Invariance Analysis of Williamson Model Using the Method of Satisfaction of Asymptotic Boundary Conditions

Nita Jain and M.G. Timol

***Abstract*—This paper is focused on developing the deductive group-theoretic transformations for the similarity solution of steady, laminar, incompressible two dimensional boundary layer flow governing Williamson fluid. The applications of one parameter deductive group transformation are applied for simultaneous elimination of more than one variable. And consequently the system of governing highly non-linear partial differential equations with auxiliary conditions reduces to a non-linear ordinary differential equation with appropriate auxiliary conditions. The numerical solution for the considered Williamson fluid is derived systematically in dimensionless form as an application of engineering with MATLAB using MSABC.**

***Index Terms*—Williamson fluids, Deductive group-theoretic method, similarity solution, MSABC, skin friction**

***MSC 2010 Codes* – 76A05, 76M55, 54H15**

I. Introduction

There are several types of non-Newtonian fluid models which are proposed by scientists working in this area. Several empirical models are used to approximate the experimental data. Calculations on non-Newtonian flow present a new challenge in flow analysis. Simulating these types of flows in order to calculate pipe and pump sizes presents a significant challenge to the engineers.

Non-Newtonian fluids exhibit a complex rheology involving a non-linear relationship between the shear rate and the applied shear stress. Different rheological models have been proposed in literature to represent their behavior e.g. Power-law, Prandtl-Eyring, Williamson and Powel Eyring, Shutterby etc. The aim of the research is to study the behavior of these fluids in industrial and environmental applications.

As the boundary-layer assumption is an asymptotic approximation, certain terms seems to be assumed or neglected such as the thermal boundary layer is assumed much thinner than the velocity boundary layer, the momentum transport terms in the equation of motion could be neglected. The possibility of solving these equations without such assumptions, from a mathematical viewpoint, is a desirable goal.

Williamson [1] discussed the flow of pseudo plastic materials and proposed a model equation to describe the flow of pseudo plastic fluids and experimentally verified the results. Cramer et al. [2] showed that this model fits the experimental data of polymer solutions and particle suspensions better than other models. Lyubimov and Perminov [3] discussed the flow of a thin layer of a Williamson fluid over an inclined surface in the presence of a gravitational field. Dapra and Scarpi [4] developed the perturbation solution for a Williamson fluid injected into a rock fracture. Peristaltic flow of a Williamson fluid has been discussed by Nadeem et al. [5]. Vasudev et al. [6] studied the peristaltic pumping of a Williamson fluid through a porous medium considering heat transfer.

The applications of one-parameter deductive group transformation is applied for simultaneous elimination of more than one variable . The similarity methods help to analyze the most physical systems for possible similarity solutions. The deductive group methods can be applied to power law, non-Newtonian, boundary-layer flow systems. The major difficulty in solving non-Newtonian fluid systems is due to the nonlinearities in the equations of motion. This limits the applicability of similarity variables to the energy equation. The similarity method involves the determination of similarity variables which reduce the system to ordinary differential equations. Probably the first analysis of this type was given by Acrivos [7]. Metzner [8] contributed to the similarity transformation for the momentum equation of viscoelastic flow. They concluded that the possibility of solving these systems is limited.

Birkhoff [9] initiated the applications of group theory to fluid mechanics, opened up the way for general similarity procedures. Building on this work, Morgan [10] gave complete structure of the theory for reducing number of independent variables. Lee and Ames [11] applied the method for reducing more than one independent variable simultaneously by composing a multiparameter group. The one-parameter group method is also discussed by Hansen [12]. This method transfers the problem of searching for similarity variables to that of solving for the invariant conditions of a system of differential equations under a certain group of continuous transformations. The similarity variables are determined from the absolute invariants of the subgroup consisting of transformations of independent variables.

In most of the discussion available, the similarity transformations are applied on adhoc manner and only little attention devoted to derive it. The present paper is addressed to this problem. In the present paper the deductive group method based on general group transformation is applied to derive similarity solutions for steady, incompressible, laminar

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two dimensional boundary layer flows of Williamson fluids whose shearing stress is related to rate of strain by arbitrary continuous function. The similarity equations obtained are more general and systematic along with auxiliary conditions. Recently this method has been successfully applied to various non-linear problems by Abd-el- Malek et al [13] and Darji and Timol [14].

II. Governing Equations

Consider the incompressible, viscoinelastic, two dimensional steady Williamson non-Newtonian fluid. Assuming the small velocity gradient for the flow outside the boundary layer, the governing differential equations of the boundary layer flow of such fluid are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\rho\,(u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}) = \frac{\partial \tau}{\partial y} + \rho\,U_e\,\frac{dU_e}{dx} \tag{2}$$

Also, for the considered Williamson model, the shearing stress τ is related explicitly to the velocity gradient and given as:

$$\frac{\partial u}{\partial y} = \frac{\tau}{\mu_\infty + \frac{A}{B + \frac{\partial u}{\partial y}}} \tag{3}$$

with the boundary conditions :

$$u(x,0) = v(x,0) = 0 \quad and \quad u(x,\infty) = U_e(x)$$

III. Formulation of the Problem

Introducing the dimensionless quantities as:

$$\bar{x} = \frac{x}{L} \qquad \bar{y} = \sqrt{\frac{Re}{3}}\,\frac{y}{L} \qquad \bar{u} = \frac{u}{U_\infty} \qquad \bar{v} = \sqrt{\frac{Re}{3}}\,\frac{v}{U_\infty}$$

$$\bar{U_e} = \frac{U_e}{U_\infty} \qquad R_e = \frac{U_\infty L}{\nu} \qquad \bar{\mu} = \frac{\mu_0}{\mu_\infty} \qquad \bar{\tau} = \frac{\tau}{\rho {U_\infty}^2 \sqrt{\frac{Re}{3}}}$$

Also introducing the stream function ψ to integrate the continuity equation as

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}$$

Equations (2) and (3) become

$$\frac{\partial \psi}{\partial y}\,\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\,\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \tau}{\partial y} - U_e \frac{dU_e}{dx} = 0 \tag{4}$$

$$\tau = (\frac{\alpha}{1+\sqrt{\beta}\frac{\partial^2 \psi}{\partial y^2}} + 1)\;\frac{\partial^2 \psi}{\partial y^2} \tag{5}$$

subject to the boundary conditions :

$$\frac{\partial \psi}{\partial y}(x,0) = 0, \quad \frac{\partial \psi}{\partial x}(x,0) = 0, \quad \frac{\partial \psi}{\partial y}(x,\infty) = U_e(x) \tag{6}$$

where $\alpha = \frac{A}{B\mu_0}$ and $\beta = \frac{U_\infty^2 R_e}{3B^2 L^2}$ are nondimensional numbers and bars are dropped for simplicity.

IV. Method and solution of the Problem

Our method of solution depends on the application of a one-parameter deductive group of transformation to the partial differential equations (4) along with auxiliary conditions (6). Under this transformation the two independent variables will be reduced by one and the differential equations (4) will transforms into the ordinary differential equation.

A. The group systematic formulation

Introducing the group theoretic method

$$G : \Pi_\varepsilon(@) = \pounds^{@}(\varepsilon)@ + \Re^{@}(\varepsilon) \tag{7}$$

where @ stands for x, y, ψ, τ, U_e. $\pounds' s$ and $\Re' s$ are real-valued and are at least differentiable in the real argument ε.

B. The invariance analysis

For invariance invoking the group (7) in (4) ,(5) ,(6) and applying chain rule for transforming the derivatives we get

$$\frac{\partial \bar{\psi}}{\partial \bar{y}}\,\frac{\partial^2 \bar{\psi}}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{x}}\,\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} - \frac{\partial \bar{\tau}}{\partial \bar{y}} - \bar{U}_e\,\frac{d\bar{U}_e}{d\bar{x}}$$
$$= \delta_1(\varepsilon)\,(\frac{\partial \psi}{\partial y}\,\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\,\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \tau}{\partial y} - U_e\,\frac{dU_e}{dx})$$

$$(\frac{\alpha}{1+\sqrt{\beta}\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}} + 1)\;\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} \;=\; \delta_2(\varepsilon)\;(\frac{\alpha}{1+\sqrt{\beta}\frac{\partial^2 \psi}{\partial y^2}} + 1)\;\frac{\partial^2 \psi}{\partial y^2}$$

Under the one parameter group transformation (7), the above equations become

$$(\frac{\pounds^{\psi}}{\pounds^{y}})\,\frac{\partial \psi}{\partial y}\,(\frac{\pounds^{\psi}}{\pounds^{y}\pounds^{x}})\,\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\pounds^{\psi}}{\pounds^{x}})\,\frac{\partial \psi}{\partial x}\,(\frac{\pounds^{\psi}}{(\pounds^{y})^2})\,\frac{\partial^2 \psi}{\partial y^2} - (\frac{\pounds^{\tau}}{\pounds^{y}})\,\frac{\partial \tau}{\partial y} \;-$$
$$(\pounds^{U_e} U_e + \Re^{U_e})\,(\frac{\pounds^{U_e}}{\pounds^{x}})\,\frac{dU_e}{dx}$$
$$= \delta_1(\varepsilon)\,(\frac{\partial \psi}{\partial y}\,\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\,\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \tau}{\partial y} - U_e\,\frac{dU_e}{dx})$$

$$(\frac{\alpha}{1+\sqrt{\beta}\,(\frac{\pounds^{\psi}}{(\pounds^{y})^2})\,\frac{\partial^2 \psi}{\partial y^2}} + 1)\,(\frac{\pounds^{\psi}}{(\pounds^{y})^2})\,\frac{\partial^2 \psi}{\partial y^2}$$
$$= \delta_2(\varepsilon)\,(\frac{\alpha}{1+\sqrt{\beta}\;\frac{\partial^2 \psi}{\partial y^2}} + 1)\;\frac{\partial^2 \psi}{\partial y^2}$$

For the invariance of above equations

$$\frac{{\pounds^{\psi}}^2}{\pounds^{x}(\pounds^{y})^2} = \frac{\pounds^{\tau}}{\pounds^{y}} = \frac{(\pounds^{U_e})^2}{\pounds^{x}} = \;\delta_1(\varepsilon) \quad \text{and} \quad \$_1 = 0$$

where $\quad \$_1 = \Re^{U_e}\,(\frac{\pounds^{U_e}}{\pounds^{x}})\,\frac{dU_e}{dx}$

$$\pounds^{\tau} = 1 = \frac{\pounds^{\psi}}{(\pounds^{y})^2} = \;\delta_2(\varepsilon) \quad \text{and} \quad \Re^{\tau} = 0$$

The invariance of boundary conditions give :

$$\Re^{y} = 0, \qquad \Re^{U_e} = 0 \qquad \text{and} \qquad \frac{\pounds^{\psi}}{\pounds^{y}} = \pounds^{U_e}$$

On solving these we obtained

$$\pounds^{x} = (\pounds^{y})^3, \qquad \pounds^{\psi} = (\pounds^{y})^2, \qquad \pounds^{U_e} = \pounds^{y}, \qquad \pounds^{\tau} = 1$$

$$\Re^{y} = \Re^{U_e} = \Re^{\tau} = 0$$

Finally, we get the one-parameter group $\bar{G}$, which transforms invariantly the differential equation (4) and the auxiliary conditions (6).

The group $\bar{G}$ is of the form :

$$\bar{G}: \quad \begin{aligned} \bar{x} &= (\pounds^{y})^3(\varepsilon)\;\; x + \Re^{x}(\varepsilon) \\ \bar{y} &= \pounds^{y}(\varepsilon)\;\; y \\ \bar{\psi} &= (\pounds^{y})^2(\varepsilon)\;\; \psi + \Re^{\psi}(\varepsilon) \\ \bar{\tau} &= \tau \\ \bar{U_e} &= \pounds^{y}(\varepsilon)\;\; U_e \end{aligned} \tag{8}$$

C. The complete set of absolute invariants

Now, We proceed in our analysis to obtain a complete set of absolute invariants so that the original problem will transformed into an ordinary differential equation (similarity representation) in a similarity variable via group theoretic method. We have applied HAMAD [15] formulations for PDEs of 2- independent variables.

By considering $x_1 = x,\ x_2 = y,\ y_1 = \psi,\ y_2 = \tau,\ y_3 = U_e$ and the definitions of α_i , β_i ; i = 1 to 5 we get

$$\alpha_i = \frac{\partial \pounds^i}{\partial \varepsilon}|_{\varepsilon=\varepsilon_0} \quad \text{and} \quad \beta_i = \frac{\partial \Re^i}{\partial \varepsilon}|_{\varepsilon=\varepsilon_0} \quad ; i = 1 \ \ to \ \ 5$$

Where ε_0 denotes the value of ε which yield the identity element of the group. The generator is given by

$$X = (\alpha_1\, x_1 + \beta_1)\frac{\partial g}{\partial x_1} + (\alpha_2\, x_2 + \beta_2)\frac{\partial g}{\partial x_2} + (\alpha_3\, y_1 + \beta_3)\frac{\partial g}{\partial y_1} + (\alpha_4\, y_2 + \beta_4)\frac{\partial g}{\partial y_2} + (\alpha_5\, y_3 + \beta_5)\frac{\partial g}{\partial y_3}$$

Hence characteristic equ. becomes

$$\frac{dx}{\alpha_1\, x + \beta_1} = \frac{dy}{\alpha_2\, y} = \frac{d\psi}{\alpha_3\, \psi + \beta_3} = \frac{d\tau}{0} = \frac{dU_e}{\alpha_5\, U_e}$$

On solving this with the help of the relations between α_i and β_i from equ. (8) we obtained similarity variables as follow :

$$\eta = y\,(x+\beta)^{-\frac{1}{3}} \quad where \quad \beta = \frac{\beta_1}{\alpha_1}$$

$$\psi = (x+\beta)^{\frac{2}{3}}\ F(\eta) - \frac{\beta_3}{\alpha_3} \qquad (9)$$

$$\tau = H(\eta)$$

$$U_e = (x+\beta)^{\frac{1}{3}}\ F_1(\eta)$$

D. The reduction to an ordinary differential equation

The similarity transformations (9) maps equs. (4) to (6) into the following nonlinear ordinary differential equations :

$$F'^2 - 2FF'' - 3H' - 1 = 0 \qquad (10)$$

$$H = \left(\frac{\alpha}{(1+\sqrt{\beta}\ F'')^2} + 1\right)\ F'' \qquad (11)$$

$$F(0) = 0, \quad F'(0) = 0, \quad F'(\infty) = 1 \qquad (12)$$

Without loss of generality we assume $F_1(\eta) = 1$.

V. The Numerical Solution

Substituting the equ. (11) in (10) we obtained the non-linear ordinary differential equation as :

$$F''' = \frac{\frac{1}{\alpha}\,(1+\sqrt{\beta}\ F'')^2}{1+\frac{1}{\alpha}\,(1+\sqrt{\beta}\ F'')^2}\ (F'' - 2FF' - 1) \qquad (13)$$

The numerical method applied to solve equ. (13) with the boundary conditions (12) is due to Nachtsheim and Swigert [16] based on the least square convergence criterion. The asymptotic boundary conditions are satisfied at the edge of the boundary layer by adjusting the initial conditions so that the mean square error between the computed variables and asymptotic values is minimized. Here to solve equation (13) with boundary condition (12) is equivalent to the problem of finding a value of $F''(0)$ for which the boundary condition at the edge of boundary layer is satisfied. That is the solution of non-linear equation $F'_{edge}[F''(0)] = 1$ is to be determined where $F'_{edge} = F'(\eta_{edge})$. Denoting $F''(0) = x$ and following the asymptotic boundary conditions method ,

$$\aleph = (1+\sqrt{\beta}\ \ F'')^2$$

$$F''' = \frac{\frac{1}{\alpha}\ \aleph\ (F''-2FF''-1)}{1+\frac{1}{\alpha}\ \aleph} \qquad (14)$$

$$F(0) = 0, \quad F'(0) = 0, \quad F''(0) = x \qquad (15)$$

The perturbation differential equation for the x derivative is written as

$$F'''(x) = \frac{\frac{2}{\alpha}\ \sqrt{\beta\,\aleph}\ F_x'''\ (F'^2-2FF''-1)}{1+\frac{1}{\alpha}\ \aleph} + \frac{\frac{2}{\alpha}\ \aleph\ (F'F_x'-F_xF''-FF_X'')}{1+\frac{1}{\alpha}\ \aleph} - \frac{\frac{2}{\alpha}\ \sqrt{\beta\,\aleph^3}\ F_x'''\ (F'^2-2FF''-1)}{(1+\frac{1}{\alpha}\ \aleph)^2}$$
(16)

$$F_x(0) = 0, \quad F_x'(0) = 0, \quad F_x''(0) = 1 \qquad (17)$$

Following the principal of least squares principal, $\triangle x$ is given by

$$\triangle x = \frac{F_x' - F'F_x' - F''F_x''}{F_x'^2 + F_x''^2} \qquad (18)$$

The error E between the asymptotic conditions and the computed values at $\eta = \eta_{stop}$ is given by

$$E = \frac{(1-F')^2}{F''^2} \qquad (19)$$

Assuming $x = 1$ the equations (14) and (16) along with their boundary conditions (15) and (17) respectively are integrated using the Adams-Moulton procedure. Integration is carried out using MATLAB ode solver with the step size 0.05. For each value of η the tolerance value of $\frac{\triangle x}{x}$ is specified. Starting from $\eta = 0, h = 0.05$ integration is performed until η_{stop} is equal to specified value. At this step, the value of $\frac{\triangle x}{x}$ is checked. If it is found to be more than the specified tolerance the procedure is repeated with $x + \triangle x$. When value of $\frac{\triangle x}{x}$ is within the tolerance range, error value E is computed and tested against the tolerance value specified which is taken to be 10^{-8}. If the value of computed E exceeds the specified value, η_{stop} is increased and the complete procedure is repeated. Controlling the non-dimensional numbers $\alpha = 0.1$ and then for $\beta = 5{*}10^3$; $\beta = 5{*}10^4$;$\beta = 5{*}10^5$ the velocity profile and the slope of velocity profile are generated. (See figure-1 and figure-2). Both α and β have great influence on the velocity of the Williamson fluids. It should be observed that for fix value of α, the velocity of fluid is increases rapidly and approaches to one as β increases. Also, the slope of velocity profile in figure (2) is found always decreases fast and approaches to zero as β increases. The physical quantity of interest is the coefficient of local skin friction C_f which is given by the equation

$$\tau_w = \left(\frac{\alpha}{1+\sqrt{\beta}\ F''(0)} + 1\right)F''(0)$$

$$\frac{\sqrt{3}}{2}\ C_f\ \sqrt{R_e} = \tau_w$$

This skin friction (fig. - 3) is plotted for different values of α keeping β fixed and vice versa. All Figures 1 - 3 are plotted in terms of dimensionless parameters.

VI. Conclusion

Numerical solution is generated for non-Newtonian Williamson fluids flowing over a 90^0 wedge. Numerical results are presented in the form of graphs. The method appears to be insensitive to initial guesses and converges quickly to the solution. By this method the implicit study of differential equations obtaining a parameter has also been reduced to an automatic technique.

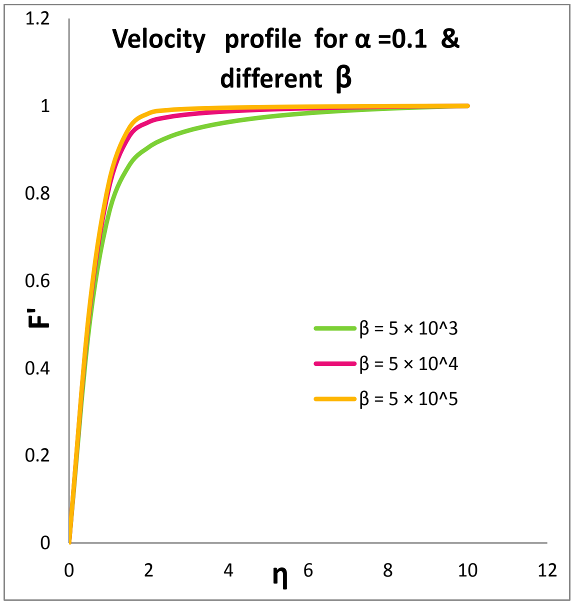

Figure – 1

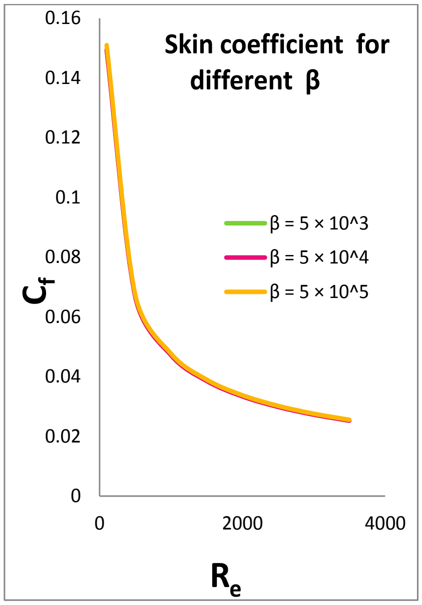

Figure – 3a

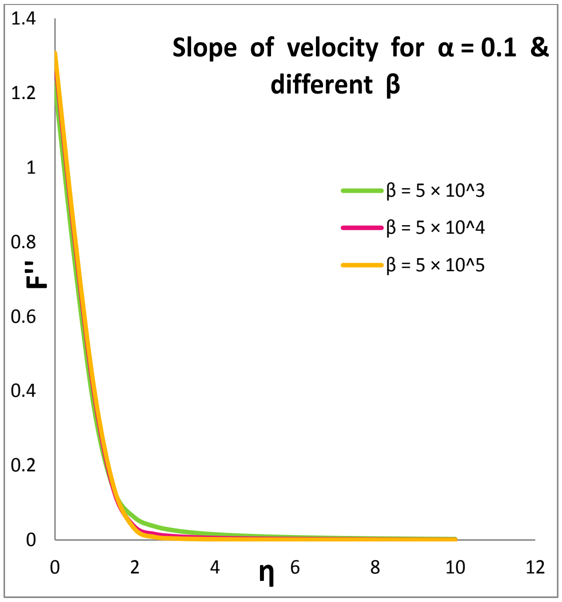

Figure –2

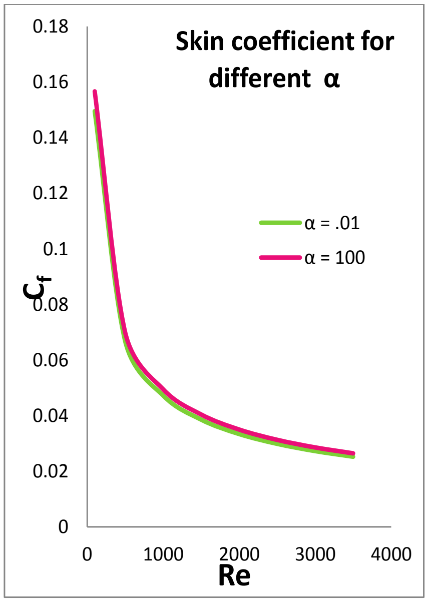

Figure – 3b

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