Dominator Coloring of Some Degree Splitting Graphs

Samir K. Vaidya and Minal S. Shukla

***Abstract*—A dominator coloring of a graph is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The minimum number of colors required for dominator coloring of G is called the dominator chromatic number of G and is denoted by $\chi_d(G)$. We investigate some results on dominator coloring in the context of degree splitting graph of P_n and S_n.**

***Index Terms*—Coloring, domination number, dominator coloring, degree splitting graph.**

***MSC 2010 Codes* – 05C15, 05C38, 05C76, 05C69**

I. Introduction

We begin with simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. A coloring of vertices of G is a mapping $f : V(G) \to \mathbb{N}$. For every vertex v the integer $f(v)$ is called the color of v. A coloring is proper if any two adjacent vertices have different colors. The chromatic number $\chi(G)$ of graph G is the smallest integer k such that G admits a proper coloring using k colors. The set of vertices with particular color is called color class.

The open neighbourhood of $v \in V(G)$ is the set $N(v) = \{u \in V(G)/uv \in E(G)\}$ while the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V(G)$ is called dominating set if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S. A dominating set S is a minimal dominating set if no proper subset S' of S is a dominating set. The domination number $\gamma(G)$ is the minimum cardinality of minimal dominating set of G.

A dominator coloring of a graph G is a proper coloring of graph such that every vertex of $V(G)$ dominates all the vertices of at least one color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of G. This concept was introduce by Gera *et al.* [1].

The concept of dominator coloring has been extensively studied by Kavitha and David [2–4]. The relation between dominator coloring and domination number of some classes of graphs were studied by Arumugam *et al.* [5] while dominator coloring in bipartite graphs is discussed by Gera [6]. The dominator chromatic number of some cycle related graphs have investigated by Vaidya and Shukla [7]

Dr. Samir K. Vaidya is a Professor at the Department of Mathematics, Saurashtra University, Rajkot, Gujarat-360005, India. (E-mail: samirkvaidya@yahoo.co.in)

Minal S. Shukla is a Assistant Professor at the Department of Mathematics, Atmiya Institute of Technology and Science, Rajkot, Gujarat-360005, India. (E-mail: shuklaminal19@gmail.com).

Definition 1.1 A *shell* S_n is the graph obtained by taking $n-3$ concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex. That is, $S_n = P_{n-1} + K_1$.

Proposition 1.2 [8] For complete graph K_n, $\chi_d(K_n) = n$.

Proposition 1.3 [9]

$$\chi[DS(P_n)] = \begin{cases} 2, & n = 3 \\ 3, & n \neq 3 \end{cases}$$

Proposition 1.4 [10]

$$\gamma[DS(P_n)] = 2,\ n \geqslant 3$$

Proposition 1.5 [9]

$$\chi[DS(S_n)] = \begin{cases} 4, & n = 3 \\ 3, & n \neq 3 \end{cases}$$

Proposition 1.6 [11] For any graph G, $\chi(G) \geqslant 3$ if and only if G has an odd cycle.

Proposition 1.7 [5] For the cycle C_n,

$$\chi_d[C_n] = \begin{cases} \lceil \frac{n}{3} \rceil & if \quad n = 4 \\ \lceil \frac{n}{3} \rceil + 1 & if \quad n = 5 \\ \lceil \frac{n}{3} \rceil + 2 & if \quad otherwise \end{cases}$$

II. Main Results

Definition 2.1 Let G be a graph with $V(G) = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of all the vertices of the same degree with at least two elements $T = V(G) \setminus \bigcup_{i=1}^{t} S_i$. The *degree splitting* graph of G, denoted by $DS(G)$, is obtained from G by adding vertices $w_1,\ w_2, \ldots, w_t$ and joining w_i to each vertex of S_i for $1 \leqslant i \leqslant t$.

Theorem 2.2

$$\chi_d[DS(P_n)] = 2,\ n = 3$$

Proof: The path P_3 has two pendant vertices and a vertex of degree two. Thus $V(P_3) = \{v_i :\ 1 \leqslant i \leqslant 3\} = S_1 \cup S_2$ where $S_1 = \{v_1,\ v_3\}$ and $S_2 = \{v_2\}$. For obtaining $DS(P_3)$ from P_3 add one vertex w_1 corresponding to S_1. Thus $V[DS(P_n)] = V(P_n) \cup \{w_1\}$ and $E[DS(P_n)] = E(P_n) \cup \{w_1 v_i\ where\ v_i \in S_1,\ i = 1, 3\}$.

The graph $DS\ (P_3)$ is isomorphic to C_4. Then according to Proposition 1.7, $\chi_d\ [C_4] = 2 \Rightarrow \chi_d\ [DS(P_3)] = 2$.

Theorem 2.3 For $n \geqslant 4$

$$\chi_d[DS(P_n)] = \chi[DS(P_n)] + \gamma[DS(P_n)] - 1$$

Proof: The path P_n has two pendant vertices and the remaining $n-2$ vertices are of degree two. Thus $V(P_n) = \{v_i;\ 1 \leqslant i \leqslant n\} = S_1 \cup S_2$ where $S_1 = \{v_1,\ v_n\}$ and $S_2 = \{v_i;\ 2 \leqslant i \leqslant n-1\}$. For obtaining $DS(P_n)$ from P_n add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V[DS(P_n)] = V(P_n) \cup \{w_1,\ w_2\}$ and $E[DS(P_n)] = E(P_n) \cup \{w_1 v_i \ where\ v_i \in S_1;\ i = 1, n\} \cup \{w_2 v_j \ where\ v_j \in S_2;\ 2 \leqslant j \leqslant n-1\}$. $|V(DS(P_n))| = n+2$ and $|E(DS(P_n))| = 2n-1$.

We color the vertices which are in γ set by using $\gamma[DS(P_n)]$ number of colors. Next we assign the colors to the remaining vertices using $\chi[DS(P_n)] - 1$ number of colors.

This proper coloring pattern give rise to a dominator coloring for the respective graphs. Hence $\chi_d[DS(P_n)] = \chi[DS(P_n)] + \gamma[DS(P_n)] - 1$.

Theorem 2.4

$$\chi_d[S_n] = \chi[S_n],\ \ n \geqslant 3$$

Proof: Let $V(S_n) = \{v,\ v_1,\ v_2, \ldots, v_{n-1}\}$ where v is the apex vertex and $E(S_n) = \{vv_i; 1 \leqslant i \leqslant n-1\} \cup \{v_i v_{i+1}; 1 \leqslant i \leqslant n-2\}$. Clearly $|V(S_n)| = n$ and $|E(S_n)| = 2n-3$. As each S_n contains at least an odd cycle C_3, then according to Proposition 1.6, $\chi(S_n) \geqslant 3$. If we assign the proper coloring as $f(v) = 1,\ f(v_{2k-1}) = 3,\ f(v_{2k}) = 2$ for $k \in \mathrm{N}$ then $\chi(S_n) = 3$. As each vertex dominates all the vertices of at least one color class. Therefore this coloring is also a dominator coloring. Hence $\chi_d[S_n] = \chi[S_n]$.

Lemma 2.5

$$\gamma[DS(S_n)] = \begin{cases} 1, & n = 3 \\ 2, & n = 4,\ 5,\ 6 \\ 3, & n \geqslant 7 \end{cases}$$

Proof: Let S_n be the graph with vertex set $\{v, v_i / 1 \leqslant i \leqslant n\}$, where v is the apex vertex and $E(S_n) = \{vv_i; 1 \leqslant i \leqslant n-1\} \cup \{v_i v_{i+1}; 1 \leqslant i \leqslant n-2\}$. There are three types of vertices in S_n

(i) vertices of degree 2, namely $v_1,\ v_n$.

(ii) vertices of degree 3, namely $v_2,\ v_3, \ldots\ v_{n-1}$.

(iii) a vertex of degree $n-1$, namely v.

Thus $V(S_n) = S_1 \cup S_2 \cup T$, where $S_1 = \{v_1,\ v_n\}$, $S_2 = \{v_2,\ v_3,\ \ldots, v_{n-1}\}$ and $T = \{v\}$. Now in order to obtain $DS(S_n)$ from S_n, we add $\{w_1, w_2\}$ corresponding to S_1, S_2. Then $V[DS(S_n)] = V(S_n) \cup \{w_1, w_2\}$ and $E[DS(S_n)] = E(S_n) \cup \{w_1 v_1,\ w_1 v_n, w_2 v_i / 2 \leqslant i \leqslant n-1\}$. So $E[DS(S_n)] = E(S_n) \cup \{w_1 v_1,\ \ w_1 v_n, w_2 v_i / 2 \leqslant i \leqslant n-1\}$. So $|V[DS(S_n)]| = n+2$.

case-1: For $n = 3$

Obviously $\gamma[DS(S_3)] = 1$ as $DS(S_3) = K_4$.

case-2: For $n = 4, 5, 6$

The graph $DS(S_4)$ has $\Delta[DS(S_4)] = 4$. Therefore $\gamma[DS(S_4)] \geqslant 2$. Consider $S_1 = \{v,\ w_2\}$ with $|S_1| = 2$. Here $N[S_1] = V[DS(S_4)]$. Thus S_1 is a dominating set of $DS(S_4)$. Hence $\gamma[DS(S_4)] = 2$.

For the graph $DS(S_5)$, $\Delta[DS(S_5)] = 4$. Therefore $\gamma[DS(S_5)] \geqslant 2$. Consider $S_1 = \{v_1,\ v_3\}$ with $|S_1| = 2$. Also $N[S_1] = V[DS(S_5)]$. Then S_1 is a dominating set of $DS(S_5)$. Hence $\gamma[DS(S_5)] = 2$.

In the graph $DS(S_6)$, $\Delta[DS(S_6)] = 5$. Then $\gamma[DS(S_6)] \geqslant 2$. Consider $S_1 = \{v_2,\ v_5\}$ with $|S_1| = 2$. Since $N[S_1] = V[DS(S_6)]$. Then S_1 is a dominating set of $DS(S_6)$. Hence $\gamma[DS(S_6)] = 2$.

case-3: For $n \geqslant 7$

None of the vertices of the graph $DS(S_n)$ is of degree $n+1$. This implies that $\gamma[DS(S_n)] \geqslant 2$. As $N[w_1] = \{w_1, v_2, \ldots\ v_{n-1}\}$, $N[w_2] = \{w_2, v_1, v_n\}$ and $N[v] = \{v,\ v_1, \ldots\ v_n\}$. Consider the dominating set $S = \{v,\ w_1,\ w_2\}$. This implies that $N[S] = V[DS(S_n)]$. Therefore S is dominating set with $|S| = 3$. Hence $\gamma[DS(S_n)] = 3$.

Theorem 2.6.

$$\chi_d[DS(S_n)] = \begin{cases} 4, & n = 3 \\ \chi[DS(S_n)] + \gamma[DS(S_n)] - 1, & n = 4,\ 5 \\ \chi[DS(S_n)] + \gamma[DS(S_n)], & n = 6 \end{cases}$$

Proof: We continue with the terminology and notations used in Lemma 2.5.

The graph $DS(S_3)$ is isomorphic to K_4, by Proposition 1.2, $\chi_d[DS(S_3)] = 4$.

For $n = 4, 5$

We color the vertices which are in γ set by using $\gamma[DS(S_n)]$ number of colors. Next we assign the colors to the vertices of $N(v)$ using $\chi[DS(S_n)] - 1$ number of colors. As each vertex dominates all the vertices of at least one color class. Therefore this proper coloring procedure give rise to a dominator coloring for the respective graphs. Thus $\chi_d[DS(S_n)] = \chi[DS(S_n)] + \gamma[DS(S_n)] - 1$.

For $n = 6$, we color the vertices which are in γ set by using $\gamma[DS(S_6)]$ number of colors. Next we assign the colors to the remaining vertices using $\chi[DS(S_6)]$ number of colors. Thus $\chi_d[DS(S_6)] = \chi[DS(S_6)] + \gamma[DS(S_6)]$.

Theorem 2.7.

$$\chi_d[DS(S_n)] = \chi[DS(S_n)] + \gamma[DS(S_n)],\ n \geqslant 7$$

Proof: We continue with the terminology and notations used in Lemma 2.5.

We color the vertices which are in γ set by using $\gamma[DS(S_n)]$ number of colors. Next we assign the colors to the vertices $N(v)$ using $\chi[DS(S_n)]$ number of colors. As each vertex dominates all the vertices of at least one color class. Therefore this proper coloring pattern give rise to a dominator coloring for the respective graphs. Hence $\chi_d[DS(S_n)] = \gamma[DS(S_n)] + \chi[DS(S_n)]$.

III. Conclusion

Domination number for degree splitting graph of some standard graphs is reported in Basvanagoud *et al.* [9] while we investigate dominator coloring for degree splitting graphs of path and shell. To investigate dominator coloring for various graph families is an open area of research.

Acknowledgement

Our thanks are due to the anonymous referee for careful reading of the paper.

REFERENCES

[1] R. Gera, S. Horton and C. Rasmussen, “Dominator Colorings and Safe Clique Partitions”, *Congressus Numerantium*, vol. 181, pp. 19-32, 2006.

[2] K. Kavitha and N. G. David, “Dominator Coloring of Some Classes of Graphs”, *International Journal of Mathematical Archive*, vol. 3, no. 11, pp. 3954-3957, 2012.

[3] K. Kavitha and N. G. David, “Dominator Coloring on Star and Double star Graph Families”, *International Journal of Computer Applications*, vol. 48, no. 3, pp. 22-25, 2012.

[4] K. Kavitha and N. G. David, “Dominator Coloring of Central Graphs”, *International Journal of Computer Applications*, vol. 51, no. 12, pp. 11-14, 2012.

[5] S. Arumugam, J. Bagga and K Raja Chandrasekar, “On Dominator Coloring in Graphs”, *Proc. Indian Acad Sci. (Math. Sci.)*, vol. 122, no. 4, pp. 561-571, 2012.

[6] R. Gera, “On Dominator Coloring in Bipartite Graphs”, *IEEE Computer Society*, pp. 1-6, 2007.

[7] S. K. Vaidya and M. S. Shukla, “Dominator coloring of some cycle related graphs”, Communicated, 2015.

[8] R. Gera, “On Dominator Coloring in Graphs”, *Graph Theory Notes of New York LII*, pp. 25-30, 2007.

[9] S. K. Vaidya and Rakhimol V. Isaac, “The b-chromatic number of some degree splitting Graphs”, *Malaya Journal of Matematik* , vol. 2, no. 3, pp. 249-253, 2014.

[10] B. Basvanagoud, Prashant V. Patil and Sunilkumar M. Hosamani, “Domination in degree splitting graphs”, *Journal of Analysis and Computation*, vol. 8, no. 1, pp. 1-8, 2012.

[11] J. A. Clark and D. A. Holton, *A First Look at Graph Theory*, World Scientific, 1969.