Reducibility Number in Some Classes of Finite Lattices and Posets

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***Abstract*—In this paper we characterize deletable elements of modular lattices with respect to upper/lower semimodular lattices such that the obtain lattice is not modular, we characterize deletable elements of distributive lattices with respect to pseudocomplemented lattices such that the obtain lattice is not distributive and we characterize deletable elements in complemented posets/lattices also we study reducibility number in some classes of finite posets/lattices.**

***Index Terms*—Lattice, Poset, Deletable, Reducibility number.**

***MSC 2010 Codes* – 06A06, 06C15, 06C10, 06D15**

I. Introduction

ALL the posets/lattices considered here are finite. The notion of reducibility in finite lattices is studied by Bordalo and Monjardet [1]. This notion for finite posets is studied by Kharat and Waphare [2]. The notion of reducibility number is introduced and studied by Kharat, Waphare and Thakare [3]. An element x of a (finite) poset/lattice P satisfying certain properties is *deletable* if $(P-x)$ is a poset/lattice satisfying the same properties of P. A class of posets/lattices is *reducible* if each poset/lattice of this class admits (at least) one deletable element. Equivalently, a class of posets/lattices is *reducible* if one can go from any poset/lattice in this class to the trivial lattice by a sequence of posets/lattices of the class obtained by deleting one element in each step. This notion, however, is different from the notion of dismantlability for lattices; see [4]. It is known that some classes of posets/lattices are not reducible; see [1] and [2]. A class of posets/lattices is not reducible means there is a member of the class which does not have any deletable element. Let $\mathcal{P}$ be a class of posets and $P \in \mathcal{P}$. We say that a non-empty subset S of P is *deletable* if $P-S \in \mathcal{P}$. A positive number r is called reducibility number of P with respect to the class $\mathcal{P}$, denoted by red $(P,\mathcal{P})$, if there exists a deletable subset S of P with $|S|=r$ and no non-empty subset T of P with $|T|<r$ is deletable [3].

In section III we characterize deletable elements of modular lattices with respect to upper/lower semimodular lattices such that the obtain lattice is not modular lattice, we characterize deletable elements of distributive lattices with respect to pseudocomplemented lattices such that the obtain lattice is not distributive and also we characterize deletable elements in complemented lattices and posets. In section IV we study reducibility number in some classes of finite lattices and posets.

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II. Some Preliminary Results

In this section, some definitions and preliminary results are given which will be used in this paper.
For elements a,b of a poset/lattice P, we say that a is covered by b (or b covers a) if $a<b$ and there is no c such that $a<c<b$ and this fact is denoted by $a \prec b$ [5]. For $x \in P, \{x\}^- = \{y \in P : y \prec x\}$ and $\{x\}^+ = \{z \in P : x \prec z\}$.

An element x of a poset/lattice is *join-irreducible* (respectively *meet-irreducible*) if x covers (*respectively is covered by*) a unique element. J (*respectively* M) shall denote the set of all join-irreducible elements (*respectively set of all meet-irreducible elements*) of given poset/lattice. The set $J \cap M$ is the set of all doubly irreducible elements of given poset/lattice and the set $J \cup M$ is the set of all irreducible elements of given poset/lattice [6].

It is obvious that if $x \in J$ then, $\{x\}^-$ is a singleton set and we denote it by x^-. Also if $x \in M$ then, $\{x\}^+$ is a singleton set and we denote it by x^+.

An element j of a lattice L is join-prime if $j \leq a \vee b$ implies $j \leq a$ or $j \leq b$; dually an element m of L is meet-prime if $m \geq a \wedge b$ implies $m \geq a$ or $m \geq b$. It is well known that such elements are - respectively - join-irreducible and meet-irreducible.

Theorem 2.1. [1] Let x be an element of an upper semimodular lattice. Then x is *USM-deletable* if and only if either $x \in J \cap M$ or $x \in J - M$ and, if $x^- \notin M, x \prec y$ implies $y \notin J$, or $x \in M - J$ and $y \prec x$ implies $y \in M$. □

Theorem 2.2. [1] Let x be an element of an upper semimodular lattice such that $x \in M - J$ and $y \prec x$ implies $y \in M$. Then $x^+ \in J$. □

Theorem 2.3. [1] Let x be an element of a modular lattice L then x is M*-deletable* if and only if either $x \in J \cap M$ or $x \in M - J$ and $x^+ \in J$ or $x \in J - M$ and $x^- \in M$. □

Theorem 2.4. [1] Let L be a distributive (hence modular) lattice. The $\mathcal{D}$*-deletable* elements of L are exactly its $\mathcal{M}$*-deletable* elements. Thus $L' = L - x$ is distributive if and only if L' is modular. □

Lemma 2.1. [7] The following properties are equivalent for a lattice L:

1) L is a pseudocomplemented lattice.
2) Each atom of L has a pseudocomplement.
3) Each atom of L is join-prime.

□

Proposition 2.1. [1] Let x be an irreducible element of a pseudocomplemented lattice L. Then x is a $\mathcal{PC}$*-deletable* element if and only if x does not satisfy any of the two following properties:

1) x is an atom a of L, there exists a join-irreducible element j of L covering a and the set $\{t \in L-a : t \wedge j = a\}$ is non-empty and has no greatest element or has a greatest element $\ngeq a^{*+}$.
2) x is the pseudocomplement a^* of an atom a of L and x is not a join-irreducible element of L.

III. Some Properties of Deletable Elements in Finite Lattices and Posets

Theorem 3.1. Let x be an element of a modular lattice then x is *USM-deletable* but x is not $\mathcal{M}$*-deletable* iff $x \in J - M$ and $x^- \notin M, x \prec y$ implies $y \notin J$.

Proof: Let x be *USM-deletable* then by using theorem 2.1 either $x \in J \cap M$ or $x \in J - M$ and, if $x^- \notin M, x \prec y$ implies $y \notin J$, or $x \in M - J$ and $y \prec x$ implies $y \in M$.

If $x \in J \cap M$ then by using theorem 2.3 x is $\mathcal{M}$*-deletable* which is a contradiction to hypothesis.

If $x \in M - J$ and $y \prec x$ implies $y \in M$ then by using theorem 2.2, $x^+ \in J$ then by theorem 2.3 x is $\mathcal{M}$*-deletable* which is again a contradiction to hypothesis.

If $x \in J - M$ and if $x^- \in M$ implies that x is $\mathcal{M}$*-deletable* which is a contradiction to hypothesis so $x^- \notin M$ and $x \prec y$ implies $y \notin J$.

Conversely, let $x \in J - M$ and $x^- \notin M, x \prec y$ implies $y \notin J$ so by using theorem 2.1 and theorem 2.3, x is *USM-deletable* but x is not $\mathcal{M}$*-deletable*. □

By using concept of duality, we have dual of Theorem 3.1.

Theorem 3.2. Let x be an element of a modular lattice then x is *LSM-deletable* but x is not $\mathcal{M}$*-deletable* iff $x \in M - J$ and $x^+ \notin J, y \prec x$ implies $y \notin M$. □

Corollary 3.1. Let x be an irreducible element of a distributive lattice L. Then x is a $\mathcal{PC}$*-deletable* but x is not $\mathcal{D}$*-deletable* if and only if either $x \in J - M$ and $x^- \notin M$ or $x \in M - J$ and $x^+ \notin J$, also x does not satisfy any of the two following properties:

1) x is an atom a of L, there exists a join-irreducible element j of L covering a and the set $\{t \in L-a : t \wedge j = a\}$ is non-empty and has no greatest element or has a greatest element $\ngeq a^{*+}$.
2) x is the pseudocomplement a^* of an atom a of L.

Proof: Let x be $\mathcal{PC}$*-deletable* but x is not $\mathcal{D}$*-deletable*. Since x is not $\mathcal{D}$*-deletable* then by theorems 2.3 and 2.4 either $x \in J - M$ and $x^- \notin M$ or $x \in M - J$ and $x^+ \notin J$, since x is $\mathcal{PC}$*-deletable* by proposition 2.1 x does not satisfy to property (1) and if x is the pseudocomplement a^* of an atom a of L then x is a join-irreducible element but by lemma 2.1, x must be meet-irreducible element. So $x \in J \cap M$ but it is a contradiction. So x does not satisfy in property (2).

Conversely, since either $x \in J - M$ and $x^- \notin M$ or $x \in M - J$ and $x^+ \notin J$ then, by using theorems 2.3 and 2.4, x is not $\mathcal{D}$*-deletable* and since x does not satisfy in any of properties (1) and (2) then by proposition 2.1. x is $\mathcal{PC}$*-deletable*. □

Theorem 3.3. : Let x be an irreducible element of a complemented lattice. Then x is a $\mathcal{C}$*-deletable* if and only if its complements have more than one complements.

Proof: Let x be a $\mathcal{C}$*-deletable* then $L' = L - x$ is also a complemented lattice, since $x' \in L'$ then $\exists y \in L'$ s.t. $x' \wedge y = 0$ and $x \vee y = 1$ in L' therefore either $x' \wedge y = 0$ and $x' \vee y = 1$ in L or $x' \wedge y = x, x' \vee y = 1$ and $x^- = 0, x \in J - M$ in L or $x' \wedge y = 0, x' \vee y = x$ and $x^+ = 1, x \in M - J$ in L.

If $x' \wedge y = 0$ and $x' \vee y = 1$ in L then x' has more than one complements in L. In any two other cases x and x' are comparable in L but that is not possible.

Conversely, Let x' has more than one complements. We know x is complement of x' so $\exists y \in L' = L - x$ s.t. $x' \wedge y = 0$ and $x' \vee y = 1$ in L then clearly $x' \wedge y = 0$ and $x' \vee y = 1$ in L' and any other element of $L - x$ also has complement in $L' = L - x$. □

Let P be a poset with elements 0 and 1 and $a, b \in P$. If $\{a, b\}^l = \{0\}$ and $\{a, b\}^u = \{1\}$ then, we shall denote b by a' and we call it the complement of a. A complemented poset P is one in which for every element $a \in P$, the complement a' exists in P.

Theorem 3.4. Let x be an irreducible element of a complemented poset. Then x is a $\mathcal{C}$*-deletable* if and only if its complements have more than one complements.

Proof: Let x be a $\mathcal{C}$*-deletable* then $P' = P - x$ is also a complemented poset, since $x' \in P$ then $\exists y \in P'$ s.t. $\{x', y\}^l = \{0\}$ and $\{x', y\}^u = \{1\}$ in P' therefore either $\{x', y\}^l = \{0\}$ and $\{x', y\}^u = \{1\}$ in P or $x \in \{x', y\}^l$ and $\{x', y\}^u = \{1\}$ in P or $\{x', y\}^l = \{0\}$ and $x \in \{x', y\}^u$.

If $\{x', y\}^l = \{0\}$ and $\{x', y\}^u = \{1\}$ in P then x' has more than one complements in P. In any two other cases x and x' are comparable in P but that is not possible.

Conversely, Let x' has more than one complements. We know x is complement of x' so $\exists y \in P - x$ s.t. $\{x', y\}^l = \{0\}$ and $\{x', y\}^u = \{1\}$ in P then clearly $\{x', y\}^l = \{0\}$ and $\{x', y\}^u = \{1\}$ in P' and any other element of $P - x$ also has complement in $P' = P - x$. □

IV. Reducibility Number in Some Classes of Finite Lattices and Posets

Theorem 4.1. : Reducibility number of Boolean lattices with respect to the class of complemented lattices is 2.

Proof: Let L be a Boolean lattice then L is distributive and complemented therefore each element has exactly one complement so elements of L are the form of complement pairs therefore L must have even number elements. If we delete one element of L then the obtain lattice has odd number elements so it can not be Boolean lattice. Also since L is a Boolean lattice each element has only one complement so by theorem 3.3, there is not any *C-deletable* for L.

Now we claim in any Boolean lattice if we delete an atom and its complement then the obtain lattice is complemented lattice. Since each Boolean lattice is Pseudocomplemented lattice, by lemma 2.1 we have each atom, is join-prime (and thus join-irreducible) and then its pseudocomplement is meet-prime (and thus meet-irreducible) and also we know that in Boolean lattice pseudocomplement and complement of each element are same thus we delete an atom and its pseudocomplement, the obtain lattice is complemented lattice. Because let we delete x and x' and y be an arbitrary element of $L - \{x, x'\}$ then we claim y' is complement of y in $L - \{x, x'\}$, if not then let $y \wedge y' = z$ in $L - \{x, x'\}$ where $z \neq 0$ then z is a lower bound of $\{y, y'\}$ in L and $L - \{x, x'\}$ but 0 is greatest lower bound of $\{y, y'\}$ in L so $z < 0$ that is not possible similarly $y \vee y' = 1$ i.e. the obtain lattice is complemented lattice.□

Lemma 4.1. Let L be uniquely complemented lattice then complement of each atom is coatom.

Proof: Let x be an atom of L and $x' \prec y$. Since we have $x \wedge x' = 0$ and $x \vee x' = 1$ thus $x \vee y = 1$ but $x \wedge y \neq 0$. So we have $x \wedge y \leq x$ and $x \wedge y \neq 0$ and since x is an atom then $x \wedge y = x$ thus $x \leq y$ therefore $x \vee y = y$ but $x \vee y = 1$ thus $y = 1$. Therefore x' is a coatom of L. □

A uniquely complemented poset P is one in which for any element $a \in P$, it has exactly one complement.

Lemma 4.2. Let P be uniquely complemented poset then complement of each atom is coatom.

Proof: Let x be an atom of P and $x' \prec y$. Since we have $\{x, x'\}^l = \{0\}$ and $\{x, x'\}^u = \{1\}$ thus $\{x, y\}^u = 1$ but $\{x, y\}^l \neq \{0\}$. So $\exists z (\neq 0) \in \{x, y\}^l$ therefore $z \leq x$ and $z \leq y$ and $z \neq 0$ therefore $z = x$ so $x \leq y$ thus $y \in \{x, x'\}^u = \{1\}$ i.e. $y = 1$. Therefore x' is a coatom of P. □

Corollary 4.1. Reducibility number of uniquely complemented lattices with respect to the class of complemented lattices is 2.

Proof: By using proof of theorem 4.1 uniquely complemented lattices are not reducible with respect to class of complemented lattices. But since complement of each atom is a coatom so it is a meet-irreducible, similar to theorem 4.1 we can delete an atom and its complement then the obtain lattice is complemented lattice.□

Corollary 4.2. : Reducibility number of uniquely complemented posets with respect to the class of complemented posets is 2.

Proof: Proof is straightforward.□

Corollary 4.3. Reducibility number of orthocomplemented lattices is 2.

Proof: We know that in orthocomplemented lattices each element has unique orthocomplement so elements of L are the form of orthocomplement pairs therefore L must have even number elements. If we delete one element of L then the obtain lattice has odd number elements so it can not be orthocomplemented lattices. Therefore orthocomplemented lattices are not reducible. Now we claim orthocomplemente of every atom is meet-irreducible. If not then let L be an orthocomplemented lattice and a be an atom of L s.t. $a^\perp \notin M$ thus $\exists\ x, y$ s.t. $a^\perp \prec x$ and $a^\perp \prec y$ so we have $x^\perp \leq a$ and $y^\perp \leq a$ but $x^\perp \neq a$ and $y^\perp \neq a$ (since if $x^\perp = a$ by knowing $x^{\perp\perp} = x$ we have $a^\perp = x$ which is a contradiction to $a^\perp \prec x$ and similarly $y^\perp \neq a$) thus $x^\perp = 0$ and $y^\perp = 0$ that implies $x = y = 1$ so $a^\perp \in M$. Thus if we delete an atom and its orthocomplement, the obtain lattice is orthocomplemented lattice.□

References

[1] G. Bordalo and B. Monjardet, "Reducible classes of finite lattices", *Order*, vol. 13, pp. 379–390, 1996.

[2] V. S. Kharat and B. N. Waphare, "Reducibility in finite posets", *European J. Comb.,* vol. 22, no. 2, pp. 197–205, 2001.

[3] V. S. Kharat, B. N. Waphare and N. K. Thakare, "Reducibility number", *Discrete applied Mathematics,* vol. 155, no. 16, pp. 2069–2076, 2007.

[4] I. Rival, "Lattices with doubly irreducible elements,", *Can. Math. Bull.*, vol. 17, pp. 91-95, 1974.

[5] G. Birkhoff, *Lattice Theory*, Third ed., American Mathematical Society, Providence R. I., 1967.

[6] M. Stern, *Semimodular Lattices*, Cambridge University Press, 1999.

[7] C. Chameni-Nembua and B. Monjardet, "Les treillis Pseudocomplemente finis", *European J. of Combinatoriecs*, vol. 13, pp. 89–107, 1992. (1992),

[8] D. Duffus and I. Rival,"Crowns in dismantlable partially ordered sets", *Coll. Math. Soc. J. Bolyai*, vol. 18, pp. 271-292, 1976.

[9] D. Kelly and I. Rival, "Crowns, fences and dismantable lattices," *Can. J. Math.* vol. 26, pp. 1257–1271, 1974.