Fixed Points of Almost Generalized \(\alpha\)-\(\psi\)-Contractive Maps

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** Abstract **— We introduce almost generalized \(\alpha\)-\(\psi\)-contractive maps and prove the existence and uniqueness of fixed points of almost generalized \(\alpha\)-\(\psi\)-contractive maps in partially ordered sets endowed with a metric. Our results extend and generalize the results of Samet, Vetro and Vetro [16], Karapinar and Samet [8] and Cirić, Abbas, Saadati and Hussain [6]. Furthermore, we provide examples in support of our results.

** Index Terms **— Fixed points, \(\alpha\)-admissible maps, almost generalized \(\alpha\)-\(\psi\)-contractive maps.

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I. INTRODUCTION

Recently several authors studied fixed point theorems in partially ordered sets endowed with a metric. Ran and Reurings [15] and Nieto and Lopez [14] proved Banach contraction principle in partially ordered sets endowed with a metric. Agarwal, El-Gebeily and O’Regan [1] have proved some fixed point results for monotone operators in ordered metric spaces endowed with a partial order using a weak generalized contraction type assumption. For more works in this line of research we refer [2, 3, 7, 11, 12, 13].

Throughout this paper we denote by \(\Psi\) the family of nondecreasing functions \(\psi : [0, \infty) \to [0, \infty)\) which satisfies \(\sum_{n=1}^{\infty} \psi^n(t) < \infty\) for each \(t > 0\) where \(\psi^n\) is the \(n\)-th iterate of \(\psi\).

** Remark 1.1:** Any function \(\psi \in \Psi\) satisfies \(\lim_{n \to \infty} \psi^n(t) = 0\), \(\psi(t) < t\) for any \(t > 0\) and \(\psi\) is continuous at \(0\).

Let \((X, \preceq)\) be a partially ordered set and \(T : X \to X\).

We say that \(T\) is nondecreasing with respect to \(\preceq\) if \(x, y \in X\), \(x \preceq y \Rightarrow Tx \preceq Ty\).

Recently, Samet, Vetro and Vetro [16] introduced a new concept namely \(\alpha\)-\(\psi\)-contractive mappings and proved the existence of fixed points of such mappings in metric space setting.

** Definition 1.2:** [16] Let \((X, d)\) be a metric space and \(T : X \to X\). We say that \(T\) is \(\alpha\)-\(\psi\)-contractive mapping if there exist two functions \(\alpha : X \times X \to [0, \infty)\) and \(\psi \in \Psi\) such that:

\[
\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \quad \text{for all} \quad x, y \in X.
\]

** Definition 1.3:** [16] Let \((X, d)\) be a metric space, \(T : X \to X\) and \(\alpha : X \times X \to [0, \infty)\), \(\psi \in \Psi\). We say that \(T\) is \(\alpha\)-admissible if \(x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1\).

For examples on \(\alpha\)-admissible functions, we refer [15].

** Theorem 1.4:** [16] Let \((X, d)\) be a complete metric space and \(T : X \to X\). Suppose that there exist two functions \(\alpha : X \times X \to [0, \infty)\) and \(\psi \in \Psi\) such that:

\[
\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \quad \text{for all} \quad x, y \in X.
\]

Also, assume that:
(i) \(T\) is \(\alpha\)-admissible;
(ii) there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\); and
(iii) \(T\) is continuous.

Then, \(T\) has a fixed point, \(i.e.,\), there exists \(u \in X\) such that \(Tu = u\).

** Theorem 1.5:** [16] Let \((X, d)\) be a complete metric space and \(T : X \to X\). Suppose that there exist two functions \(\alpha : X \times X \to [0, \infty)\) and \(\psi \in \Psi\) such that:

\[
\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \quad \text{for all} \quad x, y \in X.
\]

Also, assume that:
(i) \(T\) is \(\alpha\)-admissible;
(ii) there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\); and
(iii) \(T\) is continuous.

Then, \(T\) has a fixed point, \(i.e.,\), there exists \(u \in X\) such that \(Tu = u\).

Recently, Karapinar and Samet [8] introduced generalized \(\alpha\)-\(\psi\)-contractive mappings and proved fixed point results.

** Definition 1.6:** [8] Let \((X, d)\) be a metric space and \(T : X \to X\) be a given mapping. We say that \(T\) is a generalized \(\alpha\)-\(\psi\)-contractive mapping if there exist two functions \(\alpha : X \times X \to [0, \infty)\) and \(\psi \in \Psi\) such that:

\[
\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all} \quad x, y \in X
\]

where

\[
M(x, y) = \max\{d(x, y), \frac{(x, Tx)+d(y, Ty)}{2}, \frac{d(x, Ty)+d(y, Tx)}{2}\}
\]

** Theorem 1.7:** [8] Let \((X, d)\) be a complete metric space and \(T : X \to X\). Suppose that there exist two functions \(\alpha : X \times X \to [0, \infty)\) and \(\psi \in \Psi\) such that:

\[
\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all} \quad x, y \in X
\]

where

\[
M(x, y) = \max\{d(x, y), \frac{(x, Tx)+d(y, Ty)}{2}, \frac{d(x, Ty)+d(y, Tx)}{2}\}
\]

Also, assume that the following conditions are satisfied:
(i) \(T\) is \(\alpha\)-admissible;
(ii) there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\); and
(iii) \(T\) is continuous.

Then there exists \(u \in X\) such that \(Tu = u\).

** Theorem 1.8:** [8] Let \((X, d)\) be a complete metric space and \(T : X \to X\). Suppose that there exist two functions \(\alpha : X \times X \to [0, \infty)\) and \(\psi \in \Psi\) such that:

\[
\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all} \quad x, y \in X
\]

where
\[ M(x, y) = \max\{d(x, y), \frac{(x, Tx)+d(y, Ty)}{2}, \frac{(x, Ty)+d(y, Tx)}{2}\}. \]

Also, assume that the following conditions are satisfied:

(i) \( T \) is \( \alpha \) - admissible;
(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \); and
(iii) if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \) and \( x_n \rightarrow x \) as \( n \rightarrow \infty \), then there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n_k}, x) \geq 1 \) for all \( k \).

Then there exists \( u \in X \) such that \( Tu = u \).

For more works in this line of research we refer [9, 10].

In 2004, Berinde [5] introduced ‘weak contraction maps’ which are named as ‘almost contraction maps’ as a generalization of contraction maps and proved fixed point results in complete metric spaces.

**Definition 1.9:** [5] Let \((X, d)\) be a metric space. A map \(T : X \rightarrow X\) is called an ‘almost contraction’ if there exists a constant \( \delta \in (0, 1) \) and \( L \geq 0 \) such that

\[ d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \quad \text{for all} \quad x, y \in X. \]

In 2008, Babu, Sandhya and Kameshwari [4] modified the above definition by introducing ‘condition \( B\)’ and proved a fixed point theorem in complete metric spaces.

**Definition 1.10:** [4] Let \((X, d)\) be a metric space a map \(T : X \rightarrow X\) is said to satisfy ‘condition \( B\)’ if there exist a \( 0 < \delta < 1 \) and \( L \geq 0 \) such that

\[ d(Tx, Ty) \leq \delta d(x, y) + \min\{d(x, Tx), d(y, Ty)\}, \]

\[ d(x, Ty), d(y, Tx) \quad \text{for all} \quad x, y \in X. \]

In 2011, Ciric, Abbas, Saadati and Hussain [6] proved the following fixed point results of an almost generalized contractive maps in ordered metric spaces.

**Theorem 1.11:** [6] Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Let \( T : X \rightarrow X\) be a strictly increasing continuous mapping with respect to \( \preceq \). Suppose that there exists \( \delta \in (0, 1) \) and \( L \geq 0 \) such that

\[ d(Tx, Ty) \leq \delta M(x, y) + L\min\{d(x, Tx), d(y, Ty)\}, \]

\[ d(x, Ty), d(y, Tx) \quad \text{for all} \quad x, y \in X \]

where

\[ M(x, y) = \max\{d(x, y), (x, Tx), d(y, Ty)\}, \]

\[ d(x, Ty), d(y, Tx) \]

If there exists \( x_0 \in X \) such that \( x_0 \preceq Tx_0 \) and for an increasing sequence \( \{x_n\} \) in \( X \) converging to \( x \in X \) we have \( x_n \preceq x \) for all \( n \). Then \( T \) has a fixed point in \( X \).

In this paper, we introduce almost generalized \( \alpha-\psi\)-contractive maps and prove the existence and uniqueness of fixed points in partially ordered sets endowed with a metric. Our results extend and generalize the results of Samet, Vetro and Vetro [16] and that of Karapinar and Samet [8]and Ciric, Abbas, Saadati and Hussain [6]. Furthermore, we provide examples in support of our results.

**II. MAIN RESULTS**

**Theorem 2.1:** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Let \( T : X \rightarrow X\) be a nondecreasing map with respect to \( \preceq \). Suppose that there exist two functions \( \alpha : X \times X \rightarrow [0, \infty) \) and \( \psi \in \Psi \) and \( L \geq 0 \) such that

\[ \alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) + LN(x, y) \] (2.1.1)

for all \( x, y \in X \) with \( x \nless y \) where

\[ M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)+d(y, Tx)}{2}\} \]

and

\[ N(x, y) = \min\{d(x, Ty), d(y, Tx)\}. \]

Also, assume that

(i) \( T \) is \( \alpha \) - admissible;
(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) with \( x_0 \nless Tx_0 \); and
(iii) \( T \) is continuous.

Then \( T \) has a fixed point, i.e., there exists \( x^* \in X \) such that \( x^* = Tx^* \).

**Proof.** By (ii), suppose that there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) with \( x_0 \nless Tx_0 \). We define a sequence \( \{x_n\} \) in \( X \) by \( x_{n+1} = Tx_n \) for \( n \in \{0, 1, 2, \ldots\} \). (2.1.2)

If \( x_n = x_{n+1} \) for some \( n \), then \( x_n = Tx_n \) and hence \( x_n \) is a fixed point of \( T \).

Now assume that \( d(x_n, x_{n+1}) > 0 \) for all \( n \). Since \( \alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \), by (i) it follows that \( \alpha(Tx_0, x_1) = \alpha(x_1, x_2) \geq 1 \).

Inductively, we have

\[ \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all} \quad n \geq 0. \] (2.1.3)

Since \( T \) is nondecreasing and \( x_0 \nless Tx_0 = x_1 \) we have

\[ x_1 = Tx_0 \nless Tx_1 = x_2, \quad \text{i.e.,} \quad x_1 \nless x_2. \] (2.1.4)

Inductively we have \( x_n \nless x_{n+1} \) for all \( n \geq 0 \).

Hence we have \( x_0 \nless x_1 \nless x_2 \nless \cdots \nless x_n \nless x_{n+1} \nless \cdots \).

Now, from (2.1.1), (2.1.3) and (2.1.4) we have
\[ d(x_n, x_{n+1}) = d(Tx_n, x_n) \]
\[ \leq \alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \]
\[ \leq \psi(d(M(x_{n-1}, x_n)) + LN(x_{n-1}, x_n) \] (2.1.5)

where

\[ M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_n, Tx_{n-1})+d(x_{n-1}, Tx_n)}{2}\}. \]
\[ d(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty) \} \]

and

\[ N(x, y) = \min \{d(x, Ty), d(y, Tx) \}. \]

Also, assume that

(i) \( T \) is \( \alpha^* \)-admissible;

(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) with \( x_0 \leq Tx_0 \); and

(iii) if \( \{x_n\} \) is a nondecreasing sequence in \( X \) such that

\[ x_n \to x^* \text{ then } x_n \leq x^*, \] also if \( \alpha(x_n, x_{n+1}) \geq 1 \), then

\[ \alpha(x_n, x^*) \geq 1 \text{ for all } n. \]

Then \( T \) has a fixed point, i.e., there exists \( x^* \in X \) such that

\[ x^* = Tx^*. \]

**Proof.** From the proof of the Theorem 2.1, the sequence \( \{x_n\} \) defined by \( x_{n+1} = Tx_n \) is Cauchy in \( X \). Since \( X \) is complete there exists \( x^* \in X \) such that \( x^* = \lim_{n \to \infty} x_n. \)

From \( \alpha(x_n, x_{n+1}) \geq 1 \) we have \( \alpha(x_n, x_*) \geq 1 \) for all \( n \) and also \( x_n \leq x^* \). Now, we show that \( Tx^* = x^* \). Assume that \( d(Tx^*, x^*) > 0 \).

On using (2.2.1), we have

\[ d(x_{n+1}, Tx^*) = d(x_{n+1}, x^*) \leq \alpha(x_n, x^*) d(x_n, Tx^*) \]

\[ \leq \psi(M(x_n, x^*)) + LN(x_n, x^*) \]

\[ \leq \psi(M(x_n, x^*)) + LN(x_n, x^*) \]

and

\[ N(x_n, x^*) = \min \{d(x_n, Tx^*), d(x^*, x_{n+1}) \}. \]

On taking limits as \( n \to \infty \) in (2.2.3) and (2.2.4) we have

\[ \lim_{n \to \infty} M(x_n, x^*) = \lim_{n \to \infty} \max \{d(x_n, x^*), d(x_n, x_{n+1}) \}, \]

\[ \lim_{n \to \infty} d(x^*, Tx^*) = \lim_{n \to \infty} \frac{d(x^*, Tx^*) + d(x^*, x_{n+1})}{2} \]

\[ = \max \{0, d(x^*, Tx^*) \}, \]

\[ = d(x^*, Tx^*). \]

and

\[ \lim_{n \to \infty} N(x_n, x^*) = \lim_{n \to \infty} \min \{d(x_n, Tx^*), d(x^*, x_{n+1}) \}, \]

\[ = \min \{d(x^*, Tx^*), 0 \} = 0. \]

Now, since

\[ d(x^*, Tx^*) \leq M(x_n, x^*) \text{ and } \lim_{n \to \infty} M(x_n, x^*) = d(x^*, Tx^*) \]

we conclude that

\[ M(x_n, x^*) \to d(x^*, Tx^*)^+ \text{ as } n \to \infty. \]

Since \( \psi \) satisfies the property \( \lim_{t \to t^*} \psi(t) < t \) for all \( t > 0 \), on taking limits as \( n \to \infty \) in (2.2.2), by using (2.2.5), (2.2.6) and (2.2.7), we have

\[ d(x^*, Tx^*) \leq \lim_{n \to \infty} \psi(M(x_n, x^*)) < d(x^*, Tx^*), \]

a contradiction. Therefore \( d(x^*, Tx^*) = 0 \), i.e., \( x^* = Tx^* \).

In order to obtain the uniqueness of fixed points of almost generalized \( \alpha^*-\psi^* \)-contractive mappings we use the following hypotheses:
(H): for all \( x, y \in X \) there exists \( z \in X \) such that \( z \) is comparable to \( x \) and \( y \) and \( \alpha(x, z) \geq 1 \) and \( \alpha(y, z) \geq 1 \) and also \( z \leq Tz, \alpha(z, Tz) \geq 1 \). Moreover, we replace
\[
N(x, y) = \min \{d(x, Ty), d(y, Tx)\} \text{ of inequality (2.1.1(respectively (2.2.1)) by}
\]
\[
N'(x, y) = \min \{d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\}.
\]

**Theorem 2.3:** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Let \( T : X \to X \) be a nondecreasing map with respect to \( \preceq \). Suppose that there exist two functions \( \alpha : X \times X \to [0, \infty) \) and \( \psi \in \Psi \) with \( \lim_{r \to t^+} \psi(r) < t \) for all \( t > 0 \) and \( L \geq 0 \) such that
\[
\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) + LN'(x, y)
\]
for all \( x, y \in X \) with \( x \preceq y \)

and
\[
\alpha(x, y)d(Tx, Ty) = \max \{d(x, Ty), d(y, Ty), d(x, Ty), d(y,Tx)\}
\]

Also, assume that
(i) \( T \) is \( \alpha \)-admissible;
(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \)
with \( x_0 \leq Tx_0 \);
(iii) \( T \) is continuous; and
(iv) condition (H) holds.

Then \( T \) has a unique fixed point.

**Proof.** Since the inequality (2.3.1) implies the inequality (2.1.1), by the proof of Theorem 2.1 the set of fixed points of \( T \) is non-empty. Suppose that \( x^* \) and \( y^* \) are two distinct fixed points of \( T \). By our assumption (H), there exists \( z \in X \) such that \( z \) is comparable to \( x^* \) and \( y^* \) and \( \alpha(x^*, z) \geq 1 \) and \( \alpha(y^*, z) \geq 1 \). Now, put \( z = z_0 \) and choose \( z_1 \in X \) such that \( z_1 = Tx_0 \).

We define sequence \( \{z_n\} \) in \( X \) by \( z_{n+1} = Tz_n \) for all \( n \geq 0 \). Since \( z \) is comparable to \( x^* \) and \( y^* \) it follows that \( x^* \preceq z_n \) and \( y^* \preceq z_n \). Inductively, we can show that \( x^* \preceq z_n \) and \( y^* \preceq z_n \) (2.3.2)
for all \( n \geq 1 \).

Since \( \alpha(x^*, z) \geq 1 \) and \( \alpha(y^*, z) \geq 1 \) and \( T \) is \( \alpha \)-admissible we have
\[
\alpha(Tx^*, Tz) = \alpha(x^*, z_1) \geq 1 \text{ and } \alpha(Ty^*, Tz) = \alpha(y^*, z_1) \geq 1.
\]
Inductively, we can show that
\[
\alpha(x^*, z_n) \geq 1 \text{ and } \alpha(y^*, z_n) \geq 1.
\]
Now, on taking \( x = x^* \), \( y = z_n \) and using (2.3.2) and (2.3.3) in (2.3.1), we have
\[
d(x^*, z_{n+1}) = d(Tx^*, Tz_n) \leq \alpha(x^*, z_n)d(Tx^*, Tz_n)
\]
\[
\leq \psi(M(x^*, z_n)) + LN(x^*, z_n)
\]
where
\[
M(x^*, z_n) = \max \{d(x^*, z_n), d(x^*, Tx^*), d(z_n, Tz_n),
\]
and
\[
N'(x^*, z_n) = \min \{d(x^*, Tx^*), d(z_n, z_{n+1}), d(x^*, z_{n+1}),
\]
\[
d(z_n, x^*)\}
\]
\[
= \max \{d(x^*, z_n), d(z_n, z_{n+1}), d(x^*, z_{n+1}),
\]
\[
d(z_n, x^*)\}
\]
\[
\leq \max \{d(x^*, z_n), d(z_n, z_{n+1}), d(x^*, z_{n+1}),
\]
\[
d(z_n, x^*)\}
\]
\[
= \min \{0, d(z_n, z_{n+1}), d(x^*, z_{n+1}),
\]
\[
d(z_n, x^*)\} = 0.
\]
Thus by the monotonicity of \( \psi \) we have
\[
d(x^*, z_{n+1}) \leq \psi(\max \{d(x^*, z_n), d(z_n, z_{n+1}), d(x^*, z_{n+1})\}).
\]
We assume without loss of generality that \( M(x^*, z_n) > 0 \) for all \( n \). Now, we consider the following three cases:

Case(1): \( \max \{d(x^*, z_n), d(x^*, z_{n+1}), d(z_n, z_{n+1})\} = d(x^*, z_{n+1}). \)

In this case, we have
\[
d(x^*, z_{n+1}) \leq \psi(d(x^*, z_{n+1})) < d(x^*, z_{n+1}), \text{ a contradiction.}
\]

Case(2): \( \max \{d(x^*, z_n), d(x^*, z_{n+1}), d(z_n, z_{n+1})\} = d(x^*, z_{n}). \)

Then
\[
d(x^*, z_{n+1}) \leq \psi(d(x^*, z_{n})) \leq \psi^2(d(x^*, z_{n-1})
\]
\[
\leq \psi^3(d(x^*, z_{n-2}) \leq \cdots \leq \psi^n(d(x^*, z_{1}) \to 0
\]
as \( n \to \infty \).

Therefore, \( d(x^*, z_{n+1}) \to 0 \) as \( n \to \infty \).

Similarly, \( d(y^*, z_{n+1}) \to 0 \) as \( n \to \infty \). Hence \( x^* = y^* \).

Case(3): \( \max \{d(x^*, z_n), d(x^*, z_{n+1}), d(z_n, z_{n+1})\} = d(z_n, z_{n+1}). \)

In this case, we have
\[
d(x^*, z_{n+1}) \leq \psi(d(z_n, z_{n+1})) \leq \psi^2(d(z_{n-1}, z_n)
\]
\[
\leq \psi^3(d(z_{n-2}, z_{n-3}) \leq \cdots \leq \psi^n(d(z_{0}, z_{1}) \to 0 \text{ as } n \to \infty.
\]

Therefore, \( d(x^*, z_{n+1}) \to 0 \) as \( n \to \infty \).

Similarly, \( d(y^*, z_{n+1}) \to 0 \) as \( n \to \infty \). Hence \( x^* = y^* \).

**Theorem 2.4:** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Let \( T : X \to X \) be a nondecreasing map with respect to \( \preceq \). Suppose that there exist two functions \( \alpha : X \times X \to [0, \infty) \) and \( \psi \in \Psi \) with \( \lim_{r \to t^+} \psi(r) < t \) for all \( t > 0 \) and \( L \geq 0 \) such that
\( \alpha(x, y) d(Tx, Ty) \leq \psi(M(x, y)) + LN'(x, y) \) (2.4.1)

for all \( x, y \in X \) with \( x \preceq y \)

where

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx) \}
\]

and

\[
N'(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx) \}.
\]

Also, assume that

(i) \( T \) is \( \alpha \)-admissible;

(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \)

with \( x_0 \preceq Tx_0 \);

(iii) if \( \{x_n\} \) is a nondecreasing sequence in \( X \) such that

\[ x_n \rightarrow x^* \] then \( x_n \preceq x^* \), also if \( \alpha(x_n, x_{n+1}) \geq 1 \), then

\[ \alpha(x_n, x^*) \geq 1 \] for all \( n \); and

(iv) condition (H) holds.

Then \( T \) has a unique fixed point.

Proof. Runs on the same lines as that of Theorem 2.3.

III. COROLLARIES AND EXAMPLES

Corollary 3.1: Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Let \( T : X \rightarrow X \) be a nondecreasing map with respect to \( \preceq \). Suppose that there exist two functions \( \alpha : X \times X \rightarrow [0, \infty) \) and \( \psi \in \Psi \) and \( L \geq 0 \) such that

\[
\alpha(x, y)d(Tx, Ty) \leq \psi(M'(x, y)) + LN(x, y)
\]

for all \( x, y \in X \) with \( x \preceq y \).

where

\[
M'(x, y) = \max\{d(x, y), d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx) \}
\]

and

\[
N(x, y) = \min\{d(x, Ty), d(y, Tx) \}.
\]

Also, assume that

(i) \( T \) is \( \alpha \)-admissible;

(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) with \( x_0 \preceq Tx_0 \); and

(iii) \( T \) is continuous.

Then \( T \) has a fixed point, i.e., there exists \( x^* \in X \) such that

\[ x^* = Tx^* \]

for all \( x, y \in X \) with \( x \preceq y \)

where

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx) \}, \quad \frac{d(x, Ty) + d(y, Tx)}{2}
\]

Also, assume that

(i) \( T \) is \( \alpha \)-admissible;

(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) with \( x_0 \preceq Tx_0 \); and

(iii) \( T \) is continuous.

Then \( T \) has a fixed point, i.e., there exists \( x^* \in X \) such that

\[ x^* = Tx^* \]

for all \( x, y \in X \) with \( x \preceq y \)

where

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx) \}, \quad \frac{d(x, Ty) + d(y, Tx)}{2}
\]

Also, assume that

(i) \( T \) is \( \alpha \)-admissible;

(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) with \( x_0 \preceq Tx_0 \); and

(iii) \( T \) is continuous.

Then \( T \) has a fixed point, i.e., there exists \( x^* \in X \) such that

\[ x^* = Tx^* \]
Remark 3.7: Theorem 1.11 follows as a corollary to Theorem 2.1, since the inequality (1.1.1) follows from the inequality (2.1.1) with $\psi(t) = \delta t$, $\delta \in [0, 1)$, $t \geq 0$; and $\alpha(x, y) = 1$ for all $x, y \in X$.

By choosing $\psi(t) = kt$, $0 \leq k < 1$ in Theorem 2.2 we get the following:

Corollary 3.8: Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist a function $\alpha : X \times X \to [0, \infty)$, and $L \geq 0$ such that
\[\alpha(x, y)d(Tx, Ty) \leq kM(x, y) + LN(x, y)\]
for all $x, y \in X$ with $x \preceq y$ where
\[M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}\]
and
\[N(x, y) = \min\{d(x, Ty), d(y, Tx)\} .\]

Also, assume that
(i) $T$ is $\alpha$-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) if $\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \preceq x^*$ then $x_n \preceq x^*$, also if $\alpha(x_n, x_{n+1}) \geq 1$, then $\alpha(x_n, x^*) \geq 1$ for all $n$.

Then $T$ has a fixed point, i.e., there exists $x^* \in X$ such that $x^* = Tx^*$.

Remark 3.9: Theorem 1.12 follows as a corollary to Theorem 2.2, since the inequality (1.1.1) follows from the inequality (2.2.1) with $\psi(t) = \delta t$, $\delta \in [0, 1)$, $t \geq 0$; and $\alpha(x, y) = 1$ for all $x, y \in X$.

The following result is an immediate consequence of Corollary 3.3.

Corollary 3.10: Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist a function $\alpha : X \times X \to [0, \infty)$, and $L \geq 0$ such that
\[\alpha(x, y)d(Tx, Ty) \leq kM(x, y) + LN(x, y)\]
for all $x, y \in X$ with $x \preceq y$.

Also, assume that
(i) $T$ is $\alpha$-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) $T$ is continuous.

Then $T$ has a fixed point, i.e., there exists $x^* \in X$ such that $x^* = Tx^*$.

Remark 3.11: If we define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ for all $x, y \in X$ in Corollary 3.10, we get Theorem 2.1 of [14].
\( \alpha(2, 4) d(T_2, T_4) = 2 \leq \psi(M(2, 4)) = \psi(2) \), which is absurd for any \( \psi \in \Psi \) and any \( \alpha \)-admissible function \( T \) with \( \alpha(2, 4) = 1 \).

Hence the inequality (2.1.1) fails to hold when \( L = 0 \) for any \( \alpha \)-admissible function \( T \) with \( \alpha(x, y) \geq 1 \), \( x, y \in X \) and \( \psi \in \Psi \). This example indicates the importance of \( L \) in the inequality (2.1.1) of Theorem 2.1.

Here, we observe that the inequality (1.11.1) fails to hold at \( x = 2 \) and \( y = 4 \) for any \( \delta \in [0, 1) \) and \( L \geq 0 \), for
\[
d(T_2, T_4) = 2 \leq 2\delta = \delta M(2, 4) + L_0 = \delta M(2, 4) + L\min\{d(2, T_2), d(4, T_4), d(2, T_4), d(4, T_2)\}
\]
which is absurd. Hence Theorem 1.11 is not applicable. So by Remark 3.7 and Example 3.12, it follows that Theorem 2.1 is a generalization of Theorem 1.11. Also, we observe that under the setting of this example, the inequalities (1.4.1) and (1.7.1) fail to hold at \( x = 2 \) and \( y = 4 \). Hence Theorem 1.4 and Theorem 1.7 are also not applicable. This phenomenon suggests that Theorem 2.1 is also a generalization of Theorem 1.4 and Theorem 1.7.

The following is also an example in support of Theorem 2.1.

**Example 3.13:** Let \( X = \{0, 1, 2, 3, 4\} \) with the usual metric.

We define a partial ordering on \( X \) as follows
\[
\preceq := \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4), (3, 4), (2, 3)\}
\]
Then \( (X, \preceq) \) is a partially ordered set.

Let \( A = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 4), (2, 0)\} \)
\( B = \{(1, 3), (2, 1), (3, 1), (3, 2), (3, 0), (3, 4)\} \)
\( C = \{(1, 0), (4, 0), (4, 1), (4, 2), (2, 4), (2, 3), (4, 3)\} \)
We define \( T : X \to X \) and \( \alpha : X \times X \to [0, \infty) \) by
\[
T_0 = T_1 = 0, T_2 = 3, T_3 = 4 \text{ and } T_4 = 4 \text{ and } \alpha(x, y) = \begin{cases} 3/2 & \text{if } (x, y) \in A \\ 2 & \text{if } (x, y) \in B \\ 0 & \text{if } (x, y) \in C. \end{cases}
\]

Then \( T \) is continuous, nondecreasing and \( \alpha \)-admissible. Moreover, we choose \( x_0 = 3 \in X \), clearly \( x_0 \preceq T x_0 \) and \( \alpha(x_0, T x_0) = \alpha(3, 4) = 2 \geq 1 \).

Now, we verify the inequality (2.1.1) by choosing \( \psi \in \Psi \) given by \( \psi(t) = \frac{2}{3} t \) for \( t \geq 0 \) and \( L = 2 \).

**Case (1):** \( x = 0 \) and \( y = 2 \).

In this case, \( \alpha(0, 2) d(T_0, T_2) = \frac{9}{2}, M(0, 2) = \frac{5}{2} \) and \( N(0, 2) = 2 \).

Hence, we have
\[
\alpha(0, 2) d(T_0, T_2) = \frac{9}{2} = \frac{9}{2} \leq \frac{5}{2} + 2.2 = \frac{17}{2}
\]
\[
= \psi(M(0, 2)) + LN(0, 2).
\]
**Case (2):** \( x = 0 \) and \( y = 3 \).

Then, \( \alpha(0, 3) d(T_0, T_3) = 6, M(0, 3) = \frac{7}{2} \) and \( N(0, 3) = 3 \).

Hence, we have
\[
\alpha(0, 3) d(T_0, T_3) = 6 \leq \frac{7}{2} + 2.3 = \frac{25}{4}
\]
\[
= \psi(M(0, 3)) + LN(0, 3).
\]
**Case (3):** \( x = 0 \) and \( y = 4 \).

In this case, \( \alpha(0, 4) d(T_0, T_4) = 6, M(0, 4) = 4 \) and \( N(0, 4) = 4 \). Therefore, we have
\[
\alpha(0, 4) d(T_0, T_4) = 6 \leq \frac{7}{3} + 2.4 = \frac{22}{3}
\]
\[
= \psi(M(0, 4)) + LN(0, 4).
\]
**Case (4):** \( x = 1 \) and \( y = 2 \).

Then, \( \alpha(1, 2) d(T_1, T_2) = \frac{9}{2}, M(1, 2) = 2 \) and \( N(1, 2) = 2 \).

Hence, we have
\[
\alpha(1, 2) d(T_1, T_2) = \frac{9}{2} = \frac{9}{2} \leq \frac{5}{2} + 2.2 = \frac{10}{2}
\]
\[
= \psi(M(1, 2)) + LN(1, 2).
\]
**Case (5):** \( x = 1 \) and \( y = 4 \).

In this case, \( \alpha(1, 4) d(T_1, T_4) = 6, M(1, 4) = 3 \) and \( N(1, 4) = 3 \). Hence we have
\[
\alpha(1, 4) d(T_1, T_4) = 6 \leq \frac{7}{3} + 2.3 = 8
\]
\[
= \psi(M(1, 4)) + LN(1, 4).
\]
**Case (6):** \( x = 2 \) and \( y = 4 \).

Then, \( \alpha(2, 4) d(T_2, T_4) = 0.1 = 0, M(2, 4) = 2 \) and \( N(2, 4) = 1 \).

Hence, we have
\[
\alpha(2, 4) d(T_2, T_4) = 0.1 = 0 = \psi(M(2, 4)) + LN(2, 4).
\]
**Case (7):** \( x = 3 \) and \( y = 4 \).

In this case, \( \alpha(3, 4) d(T_3, T_4) = 2.0 = 0, M(3, 4) = 1 \) and \( N(3, 4) = 0 \).

Hence, we have
\[
\alpha(3, 4) d(T_3, T_4) = 2.0 = 0 = \psi(M(3, 4)) + LN(3, 4).
\]
**Case (8):** \( x = 1 \) and \( y = 3 \).

Then, \( \alpha(1, 3) d(T_1, T_3) = 2.4 = 8, M(1, 3) = 3 \) and \( N(1, 3) = 3 \). Therefore, we have
\[
\alpha(1, 3) d(T_1, T_3) = 2.4 = 8 = \psi(M(1, 3)) + LN(1, 3).
\]
**Case (9):** \( x = 2 \) and \( y = 3 \).

In this case, \( \alpha(2, 3) d(T_2, T_3) = 0.1 = 0, M(2, 3) = 1 \) and \( N(2, 3) = 0 \).

Hence, we have
\[ \alpha(2,3)d(T2,T3) = 0 \leq \frac{4}{3}1 + 2.0 = \frac{3}{2} = \psi(M(2,3)) + LN(2,3). \]

From all the cases considered above, \( T \) satisfies the inequality (2.1.1) and hence \( T \) satisfies all the hypotheses of Theorem 2.1 and \( T \) has two fixed points 0 and 4.

We now illustrate an example in support of Theorem 2.2.

**Example 3.14:** Let \( X = [0, 4] \) with the usual metric.

We define a partial order \( \preceq \) on \( X \) by

\[ \preceq := \{(x, y) : x, y \in [0, 2), x = y \} \cup \{(x, y) : x, y \in [2, 4), x \leq y \}. \]

Then \((X, \preceq)\) is a partially ordered set.

We define \( T : X \rightarrow X \) and \( \alpha : X \times X \rightarrow [0, \infty) \) by

\[ T(x) = \begin{cases} 
\frac{2}{3} & \text{if } 0 \leq x < 1 \\
2 & \text{if } 1 \leq x < \frac{8}{3} \\
\frac{2}{3}x - 2 & \text{if } \frac{8}{3} \leq x \leq 4,
\end{cases} \]

\[ \alpha(x, y) = \begin{cases} 
1 & \text{if } 2 \leq x \leq 4 \text{ and } y = 4 \\
0 & \text{otherwise}.
\end{cases} \]

Here, we note that \( T \) is nonincreasing on \( X \) and not continuous at 1. Moreover, we choose \( x_0 = \frac{8}{3} \in X \), then

\[ \alpha(x_0, Tx_0) = \alpha\left(\frac{8}{3}, 4\right) = 1 \text{ and } \frac{8}{3} \leq T\frac{8}{3}. \]

Now, we show that \( T \) is \( \alpha \)-admissible.

**Case (1):** \( 2 \leq x < \frac{8}{3} \) and \( y = 4 \).

In this case, \( Tx = 2 \) and \( Ty = T4 = 4 \). Therefore, by the definition of \( \alpha \) we have \( \alpha(Tx, Ty) = \alpha(2, 4) = 1 \).

**Case (2):** \( x = \frac{8}{3} \) and \( y = 4 \).

Then, we have \( T\frac{8}{3} = 2 \) and \( T4 = 4 \) and

\[ \alpha(Tx, Ty) = \alpha(2, 4) = 1. \]

**Case (3):** \( \frac{8}{3} \leq x \leq y = 4 \).

In this case, \( 2 < Tx \leq 4 \) and \( Ty = T4 = 4 \). Hence, by the definition of \( \alpha \) we have \( \alpha(Tx, Ty) = \alpha(2, x, 4) = 1 \).

Therefore, \( T \) is \( \alpha \)-admissible.

Now, we verify the inequality (2.2.1) by choosing \( \psi \in \Psi \) given by \( \psi(t) = \frac{t}{2} \) for \( t \geq 0 \) and \( L = 1 \).

**Case (1):** \( 2 \leq x < \frac{8}{3} \) and \( y = 4 \).

In this case, \( \alpha(x, y) = 1, Tx = 2, Ty = 4 \) and

\[ \alpha(x, y)d(Tx, Ty) = 2, \]

\[ M(x, y) = \max\{4 - x, x - 2, 0, \frac{6-x}{2}\} = \frac{6-x}{2} \text{ and} \]

\[ N(x, y) = \min\{4 - x, 2\} = 4 - x. \]

Hence, we have

\[ 2 = \alpha(x, y)d(Tx, Ty) \leq \frac{1}{2} \frac{6-x}{2} + 1.4 - x = \psi(M(x, y)) + LN(x, y). \]

**Case (2):** \( \frac{8}{3} \leq x \leq 4 \) and \( y = 4 \).

Then, \( Tx = \frac{3}{2}x - 2 \) and \( Ty = T4 = 4 \) and \( \alpha(x, y) = 1 \).

\[ \alpha(x, y)d(Tx, Ty) = 6 - \frac{3}{2}x, \]

\[ M(x, y) = \max\{4 - x, 2 - \frac{3}{2}x, 0, \frac{20-5x}{4}\} = \frac{20-5x}{4} \text{ and} \]

\[ N(x, y) = \min\{4 - x, 6 - \frac{3}{2}x\} = 4 - x. \]

Hence, we have

\[ 6 - \frac{3}{2}x = \alpha(x, y)d(Tx, Ty) \leq 1 + 1.4 - x = \psi(M(x, y)) + LN(x, y). \]

From all the cases considered above, \( T \) satisfies the inequality (2.2.1) and hence \( T \) satisfies all the hypotheses of Theorem 2.2 and \( T \) has three fixed points 0, 2 and 4.

Here, we note that if \( L = 0 \) in the inequality (2.2.1), then for \( x = 2 \) and \( y = 4 \) we have \( \alpha(2, 4)d(T2, T4) = 2 < \psi(M(2, 4)) = \psi(2) \), which is absurd for any \( \alpha \)-admissible function \( T \) with \( \alpha(2,4) \geq 1 \), and any \( \psi \in \Psi \). Hence the inequality (2.2.1) fails to hold when \( L = 0 \) for any \( \alpha \)-admissible function \( T \) with \( \alpha(x,y) \geq 1, x, y \in X \) and \( \psi \in \Psi \). This example illustrates the importance of \( L \) in Theorem 2.2.

Here, we observe that the inequality (1.1.2.1) fails to hold at \( x = 2 \) and \( y = 4 \) for any \( \delta \in [0, 1] \) and \( L \geq 0 \), for

\[ d(T2,T4) = 2 \leq 2\delta = \delta M(2,4) + L,0 = \delta M(2,4) + L \min\{d(2,T2),d(4,T4),d(2,T4),d(4,T2)\} \]

which is absurd. Hence Theorem 1.12 is not applicable. So by Remark 3.9 and Example 3.14, it follows that Theorem 2.2 is a generalization of Theorem 1.12.

In the following, we give an example in support of Theorem 2.3.

**Example 3.15:** Let \( X = \{0, 1, 2, 3, 4\} \) with the usual metric.

We define a partial ordering on \( X \) as follows

\[ \preceq := \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 4)\}. \]

Then \((X, \preceq)\) a partially ordered set.

Let \( A = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (0, 1), (0, 2), (0, 3), (2, 1), (4, 1), (1, 0), (2, 0), (3, 0), (4, 0)\}. \)

We define \( T : X \rightarrow X \) and \( \alpha : X \times X \rightarrow [0, \infty) \) by

\[ T0 = T1 = T3 = 1, T2 = 4 \text{ and } T4 = 2. \]

\[ \alpha(x, y) = \begin{cases} 
1 & \text{if } (x, y) \in A \\
0 & \text{if } (x, y) \in X \times X \setminus A.
\end{cases} \]

Then \( T \) is continuous, nondecreasing and \( \alpha \)-admissible.

We choose \( x_0 = 1 \in X \), then \( x_0 \preceq Tx_0 \) and
\[ \alpha(x_0, Tx_0) = \alpha(1, 0) = 1. \] Also we choose \( z = 0 \in X \), then \( z \) is comparable with every \( x, y \in X \) and \( \alpha(z, Tz) = 1 \) and \( z \preceq Tz \).

Now, we verify the inequality (2.3.1) by choosing \( \psi \in \Psi \) given by \( \psi(t) = \frac{2}{3}t \) for \( t \geq 0 \) and \( L = 3 \) for comparable elements in \( X \).

We check only at \( x = 0 \) and \( y = 2 \).

In this case, \( \alpha(0, 2)d(T0, T2) = 3 \), \( M(0, 2) = \frac{5}{2} \) and \( N(0, 2) = 1 \).

Hence, we have
\[
\alpha(0, 2)d(T0, T2) = 3 \leq \frac{5}{2} + 3.1 = \frac{14}{3} = \psi(M(0, 2)) + LN(0, 2).
\]

At all the remaining points the inequality (2.3.1) holds trivially.

Hence \( T \) satisfies the inequality (2.3.1) and hence all the hypotheses of Theorem 2.3 are satisfied and \( T \) has a unique fixed point 1.

IV. CONCLUSION

In this paper, we proved the existence of fixed points for almost generalized \( \alpha-\psi \)-contractive maps (Theorem 2.1 and Theorem 2.2) in partially ordered sets endowed with a metric.

(i) Example 3.12 and Remark 3.7 suggest that Theorem 2.1 is a generalization of Theorem 1.11; Also, Theorem 2.1 is a generalization of Theorem 1.4 and Theorem 1.7.

(ii) Example 3.14 and Remark 3.9 suggest that Theorem 2.2 is a generalization of Theorem 1.12.

Uniqueness of the fixed points is also studied. Our results generalize the results of Samet, Vetro and Vetro [16], Ciric, Abbas, Saadati and Hussain [6] and Karapinar and Samet [8].

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