bounded

$\Rightarrow\ \exists\ r$ and t such that $N(x,\ t) > 1-r\ \ \forall\ x \in H$

$\Rightarrow ||x||_{1-r} \leq t\ \ \forall\ x \in H$

$\Rightarrow ||x||_{\alpha_0} \leq t$ (take $1-r = \alpha_0$).

Therefore C is bounded w.r.t α_0-fuzzy norm and hence C is bounded in $(H,\ \langle\ ,\ \rangle_{\alpha_0})$.

So C is closed, convex, bounded subset in the Hilbert space $(H,\ \langle\ ,\ \rangle_{\alpha_0})$ in which f is nonexpansive.

Thus we have by Theorem 2.2, f has a fixed point in C.

Theorem 4.2. Let $(H,\ \mu)$ be an l-fuzzy complete inner product space satisfying **(CFI-7), (CFI-8), (CFI-9)** and C be an l-fuzzy closed, convex and bounded set in H. Let $\mu(x,\ x,\ .)$ is upper semicontinuous for each $x \in H$. If $f : C \to C$ is fuzzy nonexpansive then $F(f_s)$, the set of fixed point of f_s is nonempty, where $f_s(x) = sx + (1-s)f(x);\ s \in (0,\ 1)$. Moreover f_s have the same fixed points as of f.

Proof. Since $x,\ f(x) \in C$ and C is convex set

for $s \in (0,\ 1)$, $f_s(x) = sx + (1-s)f(x) \in C$.

Thus $f_s : C \to C$. Now for each $\alpha \in (0,\ 1)$ we have

$||f_s(x_2) - f_s(x_1)||_\alpha = ||sx_2 + (1-s)f(x_2) - sx_1 - (1-s)f(x_1)||_\alpha$

$\leq s||x_2 - x_1||_\alpha + (1-s)||f(x_2) - f(x_1)||_\alpha$

$\leq s||x_2 - x_1||_\alpha + (1-s)||x_2 - x_1||_\alpha$ (since f is nonexpansive)

$= ||x_2 - x_1||_\alpha$.

Thus $f_s :\ s \in (0,\ 1)$ is nonexpansive w.r.t. each α-norm.

Since $N(x,\ .) = \mu(x,\ x,\ .)$, $x \in H$ is upper semicontinuous, it follows that f_s is fuzzy nonexpansive (by Remark 2.1).

Therefore by Theorem 2.2, we have f_s have a fixed point for each $s \in (0,\ 1)$ and hence

$F(f_s);\ s \in (0,\ 1)$ is nonempty.

Let x_0 be a fixed point of f.

Therefore $f_s(x_0) = sx_0 + (1-s)f(x_0) = sx_0 + (1-s)x_0 = x_0$.

Thus x_0 is also a fixed point of $f_s,\ s \in (0,\ 1)$.

Definition 4.2. Let $(H,\ \mu)$ be an l-fuzzy complete inner product space satisfying **(CFI-7)**. The space H is said to have fixed point property(f.p.p.) if for any sectional fuzzy continuous function $f : H \to H$, there exist $x_0 \in X$ such that $f(x_0) = x_0$.

Theorem 4.3. Let K be an l-fuzzy closed and convex set in a real l-fuzzy complete inner product space satisfying **(CFI-7), (CFI-8)**. Then K is fuzzy bounded if K has f.p.p. for fuzzy nonexpansive mapping.

Proof. Since $(H,\ \mu)$ is real l-fuzzy complete inner product space satisfying **(CFI-7), (CFI-8)** then for each $\alpha \in (0,\ 1)$, $(H,\ \langle\ ,\ \rangle_\alpha)$ are real Hilbert spaces. Since K is l-fuzzy closed it is closed w.r.t. each α-norm.

Now K has f.p.p.

$\Rightarrow$ for any sectional fuzzy continuous function $f : K \to K$, $\exists\ x_0 \in K$ such that $f(x_0) = x_0$.

$\Rightarrow \exists\ \alpha_0 \in (0,\ 1)$ such that $f : K \to K$ is continuous w.r.t. $||\ ||_{\alpha_0}$ (by Theorem 2.6) and $f(x_0) = x_0$.

Since f is a fuzzy nonexpansive f is nonexpansive w.r.t. every α-norm, for each $\alpha \in (0,\ 1)$.

Then by Theorem 2.5, K is bounded w.r.t. α_0-norm.

So $||x||_{\alpha_0} \leq M\ \ \forall\ x \in K$

$\Rightarrow N(x,\ M) \geq \alpha_0\ \ \forall\ x \in K$

$\Rightarrow N(x,\ M) > \frac{\alpha_0}{2}$.

Take $r = 1 - \frac{\alpha_0}{2}$. Then

$N(x,\ M) > 1-r$

Thus K is fuzzy bounded.

Theorem 4.4. (D. de Figueiredo). Let $(H,\ \mu)$ be an l-fuzzy complete inner product space satisfying **(CFI-7), (CFI-8), (CFI-9)** and C be an l-fuzzy closed, convex and fuzzy bounded set in H containing θ. If $T : C \to C$ is a fuzzy nonexpansive mapping, then for any x_0 in C the sequence $\{x_n\}$, with

$x_n = T_n^{n^2} x_{n-1}$, $n = 1,\ 2,\ 3, ...$ and $T_n x = \frac{n}{n+1}Tx$,

$\exists\ \alpha_0 \in (0,\ 1)$ and $\exists\ x' \in F(T)$(set of fixed point of T)

such that for any $t > 0$, $\exists\ M(t)$ such that

$\mu(x_n - x',\ x_n - x',\ t) \geq \alpha_0\ \ \forall\ n \geq M(t)$.

Proof. Since $(H,\ \mu)$ is l-fuzzy complete inner product space satisfying **(CFI-7), (CFI-8), (CFI-9)** then for each $\alpha \in (0,\ 1)$, $(H,\ \langle\ ,\ \rangle_\alpha)$ are Hilbert spaces. Since C is l-fuzzy closed so it is closed in $(H,\ \langle\ ,\ \rangle_\alpha)$, $\alpha \in (0,\ 1)$ and T is fuzzy nonexpansive implies f is nonexpansive in $(H,\ \langle\ ,\ \rangle_\alpha)$, $\alpha \in (0,\ 1)$.

Now C is fuzzy bounded

$\Rightarrow\ \exists\ r$ and t such that $N(x,\ t) > 1-r\ \ \forall\ x \in H$

$\Rightarrow ||x||_{1-r} \leq t\ \ \forall\ x \in H$

$\Rightarrow ||x||_{\alpha_0} \leq t$ (take $1-r = \alpha_0$).

Therefore C is bounded w.r.t α_0-fuzzy norm and hence C is bounded in $(H,\ \langle\ ,\ \rangle_{\alpha_0})$.

Then by Theorem 2.3, we have

$\{x_n\}$ converges to a fixed point of T w.r.t. α_0-inner product.

Let $x' \in F(T)$ where $\{x_n\}$ converges w.r.t. α_0-inner product

$\Rightarrow ||x_n - x'||_{\alpha_0} \to 0$ as $n \to \infty$.

$\Rightarrow ||x_n - x'||^2_{\alpha_0} \to 0$ as $n \to \infty$.

$\Rightarrow \langle x_n - x',\ x_n - x' \rangle_{\alpha_0} \to 0$ as $n \to \infty$.

Thus for each $t > 0$, $\exists\ M(t)$ such that

$\langle x_n - x',\ x_n - x' \rangle_{\alpha_0} < t\ \ \forall\ n \geq M(t)$

$\Rightarrow \mu(x_n - x',\ x_n - x',\ t) \geq \alpha_0\alpha_0\ \ \forall\ n \geq M(t)$.

Definition 4.3. Let $(X,\ N)$ be an l-fuzzy complete normed linear space and C be an l-fuzzy closed convex subset of X. Then $f : C \to X$ is said to be fuzzy demicompact if it has the property that whenever $\{x_n\}$ is a fuzzy bounded sequence in C and there is $\alpha \in (0,\ 1)$, for which for any $t > 0\ \exists\ M(t)$ such that $N(f(x_n) - (x_n),\ t) \geq \alpha\ \forall\ n \geq M(t)$, then $\exists$ a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is convergent.

Proposition 4.1. Let $(X,\ N)$ be an l-fuzzy complete normed linear space satisfying (N6). Let C be an l-fuzzy closed, convex subset of X and $f : C \to X$ be fuzzy demicompact. Then f is demicompact w.r.t. $||\ ||_\alpha\ \forall\ \alpha \in (0,\ 1)$ where $||\ ||_\alpha$ is an α-norm of N.

Proof. Since $(X,\ N)$ be an l-fuzzy complete normed linear space satisfying (N6), so X is complete w.r.t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$. Again since C is l-fuzzy closed thus C is closed w.r.t. $||\ ||_\alpha$ for each $\alpha \in (0,\ 1)$.

Let $\{x_n\}$ be a bounded sequence in C and $\{f(x_n) - (x_n)\}$ is

strongly convergent w.r.t. $|| \ ||_{\alpha_0}$ for $\alpha_0 \in (0,\ 1)$.
Thus $\exists\ M$ such that $||x_n||_{\alpha_0} < M \ \ \forall\ n$
$\Rightarrow N(x_n,\ M) \geq \alpha_0 \ \ \forall\ n$
$\Rightarrow N(x_n,\ M) \geq \alpha_0 > 1 - \beta_0 \ \ \forall\ n$
$\Rightarrow \{x_n\}$ fuzzy bounded.
Again suppose $x \in X$ such that $\lim_{n\to\infty} ||f(x_n) - x_n - x||_{\alpha_0} = 0$
$\Rightarrow$ for each $t > 0$, $\exists\ K(t)$ such that $||f(x_n) - x_n - x||_{\alpha_0} < t$
$\Rightarrow N(f(x_n) - x_n - x,\ t) \geq \alpha_0 \ \ \forall\ n \geq K(t)$.
Since f is fuzzy demicompact, it follows that $\exists$ a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is convergent.
i.e. $\{x_{n_i}\}$ is convergent w.r.t. $|| \ ||_{\alpha_0}$
Hence f is demicompact w.r.t. $|| \ ||_{\alpha_0}$. Since $\alpha_0 \in (0,\ 1)$ is arbitrary, f is demicompact w.r.t. each α-norm, $\alpha \in (0,\ 1)$.

Theorem 4.5. Let $f : C \to C$ be fuzzy nonexpansive and fuzzy demicompact on the l-fuzzy closed, convex and l-fuzzy bounded set C in an l-fuzzy complete inner product space $(H,\ \mu)$ satisfying **(CFI-7), (CFI-8), (CFI-9)** or strong strictly convex fuzzy normed linear space $(H,\ N)$ satisfying(N6). Then $F(f)$ is a nonempty, and convex set. Also each $s \in (0,\ 1)$, the sequence $\{x_n\}$ where $x_n = sf(x_{n-1}) + (1-s)x_{n-1}\ n = 1,\ 2,\ 3, ...$ converges strongly to a fixed point of f.

Proof. Since $(H,\ \mu)$ is l-complete fuzzy inner product space satisfying **(CFI-7), (CFI-8), (CFI-9)** then for each $\alpha \in (0,\ 1),\ \ (H,\ \langle\ ,\ \rangle_\alpha)$ are Hilbert space. Strong strictly convex satisfying(N6) implies it is strictly convex w.r.t. each α-norm. Since f is fuzzy nonexpansive it is expansive w.r.t. each α-norm. f is fuzzy demicompact implies it is demicompact w.r.t. each α-norm. C is l-fuzzy closed implies it is closed w.r.t. each α-norm. C is l-fuzzy bounded implies it is bounded w.r.t. each α-norm.
So C is closed, convex, bounded subset in the Hilbert space $(H,\ \langle\ ,\ \rangle_\alpha)$ in which f is nonexpansive and demicompact, $\forall\ \alpha \in (0,\ 1)$.
Thus we have by Theorem 2.4, $F(f)$ is nonempty, convex set. Also each $s \in (0,\ 1)$, the sequence $\{x_n\}$ where $x_n = sf(x_{n-1}) + (1-s)x_{n-1},\ n = 1,\ 2,\ 3, ...$ converges strongly to a fixed point of f w.r.t. $|| \ ||_\alpha\ \forall\ \alpha \in (0,\ 1)$ and hence $\{x_n\}$ is convergent w.r.t. N-norm (norm of strong strictly convex space or induced fuzzy norm from μ).

V. Conclusions

In recent past, numerous papers have been published in fuzzy functional analysis. But only a few works have been done on fuzzy inner product spaces. In this paper some properties of fuzzy inner product spaces are studied and some fixed point theorems are established in such spaces. There is a wide scope of research in this field and we hope that researchers will be benefitted through our work. Applications of fuzzy inner product spaces in real-world problems may be studied as a future scope of research.

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