On special circulant matrices with (k, h)-Jacobsthal sequence and (k, h)-Jacobsthal-like sequence

S.H.J. Petroudi and M. Pirouz

***Abstract*—This note is devoted to Circulant matrices involving (k, h)-Jacobsthal sequence and (k, h)-Jacobsthal-like sequence . Firstly we define (k, h)-Jacobsthal-like sequence. Then by using (k, h)-Jacobsthal-like sequence, some formulas for n^{th} term and sum of the first n terms of (k, h)-Jacobsthal and (k, h)-Jacobsthal-like sequences are derived. Moreover eigenvalues and determinants of circulant matrices involving these sequences are obtained. Finally some bounds for the spectral norm of circulant matrices involving these sequences are represented.**

***Index Terms*—Circulant matrices, (k, h)-Jacobsthal sequence, (k, h)-Jacobsthal-like sequence, determinant, spectral norm .**

***MSC 2010 Codes* – 11B37, 11B39, 15A36, 15A60.**

I. INTRODUCTION

THE (k, h)-Jacobsthal is defined by

$$T_n = kT_{n-1} + 2hT_{n-2}, \quad (1)$$

where $T_0 = 0$ and $T_1 = k$ ([1]). Bueno in [1] found a formula of n^{th} term and sum of the first n terms of this sequence. Firstly in this note we define (k, h)-Jacobsthal-like sequence and is defined by

$$P_n = kP_{n-1} + 2hP_{n-2}, \quad (2)$$

where $P_0 = 2$ and $P_1 = k$.
This note is devoted to Circulant matrices involving (k, h)-Jacobsthal sequence and (k, h)-Jacobsthal-like sequence. Firstly we define (k, h)-Jacobsthal-like sequence. Then by using (k, h)-Jacobsthal-like sequence, some formulas for n^{th} term and sum of the first n terms of (k, h)-Jacobsthal and (k, h)-Jacobsthal-like sequences are derived. Moreover eigenvalues and determinants of circulant matrices involving these sequences are obtained. Finally upper bounds and lower bounds for the spectral norm of circulant matrices involving these sequences are established. For more information about Fibonaacci sequence and some generalizations of this sequence and norm properties of particular matrices involving these sequences one can see [2]-[14].
It is known that

$$\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}, \quad (3)$$

Seyyed Hossein Jafari Petroudi is assisstant professor in the Department of Mathematics, Payame Noor University, P. O. Box 1935-3697, Tehran, Iran. (E-mail: hossein_5798@yahoo.com)

M. Pirouz is academic member in the Department of Mathematics, Guilan University, Rasht, Iran (E-mail: mpirouz60@yahoo.com).

$$\sum_{k=1}^{n-1} kx^{k-1} = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2} \quad (4)$$

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The ℓ_p norm of A is defined by

$$\|A\|_p = (\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^p)^{\frac{1}{p}} \quad (5).$$

For $p = 2$ this norm is called Frobenius or Euclidean norm and showed by $\|A\|_E$. The spectral norm of A is defined by

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i}, \quad (6)$$

where λ_i is the eigenvalue of matrix AA^H and A^H is conjugate transpose of matrix A. There is a relation between frobenius and spectral norm, that is

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E. \quad (7)$$

II. MAIN RESULTS

Theorem 2.1 Let T_n be a sequenceas in (1), then we have

$$T_n = \frac{k}{p}\left(\alpha^n - \beta^n\right),$$

where $\alpha = \frac{k+\sqrt{k^2+8h}}{2}$, $\beta = \frac{k-\sqrt{k^2+8h}}{2}$ and $p = \alpha - \beta$, $k = \alpha + \beta$.
Proof: See [1]. □

Theorem 2.2 Let P_n be a sequence as in (2), then we have

$$P_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{k+\sqrt{k^2+8h}}{2}$, $\beta = \frac{k-\sqrt{k^2+8h}}{2}$ and $p = \alpha - \beta$, $k = \alpha + \beta$.
Proof: The proof is similar to theorem 2.1. □

Theorem 2.3 Let T_n be a sequence as in (1), then we have

$$\sum_{m=0}^{n-1} T_m = \frac{T_n + 2kT_{n-1} - k}{2h + k - 1}.$$

Proof: See [1]. □

Theorem 2.4 Let T_n be a sequence as in (1), then we have $\sum_{m=0}^{n-1} T_m^2 =$

$$\frac{k^2}{p^2}\left[\frac{4h^2P_{2n-2} - P_{2n} - P_2 + 2}{4h^2 - P_2 + 1} - 2 \cdot \frac{1 - (-2h)^n}{1 + 2h}\right].$$

Proof: By theorem2.1 we have

$$\sum_{m=0}^{n-1} T_m^2 = \sum_{m=0}^{n-1} \left[\frac{k}{p}(\alpha^m - \beta^m)\right]^2$$

$$= \frac{k^2}{p^2} \sum_{m=0}^{n-1} \left(\alpha^{2m} + \beta^{2m} - 2(\alpha\beta)^m\right)$$

$$= \frac{k^2}{p^2} \left[\sum_{m=0}^{n-1} (\alpha^2)^m + \sum_{m=0}^{n-1} (\beta^2)^m - 2\sum_{m=0}^{n-1} (\alpha\beta)^m\right].$$

By using (3) and (4) we get

$$\sum_{m=0}^{n-1} T_m^2 = \frac{k^2}{p^2} \left[\frac{\alpha^{2n}-1}{\alpha^2-1} + \frac{\beta^{2n}-1}{\beta^2-1} - 2\frac{(\alpha\beta)^n-1}{\alpha\beta-1}\right]$$

$$= \frac{k^2}{p^2} \left[\frac{\alpha^2\beta^2(\alpha^{2n-2}+\beta^{2n-2}) - (\alpha^{2n}+\beta^{2n}) - (\alpha^2+\beta^2) + 2}{(\alpha\beta)^2 - (\alpha^2+\beta^2) + 1}\right]$$

$$+ \frac{k^2}{p^2} \left[2\frac{(-h)^n - 1}{h+1}\right].$$

So by theorems2.1 and 2.2 we deduce that

$$\sum_{m=0}^{n-1} T^2 = \frac{k^2}{p^2} \left[\frac{4h^2 P_{2n-2} - P_{2n} - P_2 + 2}{4h^2 - P_2 + 1} + 2 \cdot \frac{1-(-2h)^n}{1+2h}\right].$$

□

Theorem 2.5 Let P_n be a sequence as in (2), then we have

$$\sum_{m=0}^{n-1} P_m = \frac{P_n + 2hP_{n-1} + k - 2}{k + 2h - 1}.$$

Proof: The proof is similar to theorem2.3. □

Theorem 2.6 Let P_n be a sequence as in (2), then we have

$$\sum_{m=0}^{n-1} P_m^2 =$$

$$\left[\frac{2 - P_2 + 4h^2 P_{2n-2} - P_{2n}}{1 - P_2 + 4h^2}\right] + 2\left[\frac{1-(-2h)^n}{1+2h}\right].$$

Proof: The proof is similar to theorem2.4. □

III. Circulant matrices

A matrix $C = [c_{i,j}] \in M_n$ is called a Circulant matrix if it is of the form

$c_{i,j} = a_{j-i}$ for $\quad j \geq i$,

and

$c_{i,j} = a_{n+j-i}$ for $\quad j < i$.

Theorem 3.1 Let $C = circ(c_0, c_1, c_2, \cdots, c_{n-1})$ is an $n \times n$ circulant matrix. The eigenvalues of C are

$$\lambda_j = \sum_{k=0}^{n-1} c_k w^{jk}, \quad j = 0, 1, \cdots, n-1$$

where $w = exp(\frac{2\pi i}{n})$ and $i = \sqrt{-1}$.

Proof: see [2]. □

We define the $n \times n$ circulant matrices C_n and D_n with (k,h)-Jacobsthal sequence and (k,h)-Jacobsthal-like sequence respectively by

$$C_n = circ(T_0, T_1, T_2, \cdots, T_{n-1}), \quad (8)$$

and

$$D_n = circ(P_0, P_1, P_2, \cdots, P_{n-1}), \quad (9)$$

where T_n is the n^{th}term of (k,h)-Jacobsthal sequence and P_n is the n^{th} term of (k,h)-Jacobsthal-like sequence.

Theorem 3.2 Let C_n be a circulant matrix as in (8), then the eigenvalues of C_n are:

for j=0 we have

$$\lambda_0 = \sum_{m=0}^{n-1} T_m = \frac{T_n + 2kT_{n-1} - k}{2h + k - 1},$$

and for $j \geq 1$ we have

$$\lambda_j = \frac{(k + 2hT_{n-1})w^j - T_n}{1 - (k + 2hw^j)w^j}.$$

where $w = exp(\frac{2\pi i}{n})$ and $i = \sqrt{-1}$.

Proof: For $j = 0$ the results follows from theorem3.1 and theorem2.3.

For $j \geq 1$, by theorem3.1 and theorem2.1 we have

$$\lambda_j = \sum_{m=0}^{n-1} T_m w^{jm} = \sum_{m=0}^{n-1} \frac{k}{p}(\alpha^m - \beta^m)\left(e^{\frac{2\pi ij}{n}}\right)^m.$$

$$= \frac{k}{p} \sum_{m=0}^{n-1} \left((\alpha e^{\frac{2\pi ij}{n}})^m - (\beta e^{\frac{2\pi ij}{n}})^m\right).$$

According to (3) we get

$$\lambda_j = \frac{k}{p}\left[\frac{1 - \alpha^n \left(e^{\frac{2\pi ij}{n}}\right)^n}{1 - \alpha e^{\frac{2\pi ij}{n}}} - \frac{1 - \beta^n \left(e^{\frac{2\pi ij}{n}}\right)^n}{1 - \beta e^{\frac{2\pi ij}{n}}}\right]$$

Thus we have

$$\lambda_j = \frac{k}{p}\left[\frac{1-\alpha^n}{1-\alpha w^j} - \frac{1-\beta^n}{1-\beta w^j}\right]$$

By some computations we obtain

$$\lambda_j = \frac{k}{p}\left[\frac{(\alpha-\beta)w^j - (\alpha^n - \beta^n) + \alpha\beta(\alpha^{n-1} - \beta^{n-1})w^j}{(1-\alpha w^j)(1-\beta w^j)}\right]$$

$$= \frac{k}{p}\left[\frac{(\alpha-\beta)w^j - (\alpha^n - \beta^n) + \alpha\beta(\alpha^{n-1} - \beta^{n-1})w^j}{1 - (\alpha+\beta)w^j + (\alpha\beta)w^{2j}}\right]$$

Consequently by theorem2.1 and theorem2.2 we get

$$\lambda_j = \frac{T_1 w^j - T_n - 2hT_{n-1}w^j}{1 - kw^j - 2hw^{2j}} = \frac{(k - 2hT_{n-1})w^j - T_n}{1 - (k + 2hw^j)w^j}.$$

Thus the proof is completed. □

Lemma 3.3 Let x and y are real variables and $w = exp(\frac{2\pi i}{n})$ then

$$\prod_{j=0}^{n-1}(x - yw^j) = x^n - y^n.$$

Proof: see[3]. □

Theorem 3.4 Let C_n be a circulant matrix as in (8), then determinant of of C_n is

$$det(C_n) = |C_n| = \frac{(k - 2hT_{n-1})^n - (T_n)^n}{1 - P_n + (-2h)^n}.$$

Proof: By [2] we have

$$det(C_n) = \prod_{j=0}^{n-1}\lambda_j = \prod_{j=0}^{n-1}\frac{(k - 2hT_{n-1})w^j - T_n}{(1 - \alpha w^j)(1 - \beta w^j)}.$$

Thus by Lemma3.3 we have

$$det(C_n) = \frac{\prod_{j=0}^{n-1}\left((k - 2hT_{n-1})w^j - T_n\right)}{\prod_{j=0}^{n-1}(1 - \alpha w^j)\prod_{j=0}^{n-1}(1 - \beta w^j)} = .$$

According to lemma3.3 we deduce that

$$det(C_n) = \frac{(k - 2hT_{n-1})^n - (T_n)^n}{(1 - \alpha^n)(1 - \beta^n)}$$

Consequently by theorem2.2 we conclude that

$$det(C_n) = \frac{(k - 2hT_{n-1})^n - (T_n)^n}{1 - P_n + (-2h)^n}.$$

□

Theorem 3.5 Let D_n be a circulant matrix as in (9), then the eigenvalues of D_n are:
for $j = 0$ we have

$$\nu_0 = \sum_{m=0}^{n-1}P_m = \frac{P_n + 2hP_{n-1} + k - 2}{k + 2h - 1},$$

and for $j \geq 1$ we have

$$\nu_j = \frac{2 - (k + 2hP_{n-1})w^j - P_n}{1 - (k + 2hw^j)w^j}.$$

where $w = exp(\frac{2\pi i}{n})$ and $i = \sqrt{-1}$.

Proof: The proof is similar to theorem3.2. □

Theorem 3.6 Let D_n be a circulant matrix as in (9), then determinant of of D_n is

$$det(D_n) = |D_n| = \frac{(2 - P_n)^n - (k + 2hP_{n-1})^n}{1 - P_n + (-2h)^n}.$$

Proof: The proof is similar to theorem3.4. □

Theorem 3.7 Let C_n be a circulant matrix as in (8), then the Euclidean norm of C_n is

$$\|C_n\|_E = \frac{\sqrt{n}k}{p}\left[\frac{4h^2P_{2n-2} - P_{2n} - P_2 + 2}{4h^2 - P_2 + 1} - 2\cdot\frac{1 - (-2h)^n}{1 + 2h}\right]^{\frac{1}{2}}.$$

Proof: By definition of Euclidean norm we have

$$\|C_n\|_E^2 = n\left(T_0^2 + T_1^2 + T_2^2 + \cdots + T_{n-1}^2\right) = n\sum_{k=0}^{n-1}T_k^2.$$

By Theorem2.4 we obtain

$$\|C_n\|_E^2 = n\frac{k^2}{p^2}\left[\frac{4h^2P_{2n-2} - P_{2n} - P_2 + 2}{4h^2 - P_2 + 1} - 2\cdot\frac{1 - (-2h)^n}{1 + 2h}\right].$$

Coseqently by taking $(\frac{1}{2})^{th}$ power from the both sides of the above equality we get the result. □

Theorem 3.8 Let C_n be a circulant matrix as in (8), then we have the following upper bound and lower bound for the spectral norm of C_n

$$\frac{k}{p}\left[\frac{4h^2P_{2n-2} - P_{2n} - P_2 + 2}{4h^2 - P_2 + 1} - 2\cdot\frac{1 - (-2h)^n}{1 + 2h}\right]^{\frac{1}{2}} \leq \|C_n\|_2 \leq \frac{\sqrt{n}k}{p}\left[\frac{4h^2P_{2n-2} - P_{2n} - P_2 + 2}{4h^2 - P_2 + 1} - 2\cdot\frac{1 - (-2h)^n}{1 + 2h}\right]^{\frac{1}{2}}.$$

Proof: It follows from (7) and theorem3.7.□

Theorem 3.9 Let D_n be a circulant matrix as in (9), then the Euclidean norm of D_n is

$$\|D_n\|_E = \sqrt{n}\left[\frac{2 - P_2 + 4h^2P_{2n-2} - P_{2n}}{4h^2 - P_2 + 1} + 2\cdot\frac{1 - (-2h)^n}{1 + 2h}\right]^{\frac{1}{2}}.$$

Proof: The proof is similar to theorem3.7.□

Theorem 3.10 Let D_n be a circulant matrix as in (9), then we have the following upper bound and lower bound for the spectral norm of D_n

$$\left[\frac{2 - P_2 + 4h^2P_{2n-2} - P_{2n}}{4h^2 - P_2 + 1} + 2\cdot\frac{1 - (-2h)^n}{1 + 2h}\right]^{\frac{1}{2}} \leq \|D_n\|_2 \leq \sqrt{n}\left[\frac{2 - P_2 + 4h^2P_{2n-2} - P_{2n}}{4h^2 - P_2 + 1} + 2\cdot\frac{1 - (-2h)^n}{1 + 2h}\right]^{\frac{1}{2}}.$$

.

Proof: It follows from (7) and theorem3.9. □

IV. Conclusion

Some new identities about determinants and eigenvalues of circulant matrices involving (k, h)-Jacobsthal sequence and (k, h)-Jacobsthal-like sequence are derived in this paper. Also in this paper upper and lower bounds for the spectral norm of circulant matrices involving these sequences are represented .

Acknowledgment

The authors are grateful to the Editor-in-Chief for his significant explanations to enrichment of this paper.

REFERENCES

[1] A. C. F. Bueno, “A note on (k,h)-Jacobsthal sequences”, *Mathematical Sciences Letters*, vol. 1, pp. 81-87, 2013.

[2] M. Bahsi and S. Solak, “On the circulant matrices with arithmetic sequence”, *International Journal on Contemporary Mathematical Sciences,* vol. 5, no. 25, pp 1213-1222, 2010.

[3] A. C. F. Bueno, “On the eigenvalues and the determinant of the right circulant matrices with Pell and Pell-Lucas numbers”, *International Journal of Mathematics and Scientific Computing*, vol. 4, no. 1, pp. 19-20, 2014.

[4] A. C. F. Bueno, “Right Circulant Matrices with Fibonacci Sequence”, *International Journal of Mathematics and Scientific Computing*, vol. 2, no. 2, pp. 8-9, 2012.

[5] A. C. F. Bueno, “On Arithmetic Right Circulant Matrix Sequences”, *International Journal of Mathematics and Scientific Computing*, vol. 4, no. 1, pp. 25-27, 2014.

[6] D. Bozkurt, “A note on the spectral norms of the matrices connected integer numbers sequence”, *General Mathematics,* vol. 1, pp. 171-190, 2011.

[7] H. Civciv and R. Turkmen, “On the bounds for the spectral and ℓ_p norms of the Khatri-Rao products of Cauchy-Hankel matrices”, *Journal of Inequalities in Pure and Applied Mathematics,* vol. 7, pp. 365-380, 2006.

[8] E. Dupree and B. Mathes, “Singular values of k-Fibonacci and k-Lucas Hankel matrix”, *International Journal on Contemporary Mathematical Sciences,* vol. 7, pp. 2327-2339, 2012.

[9] A. D. Godase and M. B. Dhakne, “On the properties of generalized multiplicative coupled fibonacci sequence of rth order”, *International Journal of Advances in Applied Mathematics and Mechanics,* vol. 2, no. 3, pp. 252-257, 2015.

[10] A. Nalli and M. Sen, “On the norms of circulant matrices with generalized Fibonacci numbers”, *Selçuk Journal of Applied Mathematics,* vol. 1 , pp. 107-116, 2010.

[11] S. H. J. Ptroudi and B. Pirouz, “On some properties of (k,h)-Pell sequence and (k,h)-Pell-Lucass sequence”, *International Journal of Advances in Applied Mathematics and Mechanics,* vol. 3 no. 1, pp 98-101, 2015.

[12] S. Q. Shen, “The spectral norms of circulant matrices nvolving (k,h)-Fibonacci and (k,h)-Lucas numbers”, *International Journal on Contemporary Mathematical Sciences,* vol. 9, no. 14, pp 661-665, 2014.

[13] S. Solak and M. Bahsi, “A particular matrix and its some properties”, *Scientific Research and Essays*, vol. 8, no. 1, pp. 1-5, 2013.

[14] S. Solak and M. Bahsi, “On the spectral norms of Toeplitz matrices with Fibonacci and Lucas numbers”, *Hacettepe Journal of Mathematics and Statistics*, vol. 42, no. 1, 15-19, 2013.