# Fixed Points of Almost Generalized $\alpha$-$\psi$-Contractive Maps

G. V. R. Babu* and K. T. Kidane**

***Abstract*—We introduce almost generalized $\alpha$-$\psi$- contractive maps and prove the existence and uniqueness of fixed points of almost generalized $\alpha$-$\psi$- contractive maps in partially ordered sets endowed with a metric. Our results extend and generalize the results of Samet, Vetro and Vetro [16], Karapinar and Samet [8] and Ciric, Abbas, Saadati and Hussain [6]. Furthermore, we provide examples in support of our results.**

***Index Terms*—Fixed points, $\alpha$- admissible maps, almost generalized $\alpha$-$\psi$- contractive maps.**

***MSC 2010 Codes* – 47H10, 54H25.**

## I. INTRODUCTION

Recently several authors studied fixed point theorems in partially ordered sets endowed with a metric. Ran and Reurings [15] and Nieto and Lopez [14] proved Banach contraction principle in partially ordered sets endowed with a metric. Agarwal, El-Gebeily and O'Regan [1] have proved some fixed point results for monotone operators in ordered metric spaces endowed with a partial order using a weak generalized contraction type assumption. For more works in this line of research we refer [2, 3, 7, 11, 12, 13].

Throughout this paper we denote by $\Psi$ the family of nondecreasing functions $\psi : [0,\ \infty) \to [0,\ \infty)$ which satisfies $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$ where $\psi^n$ is the $n^{th}$ iterate of $\psi$.

*Remark 1.1:* Any function $\psi \in \Psi$ satisfies $\lim\limits_{n\to\infty} \psi^n(t) = 0$, $\psi(t) < t$ for any $t > 0$ and $\psi$ is continuous at 0.

Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$.

We say that $T$ is nondecreasing with respect to $\preceq$ if $x,\ y \in X$, $x \preceq y \Rightarrow Tx \preceq Ty$.

Recently, Samet, Vetro and Vetro [16] introduced a new concept namely $\alpha$-$\psi$- contractive mappings and proved the existence of fixed points of such mappings in metric space setting.

*Definition 1.2:* [16] Let $(X,\ d)$ be a metric space and $T : X \to X$. We say that $T$ is $\alpha$-$\psi$- contractive mapping if there exist two functions $\alpha : X \times X \to [0,\ \infty)$ and $\psi \in \Psi$ such that

$\alpha(x,\ y)d(Tx,\ Ty) \le \psi(d(x,\ y))$ for all $x,\ y \in X$.

*Definition 1.3:* [16] Let $(X,\ d)$ be a metric space, $T : X \to X$ and $\alpha : X \times X \to [0,\ \infty)$. We say that $T$ is $\alpha$- admissible if $x,\ y \in X$, $\alpha(x,\ y) \ge 1 \Rightarrow \alpha(Tx,\ Ty) \ge 1$.

*G. V. R. Babu, Department of Mathematics, Andhra University, Visakhapatnam-530 003, INDIA. e-mail: gvr_babu@hotmail.com

**K. T. Kidane, Department of Mathematics, Andhra University, Visakhapatnam-530 003, INDIA. *Permanent address*: Department of Mathematics, Jimma University, Jimma , p.O.Box 378, Ethiopia. e-mail: kidanekoyas@yahoo.com

For examples on $\alpha$- admissible functions, we refer [15].

*Theorem 1.4:* [16] Let $(X,\ d)$ be a complete metric space and $T : X \to X$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$\alpha(x, y)d(Tx, Ty) \le \psi(d(x, y))$ for all $x,\ y \in X$. (1.4.1)

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,\ Tx_0) \ge 1$; and
(iii) $T$ is continuous.

Then, $T$ has a fixed point, $i.e.$, there exists $u \in X$ such that $Tu = u$.

*Theorem 1.5:* [16] Let $(X,\ d)$ be a complete metric space and $T : X \to X$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$\alpha(x, y)d(Tx, Ty) \le \psi(d(x, y))$ for all $x,\ y \in X$. (1.5.1)

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,\ Tx_0) \ge 1$; and
(iii) if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n,\ x_{n+1}) \ge 1$ for all $n$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,\ x) \ge 1$ for all $n$.

Then, $T$ has a fixed point, $i.e.$, there exists $u \in X$ such that $Tu = u$.

Recently, Karapinar and Samet [8] introduced generalized $\alpha$-$\psi$ contractive mappings and proved fixed point results.

*Definition 1.6:* [8] Let $(X,\ d)$ be a metric space and $T : X \to X$ be a given mapping. We say that $T$ is a generalized $\alpha$-$\psi$- contractive mapping if there exist two functions $\alpha : X \times X \to [0,\ \infty)$ and $\psi \in \Psi$ such that

$\alpha(x,\ y)d(Tx,\ Ty) \le \psi(M(x,\ y))$ for all $x,\ y \in X$ where

$M(x,\ y) = \max\{d(x,\ y), \frac{(x,\ Tx)+d(y,\ Ty)}{2}, \frac{d(x,\ Ty)+d(y,\ Tx)}{2}\}$.

*Theorem 1.7:* [8] Let $(X,\ d)$ be a complete metric space and $T : X \to X$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$\alpha(x, y)d(Tx, Ty) \le \psi(M(x, y))$ for all $x,\ y \in X$. (1.7.1)

where

$M(x,\ y) = \max\{d(x,\ y), \frac{(x,\ Tx)+d(y,\ Ty)}{2}, \frac{d(x,\ Ty)+d(y,\ Tx)}{2}\}$.

Also, assume that the following conditions are satisfied:

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,\ Tx_0) \ge 1$ ; and
(iii) $T$ is continuous.

Then there exists $u \in X$ such $Tu = u$.

*Theorem 1.8:* [8] Let $(X,\ d)$ be a complete metric space and $T : X \to X$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$\alpha(x, y)d(Tx, Ty) \le \psi(M(x, y))$ for all $x,\ y \in X$. (1.8.1)

where

$M(x,\ y) = \max\{d(x,\ y),\ \frac{(x,\ Tx)+d(y,\ Ty)}{2},\ \frac{d(x,\ Ty)+d(y,\ Tx)}{2}\}.$

Also, assume that the following conditions are satisfied:

(i) $T$ is $\alpha$ - admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0,\ Tx_0) \geq 1$ ; and
(iii) if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n,\ x_{n+1}) \geq 1$ for all $n$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},\ x) \geq 1$ for all $k$.

Then there exists $u \in X$ such $Tu = u$.

For more works in this line of research we refer [9, 10].

In 2004, Berinde [5] introduced 'weak contraction maps' which are named as 'almost contraction maps' as a generalization of contraction maps and proved fixed point results in complete metric spaces.

*Definition 1.9:* [5] Let $(X, d)$ be a metric space. A map $T : X \to X$ is called an 'almost contraction' if there exists a constant $\delta \in (0, 1)$ and $L \geq 0$ such that

$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx)$ for all $x, y \in X$.

In 2008, Babu, Sandya and Kameshawari [4] modified the above definition by introducing 'condition $(B)$' and proved a fixed point theorem in complete metric spaces.

*Definition 1.10:* [4] Let $(X, d)$ be a metric space a map

$T : X \to X$ is said to satisfy 'condition $(B)$' if there exist a $0 < \delta < 1$ and $L \geq 0$ such that

$$d(Tx,\ Ty) \leq \delta d(x,\ y) + L\min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}\ x, y \in X$$

In 2011, Ciric, Abbas, Saadati and Hussain [6] proved the following fixed point results of an almost generalized contractive maps in ordered metric spaces.

*Theorem 1.11:* [6] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a strictly increasing continuous mapping with respect to $\preceq$. Suppose that there exists $\delta \in [0,\ 1)$ and $L \geq 0$ such that

$$d(Tx,\ Ty) \leq \delta M(x,\ y) + L\min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \qquad (1.11.1)$$

for all comparable $x, y \in X$ where

$$M(x,\ y) = \max\{d(x,\ y),\ (x,\ Tx),\ d(y,\ Ty),\ \frac{d(x,\ Ty)+d(y,\ Tx)}{2}\}.$$

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has a fixed point.

*Theorem 1.12:* [6] Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a strictly increasing mapping with respect to $\preceq$. Suppose that there exists $\delta \in [0,\ 1)$ and $L \geq 0$ such that

$$d(Tx,\ Ty) \leq \delta M(x,\ y) + L\min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \qquad (1.12.1)$$

for all comparable $x, y \in X$ where

$$M(x,\ y) = \max\{d(x, y), (x, Tx), d(y, Ty), \frac{d(x,\ Ty)+d(y,\ Tx)}{2}\}.$$

If there exist $x_0 \in X$ such that $x_0 \preceq Tx_0$ and for an increasing sequence $\{x_n\}$ in $X$ converging to $x \in X$ we have $x_n \preceq x$ for all $n$. Then $T$ has a fixed point in $X$.

In this paper, we introduce almost generalized $\alpha$-$\psi$- contractive maps and prove the existence and uniqueness of fixed points in partially ordered sets endowed with a metric. Our results extend and generalize the results of Samet, Vetro and Vetro [16] and that of Karapinar and Samet [8]and Ciric, Abbas, Saadati and Hussain [6]. Furthermore, we provide examples in support of our results.

## II. Main Results

*Theorem 2.1:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ and $L \geq 0$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) + LN(x,\ y) \qquad (2.1.1)$$

for all $x, y \in X$ with $x \preceq y$ where

$M(x,\ y) = \max\{d(x,\ y), d(x,\ Tx), d(y,\ Ty), \frac{d(x,\ Ty)+d(y,\ Tx)}{2}\}$

and

$N(x,\ y) = \min\{d(x,\ Ty), d(y, Tx)\}$.

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) $T$ is continuous.

Then $T$ has a fixed point, $i.e.$, there exists $x^* \in X$ such that $x^* = Tx^*$.

**Proof.** By (ii), suppose that there exists $x_0 \in X$ such that $\alpha(x_0,\ Tx_0) \geq 1$ with $x_0 \preceq Tx_0$. We define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$ for $n \in \{0, 1, 2, \cdots\}$. (2.1.2)

If $x_n = x_{n+1}$ for some $n$, then $x_n = Tx_n$ and hence $x_n$ is a fixed point of $T$.

Now,we assume that $d(x_n, x_{n+1}) > 0$ for all $n$. Since $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$, by (i) it follows that $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$.

Inductively, we have

$$\alpha(x_n,\ x_{n+1}) \geq 1 \text{ for all } n \geq 0. \qquad (2.1.3)$$

Since $T$ is nondecreasing and $x_0 \preceq Tx_0 = x_1$ we have

$x_1 = Tx_0 \preceq Tx_1 = x_2$, $i.e.$, $x_1 \preceq x_2$.

Inductively we have $x_n \preceq x_{n+1}$ for all $n \geq 0$. (2.1.4)

Hence we have $x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$.

Now, from (2.1.1), (2.1.3) and (2.1.4) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \\ &\leq \psi(d(M(x_{n-1}, x_n)) + LN(x_{n-1}, x_n) \qquad (2.1.5) \end{aligned}$$

where

$$M(x_{n-1},\ x_n) = \max\ \{d(x_{n-1},\ x_n),\ d(x_{n-1},\ Tx_{n-1}), d(x_n,\ Tx_n), \frac{d(x_{n-1},\ Tx_n)+d(x_n,\ Tx_{n-1})}{2}\}$$

$$= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1}, x_{n+1})+d(x_n, x_n)}{2}\}$$
$$= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1}, x_{n+1})}{2})\}$$
$$\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1}, x_n)+d(x_n, x_{n+1})}{2}\}$$
$$= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \quad (2.1.6)$$

and

$$N(x_{n-1}, x_n) = \min\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}$$
$$= \min\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}$$
$$= \min\{d(x_{n-1}, x_{n+1}), 0\} = 0. \quad (2.1.7)$$

Since $\psi$ is nondecreasing using (2.1.6) and (2.1.7) in (2.1.5) we have

$$d(x_n, x_{n+1}) \leq \psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})) \quad (2.1.8)$$

for all $n \geq 1$. Now, if for some $n \geq 1$,

$d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, then from (2.1.8) we have

$d(x_n, x_{n+1}) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1})$,

a contradiction. Thus for all $n \geq 1$ we have

$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$.

Therefore, we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \psi^3(d(x_{n-3}, x_{n-2})) \leq \cdots \leq \psi^n(d(x_0, x_1)).$$

On using the triangle inequality, for all $k \geq 1$, we have

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}) \leq \sum_{k=n}^{n+k-1} \psi^k(d(x_0, x_1))$$
$$\leq \sum_{k=n}^{\infty} \psi^k(d(x_0, x_1)) \to 0 \text{ as } n \to \infty.$$

Therefore $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $x^* \in X$ such that $x^* = \lim_{n\to\infty} x_n$. Since $T$ is continuous, we have $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*$. Hence $x^*$ is a fixed point of $T$.

In the following, we prove fixed point results by relaxing the continuity assumption of $T$ in Theorem 2.1.

*Theorem 2.2:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ with $\lim_{r\to t^+} \psi(r) < t$ for all $t > 0$ and $L \geq 0$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)) + LN(x, y) \quad (2.2.1)$$

for all $x, y \in X$ with $x \preceq y$
where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \tfrac{d(x, Ty)+d(y, Tx)}{2}\}$$

and

$$N(x, y) = \min\{d(x, Ty), d(y, Tx)\}.$$

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) if $\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \to x^*$ then $x_n \preceq x^*$, also if $\alpha(x_n, x_{n+1}) \geq 1$, then $\alpha(x_n, x^*) \geq 1$ for all $n$.

Then $T$ has a fixed point, *i.e.*, there exists $x^* \in X$ such that $x^* = Tx^*$.

**Proof.** From the proof of the Theorem 2.1, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ is Cauchy in $X$. Since $X$ is complete there exists $x^* \in X$ such that $x^* = \lim_{n\to\infty} x_n$.
From $\alpha(x_n, x_{n+1}) \geq 1$ we have $\alpha(x_n, x*) \geq 1$ for all $n$ and also $x_n \preceq x^*$. Now, we show that $Tx^* = x^*$. Assume that $d(Tx^*, x^*) > 0$.
On using (2.2.1), we have

$$d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) \leq \alpha(x_n, x^*)d(Tx_n, Tx^*)$$
$$\leq \psi(M(x_n, x^*)) + LN(x_n, x^*) \quad (2.2.2)$$

where

$$M(x_n, x^*) = \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \tfrac{d(x_n, Tx^*)+d(x^*, Tx_n)}{2}\} \quad (2.2.3)$$

and

$$N(x_n, x*) = \min\{d(x_n, Tx^*), d(x^*, x_{n+1})\}. \quad (2.2.4)$$

On taking limits as $n \to \infty$ in (2.2.3) and (2.2.4) we have

$$\lim_{n\to\infty} M(x_n, x^*) = \lim_{n\to\infty} \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \tfrac{d(x_n, Tx^*)+d(x^*, Tx_n)}{2}\}$$
$$= \max\{0, 0, d(x^*, Tx^*), \tfrac{d(x^*, Tx^*)}{2}\}$$
$$= d(x^*, Tx^*) \quad (2.2.5)$$

and

$$\lim_{n\to\infty} N(x_n, x*) = \lim_{n\to\infty} \min\{d(x_n, Tx^*), d(x^*, x_{n+1})\}$$
$$= \min\{d(x^*, Tx^*), 0) = 0. \quad (2.2.6)$$

Now, since

$d(x^*, Tx^*) \leq M(x_n, x^*)$ and $\lim_{n\to\infty} M(x_n, x^*) = d(x^*, Tx^*)$ we conclude that

$$M(x_n, x^*) \to d(x^*, Tx^*)^+ \text{ as } n \to \infty\ . \quad (2.2.7)$$

Since $\psi$ satisfies the property $\lim_{r\to t^+} \psi(r) < t$ for all $t > 0$, on taking limits as $n \to \infty$ in (2.2.2), by using (2.2.5), (2.2.6) and (2.2.7), we have

$d(x^*, Tx^*) \leq \lim_{n\to\infty} \psi(M(x_n, x^*)) < d(x^*, Tx^*)$, a contradiction. Therefore $d(x^*, Tx^*) = 0$, $i.e., x^* = Tx^*$.

In order to obtain the uniqueness of fixed points of almost generalized $\alpha$-$\psi$- contractive mappings we use the following hypotheses:

$(H)$: for all $x,\ y \in X$ there exists $z \in X$ such that $z$ is comparable to $x$ and $y$ and $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$ and also $z \preceq Tz$, $\alpha(z,\ Tz) \geq 1$. Moreover, we replace

$N(x,\ y) = \min\{d(x,\ Ty),\ d(y,\ Tx)\}$ of inequality (2.1.1(respectively (2.2.1)) by

$$N'(x,\ y) = \min\{d(x,\ Tx),\ d(x,\ Ty),\ d(y,\ Ty),\ d(y,\ Tx)\}.$$

*Theorem 2.3:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ and $L \geq 0$ such that

$$\alpha(x,\ y)d(Tx,\ Ty) \leq \psi(M(x,\ y)) + LN'(x,\ y) \qquad (2.3.1)$$

for all $x, y \in X$ with $x \preceq y$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \tfrac{d(x,Ty)+d(y,Tx)}{2}\}$$

and

$$N'(x,y) = \min\ \{d(x,Tx), d(x,Ty), d(y,Ty), d(y,\ Tx)\}.$$

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$;
(iii) $T$ is continuous; and
(iv) condition (H) holds.

Then $T$ has a unique fixed point.

**Proof.** Since the inequality (2.3.1) implies the inequality (2.1.1), by the proof of Theorem 2.1 the set of fixed points of $T$ is non-empty. Suppose that $x^*$ and $y^*$ are two distinct fixed points of $T$. By our assumption (H), there exists $z \in X$ such that $z$ is comparable to $x^*$ and $y^*$ and $\alpha(x^*,\ z) \geq 1$ and $\alpha(y^*,\ z) \geq 1$.

Now, put $z = z_0$ and choose $z_1 \in X$ such that $z_1 = Tz_0$.

We define sequence $\{z_n\}$ in $X$ by $z_{n+1} = Tz_n$ for all $n \geq 0$. Since $z$ is comparable to $x^*$ and $y^*$ it follows that $x^* \preceq z_1$ and $y^* \preceq z_1$.

Inductively, we can show that $x^* \preceq z_n$ and $y^* \preceq z_n$ (2.3.2) for all $n \geq 1$.

Since $\alpha(x^*, z) \geq 1$ and $\alpha(y^*,\ z) \geq 1$ and $T$ is $\alpha$- admissible we have

$\alpha(Tx^*, Tz) = \alpha(x^*, z_1) \geq 1$ and $\alpha(Ty^*, Tz) = \alpha(y^*,\ z_1) \geq 1$.

Inductively, we can show that

$$\alpha(x^*,\ z_n) \geq 1 \text{ and } \alpha(y^*,\ z_n) \geq 1. \qquad (2.3.3)$$

Now, on taking $x = x^*$, $y = z_n$ and using (2.3.2) and (2.3.3) in (2.3.1), we have

$$\begin{aligned} d(x^*,\ z_{n+1}) = d(Tx^*,\ Tz_n) &\leq \alpha(x^*,\ z_n)d(Tx^*,\ Tz_n) \\ &\leq \psi(M(x^*,\ z_n)) + LN(x^*,\ z_n) \end{aligned}$$

where

$$\begin{aligned} M(x^*,\ z_n) &= \max\{d(x^*, z_n), d(x^*,\ Tx^*), d(z_n, Tz_n), \tfrac{d(x^*,Tz_n)+d(z_n,Tx^*)}{2}\} \\ &= \max\ \{d(x^*, z_n), 0, d(z_n, z_{n+1}), \tfrac{d(x^*,z_{n+1})+d(z_n,x^*)}{2}\} \\ &= \max\ \{d(x^*,\ z_n),\ d(z_n,\ z_{n+1}), \tfrac{d(x^*,\ z_{n+1})+d(z_n,\ x^*)}{2})\} \\ &\leq \max\ \{d(x^*,\ z_n),\ d(z_n,\ z_{n+1}),\ d(x^*,\ z_{n+1})\} \end{aligned}$$

and

$$\begin{aligned} N'(x^*, z_n) &= \min\{d(x^*, Tx^*), d(z_n, z_{n+1}), d(x^*, z_{n+1}), d(z_n, Tx^*)\} \\ &= \min\{d(x^*, x^*), d(z_n, z_{n+1}), d(x^*, z_{n+1}), d(z_n,\ x^*)\} \\ &= \min\{0, d(z_n, z_{n+1}), d(x^*, z_{n+1}), d(z_n, x^*)\} = 0. \end{aligned}$$

Thus by the monotonicity of $\psi$ we have

$$d(x^*, z_{n+1}) \leq \psi(\max\{d(x^*, z_n), d(z_n, z_{n+1}), d(x^*, z_{n+1})\}).$$

We assume without loss of generality that $M(x^*,\ z_n) > 0$ for all $n$. Now, we consider the following three cases:

Case(1): $\max\{d(x^*, z_n), d(x^*, z_{n+1}), d(z_n, z_{n+1})\} = d(x^*,\ z_{n+1})$.

In this case, we have

$d(x^*,\ z_{n+1}) \leq \psi(d(x^*,\ z_{n+1})) < d(x^*,\ z_{n+1})$, a contradiction.

Case(2): $\max\{d(x^*,\ z_n), d(x^*, z_{n+1}), d(z_n,\ z_{n+1})\} = d(x^*,\ z_n)$.

Then

$$\begin{aligned} d(x^*,\ z_{n+1}) &\leq \psi(d(x^*,\ z_n)) \leq \psi^2(d(x^*,\ z_{n-1}) \\ &\leq \psi^3(d(x^*,\ z_{n-2}) \leq \cdots \leq \psi^n(d(x^*,\ z_1) \to 0 \end{aligned}$$

as $n \to \infty$.

Therefore, $d(x^*,\ z_{n+1}) \to 0$ as $n \to \infty$.

Similarly, $d(y^*,\ z_{n+1}) \to 0$ as $n \to \infty$. Hence $x^* = y^*$.

Case(3): $\max\ \{d(x^*,\ z_n),\ d(x^*,\ z_{n+1}),\ d(z_n,\ z_{n+1})\} = d(z_n,\ z_{n+1})$.

In this case, we have

$$\begin{aligned} d(x^*, z_{n+1}) &\leq \psi(d(z_n, z_{n+1})) \leq \psi^2(d(z_{n-1},\ z_n) \\ &\leq \psi^3(d(z_{n-2},\ z_{n-3}) \leq \cdots \\ &\leq \psi^n(d(z_0,\ z_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $d(x^*,\ z_{n+1}) \to 0$ as $n \to \infty$.

Similarly, $d(y^*,\ z_{n+1}) \to 0$ as $n \to \infty$. Hence $x^* = y^*$.

*Theorem 2.4:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ with $\lim_{r \to t^+} \psi(r) < t$ for all $t > 0$ and $L \geq 0$ such that

$$\alpha(x,\ y)d(Tx,\ Ty) \leq \psi(M(x,\ y)) + LN^{'}(x,\ y) \qquad (2.4.1)$$

for all $x, y \in X$ with $x \preceq y$
where

$$M(x,\ y) = \max\{d(x,y), d(x,Tx), d(y,\ Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$$

and

$$N^{'}(x,\ y) = \min\ \{d(x,Tx), d(x,Ty), d(y,\ Ty), d(y,\ Tx)\}.$$

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$;
(iii) if $\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \to x^*$ then $x_n \preceq x^*$, also if $\alpha(x_n,\ x_{n+1}) \geq 1$, then $\alpha(x_n,\ x^*) \geq 1$ for all $n$; and
(iv) condition $(H)$ holds.

Then $T$ has a unique fixed point.

**Proof.** Runs on the same lines as that of Theorem 2.3.

## III. Corollaries and Examples

*Corollary 3.1:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ and $L \geq 0$ such that

$$\alpha(x,\ y)d(Tx,\ Ty) \leq \psi(M^{'}(x,\ y)) + LN(x,\ y) \qquad (3.1.1)$$

for all $x, y \in X$ with $x \preceq y$.
where

$$M^{'}(x,\ y) = \max\ \{d(x,\ ), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$$

and

$$N(x,\ y) = \min\{d(x,Ty), d(y,Tx)\}.$$

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) $T$ is continuous.

Then $T$ has a fixed point, $i.e.$, there exists $x^* \in X$ such that $x^* = Tx^*$.

**Proof.** Since the inequality (3.1.1) is a special case of the inequality (2.1.1), the conclusion of the result follows from Theorem 2.1.

*Remark 3.2:* By choosing $L = 0$ in Corollary 3.1 we get Theorem 1.7 in the setting of ordered metric space.

By choosing $L = 0$ in Theorem 2.1 we get the following.

*Corollary 3.3:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)) \qquad (3.3.1)$$

for all $x, y \in X$ with $x \preceq y$
where

$$M(x,\ y) = \max\{d(x,\ y), d(x,\ Tx),\ d(y,\ Ty), \frac{d(x,\ Ty)+d(y,\ Tx)}{2}\}.$$

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) $T$ is continuous.

Then $T$ has a fixed point, $i.e.$, there exists $x^* \in X$ such that $x^* = Tx^*$.

The following result is an immediate consequence of Corollary 3.3.

*Corollary 3.4:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,\ y)d(Tx,\ Ty) \leq \psi(d(x,\ y)) \qquad (3.4.1)$$

for all $x, y \in X$ with $x \preceq y$.

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) $T$ is continuous.

Then $T$ has a fixed point, $i.e.$, there exists $x^* \in X$ such that $x^* = Tx^*$.

*Remark 3.5:* Corollary 3.4 extends Theorem 1.4 to metric spaces endowed with partial order.

By choosing $\psi(t) = kt$, $0 \leq k < 1$ in Theorem 2.1 we get the following:

*Corollary 3.6:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist a function $\alpha : X \times X \to [0, \infty)$ and $L \geq 0$ such that

$$\alpha(x,\ y)d(Tx,\ Ty) \leq kM(x,\ y) + LN(x,\ y) \qquad (3.6.1)$$

for all $x, y \in X$ with $x \preceq y$
where

$$M(x,y) = \max\{d(x,y),\ d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$$

and

$$N(x,y) = \min\{d(x,Ty), d(y,Tx)\}.$$

Also, assume that

(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) $T$ is continuous.

Then $T$ has a fixed point, $i.e.$, there exists $x^* \in X$ such that $x^* = Tx^*$.

*Remark 3.7:* Theorem 1.11 follows as a corollary to Theorem 2.1, since the inequality (1.11.1) follows from the inequality (2.1.1) with $\psi(t) = \delta t,\ \delta \in [0,\ 1), t \geq 0$; and $\alpha(x,\ y) = 1$ for all $x,\ y \in X$.

By choosing $\psi(t) = kt$, $0 \leq k < 1$ in Theorem 2.2 we get the following:

*Corollary 3.8:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist a function $\alpha : X \times X \to [0, \infty)$, and $L \geq 0$ such that

$$\alpha(x,\ y)d(Tx,\ Ty) \leq kM(x,\ y) + LN(x,\ y) \tag{3.8.1}$$

for all $x, y \in X$ with $x \preceq y$
where

$$M(x,\ y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$$

and

$$N(x,\ y) = \min\{d(x, Ty), d(y, Tx)\}.$$

Also, assume that
(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) if $\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \to x^*$ then $x_n \preceq x^*$, also if $\alpha(x_n,\ x_{n+1}) \geq 1$, then $\alpha(x_n,\ x^*) \geq 1$ for all $n$.

Then $T$ has a fixed point, $i.e.$, there exists $x^* \in X$ such that $x^* = Tx^*$.

*Remark 3.9:* Theorem 1.12 follows as a corollary to Theorem 2.2, since the inequality (1.12.1) follows from the inequality (2.2.1) with $\psi(t) = \delta t,\ \delta \in [0,\ 1),\ t \geq 0$; and $\alpha(x,\ y) = 1$ for all $x,\ y \in X$.

The following result is an immediate consequence of Corollary 3.3.

*Corollary 3.10:* Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a nondecreasing map with respect to $\preceq$. Suppose that there exist a function $\alpha : X \times X \to [0, \infty)$ and $0 \leq k < 1$ such that

$$\alpha(x,\ y)d(Tx,\ Ty) \leq kd(x, y) \tag{3.10.1}$$

for all$x, y \in X$ with $x \preceq y$.

Also, assume that
(i) $T$ is $\alpha$- admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ with $x_0 \preceq Tx_0$; and
(iii) $T$ is continuous.

Then $T$ has a fixed point, $i.e.$, there exists $x^* \in X$ such that $x^* = Tx^*$.

*Remark 3.11:* If we define $\alpha : X \times X \to [0,\ \infty)$ by $\alpha(x,\ y) = 1$ for all $x,\ y \in X$ in Corollary 3.10, we get Theorem 2.1 of [14].

The following is an example in support of Theorem 2.1.

*Example 3.12:* Let $X = [0,\ 4]$ with the usual metric.

We define a partial order $\preceq$ on $X$ by

$$\preceq := \{(x, y) : x,\ y \in [0,\ 2), x = y\} \bigcup \{(x, y) : x,\ y \in [2,\ 4]), x \leq y\}.$$ Then $(X, \preceq)$ is a partially ordered set.

We define $T : X \to X$ and $\alpha : X \times X :\to [0,\ \infty)$ by

$$T(x) = \begin{cases} \frac{x}{2} & if\ 0 \leq x < 1 \\ \frac{3}{2}x - 1 & if\ 1 \leq x < \frac{10}{3} \\ 4 & if\ \frac{10}{3} \leq x \leq 4, \end{cases}$$

$$\alpha(x,\ y) = \begin{cases} 1 & if\ 2 \leq x \leq 4 \text{ and } y = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Then $T$ is continuous and nondecreasing on $X$. Moreover, we choose $x_0 = \frac{10}{3} \in X$, then $\alpha(x_{0,\ Tx_0}) = \alpha(\frac{10}{3},\ 4) = 1$ and $\frac{10}{3} \leq T\frac{10}{3} = 4$.

Now, we show that $T$ is $\alpha$- admissible.

<u>Case(1):</u> $2 \leq x < \frac{10}{3}$ and $y = 4$.

Then $Tx \in [2,\ 4)$ and $Ty = 4$. Therefore, by the definition of $\alpha$ we have $\alpha(Tx,\ Ty) = \alpha(Tx,\ 4) = 1$.

<u>Case(2):</u> $\frac{10}{3} \leq x \leq 4$ and $y = 4$.

In this case, $Tx = 4$ and $Ty = 4$ and hence

$\alpha(Tx,\ Ty) = \alpha(4,\ 4) = 1$.

Therefore, $T$ is $\alpha$- admissible.

Now, we verify the inequality (2.1.1) by choosing $\psi \in \Psi$ given by $\psi(t) = \frac{3}{4}t$ for $t \geq 0$ and $L = 1$.

<u>Case(1):</u> $2 \leq x < \frac{10}{3}$ and $y = 4$.

In this case $Tx = \frac{3}{2}x - 1$ and $Ty = 4$ and

$\alpha(x, y)d(Tx, Ty) = 5 - \frac{3}{2}x$,

$M(x,\ y) = \max\{4 - x,\ \frac{1}{2}x - 1,\ \frac{18-5x}{4}\} = 4 - x$ and

$N(x,\ y) = \min\{4 - x,\ 5 - \frac{3}{2}x\} = 5 - \frac{3}{2}x$.

Hence we have

$$\begin{aligned}\alpha(x,\ y)d(Tx,\ Ty) &= 5 - \tfrac{3}{2}x \leq \tfrac{3}{4}(4 - x) + 1.(5 - \tfrac{3}{2}x) \\ &= \psi(M(x,\ y)) + LN(x.\ y).\end{aligned}$$

<u>Case(2):</u> $\frac{10}{3} \leq x \leq 4$ and $y = 4$.

In this case $d(Tx,\ Ty) = d(4,\ 4) = 0$, hence we have

$$\alpha(x,\ y)d(Tx,\ Ty) = 0 \leq \psi(M(x,\ y)) + LN(x.\ y).$$

Therefore $T$ satisfies the inequality (2.1.1) and hence $T$ satisfies all the hypotheses of the Theorem 2.1 and T has three fixed points namely $0, 2$ and $4$.

If we choose $L = 0$ in the inequality (2.1.1), then for $x = 2$ and $y = 4$ we have

$\alpha(2,\ 4)d(T2,\ T4) = 2 \leq \psi(M(2,\ 4)) = \psi(2)$, which is absurd for any $\psi \in \Psi$ and any $\alpha$-admissible function $T$ with $\alpha(2,\ 4) = 1$.

Hence the inquality (2.1.1) fails to hold when $L = 0$ for any $\alpha$ admissible function $T$ with $\alpha(x,\ y) \geq 1,\ x,\ y \in X$ and $\psi \in \Psi$. This example indicates the importance of $L$ in the inequality (2.1.1) of Theorem 2.1.

Here, we observe that the inequality (1.11.1) fails to hold at $x = 2$ and $y = 4$ for any $\delta \in [0,\ 1)$ and $L \geq 0$, for

$$d(T2, T4) = 2 \leq 2\delta = \delta M(2,4) + L.0 = \delta M(2,4) + L\min\{d(2,T2), d(4,T4), d(2,T4), d(4,T2)\}$$

which is absurd. Hence Theorem 1.11 is not applicable. So by Remark 3.7 and Example 3.12, it follows that Theorem 2.1 is a generalization of Theorem 1.11. Also, we observe that under the setting of this example, the inequalities (1.4.1) and (1.7.1) fail to hold at $x = 2$ and $y = 4$. Hence Theorem 1.4 and Theorem 1.7 are also not applicable. This phenomenon suggests that Theorem 2.1 is also a generalization of Theorem 1.4 and Theorem 1.7.

The following is also an example in support of Theorem 2.1.

*Example 3.13:* Let $X = \{0,1,2,3,4\}$ with the usual metric.

We define a partial ordering on $X$ as follows

$$\preceq := \{(0,0),(0,1),(0,2),(0,3),(0,4),(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,4),(3,4),(2,3)\}.$$

Then $(X, \preceq)$ is a partially ordered set.

Let $A = \{(0,\ 0),\ (0,\ 1),\ (0,\ 2),\ (0,\ 3),\ (0,\ 4),\ (1,\ 1),\ (2,\ 2),\ (3,\ 3), (4,\ 4),\ (1,\ 2),\ (1,\ 4),\ (2,\ 0)\}$.

$B = \{(1,\ 3),\ (2,\ 1),\ (3,\ 1),\ (3,\ 2),\ (3,\ 0), (3,\ 4)\}$

$C = \{(1,\ 0),\ (4,\ 0),\ (4,\ 1),\ (4,\ 2),\ (2,\ 4),\ (2,\ 3),\ (4,\ 3)\}$.

We define $T : X \to X$ and $\alpha : X \times X \to [0,\ \infty)$ by

$T0 = T1 = 0$, $T2 = 3$, $T3 = 4$ and $T4 = 4$ and

$$\alpha(x,\ y) = \begin{cases} \frac{3}{2} & if\ (x,\ y) \in A \\ 2 & if\ (x,\ y) \in B \\ 0 & if\ (x,\ y) \in C. \end{cases}$$

Then $T$ is continuous, nondecreasing and $\alpha$- admissible. Moreover, we choose $x_0 = 3 \in X$, clearly $x_0 \preceq Tx_0$ and $\alpha(x_0, Tx_0) = \alpha(3,\ 4) = 2 \geq 1$.

Now, we verify the inequality (2.1.1) by choosing $\psi \in \Psi$ given by $\psi(t) = \frac{2}{3}t$ for $t \geq 0$ and $L = 2$.

<u>Case (1)</u>: $x = 0$ and $y = 2$.

In this case, $\alpha(0,\ 2)d(T0,\ T2) = \frac{9}{2}$, $M(0,\ 2) = \ \frac{5}{2}$ and $N(0,\ 2) = \ 2$.

Hence, we have

$$\alpha(0,\ 2)d(T0,\ T2) = \frac{9}{2} \leq= \frac{2}{3}\frac{5}{2} + 2.2 = \frac{17}{3} = \psi(M(0,\ 2)) + LN(0,\ 2).$$

<u>Case (2)</u>: $x = 0$ and $y = 3$.

Then, $\alpha(0,\ 3)d(T0, T3) = 6$, $M(0,3) = \ \frac{7}{2}$ and $N(0,3) = 3$. Hence, we have

$$\alpha(0,\ 3)d(T0,\ T3) = 6 \leq \frac{2}{3}\frac{7}{2} + 2.3 = \frac{25}{3} = \psi(M(0,\ 3)) + LN(0,\ 3).$$

<u>Case (3)</u>: $x = 0$ and $y = 4$.

In this case, $\alpha(0,\ 4)d(T0,\ T4) = 6$, $M(0,\ 4) = \ 4$ and $N(0,\ 4) = \ 4$. Therefore, we have

$$\alpha(0,4)d(T0,\ T4) = 6 \leq \frac{2}{3}.4 + 2.4 = \frac{32}{3} = \psi(M(0,4)) + LN(0,4).$$

<u>Case (4)</u>: $x = 1$ and $y = 2$.

Then, $\alpha(1,2)d(T1,T2) = \frac{9}{2}$, $M(1,2) = 2$ and $N(1,2) = 2$. Hence, we have

$$\alpha(1,2)d(T1,T2) = \frac{9}{2} \leq \frac{2}{3}.2 + 2.2 = \frac{16}{3} = \psi(M(1,2)) + LN(1,2).$$

<u>Case (5)</u>: $x = 1$ and $y = 4$.

In this case, $\alpha(1,4)d(T1,T4) = 6$, $M(1,\ 4) = 3$ and $N(1,\ 4) = 3$. Hence we have

$$\alpha(1,4)d(T1,T4) = 6 \leq \frac{2}{3}.3 + 2.3 = 8 = \psi(M(1,4)) + LN(1,4).$$

<u>Case (6)</u>: $x = 2$ and $y = 4$.

Then, $\alpha(2,4)d(T2,T4) = 0.1 = 0$, $M(2,4) = 2$ and $N(2,4) = 1$.

Hence, we have

$$\alpha(2,4)d(T2,4) = 0 \leq \frac{2}{3}.2 + 2.1 = \frac{10}{3} = \psi(M(2,4)) + LN(2,4).$$

<u>Case (7)</u>: $x = 3$ and $y = 4$.

In this case, $\alpha(3,4)d(T3,T4) = 2.0 = 0$, $M(3,4) = 1$ and $N(3,4) = 0$.

Hence, we have

$$\alpha(3,4)d(T3,T4) = 0 \leq \frac{2}{3}.1 + 2.0 = \frac{2}{3} = \psi(M(3,4)) + LN(3,4).$$

<u>Case (8)</u>: $x = 1$ and $y = 3$.

Then, $\alpha(1,3)d(T1,T3) = 2.4 = 8$, $M(1,3) = 3$ and $N(1,3) = 3$. Therefore, we have

$$\alpha(1,3)d(T1,T3) = 8 \leq \frac{2}{3}.3 + 2.3 = 8 = \psi(M(1,3)) + LN(1,3).$$

<u>Case (9)</u>: $x = 2$ and $y = 3$.

In this case, $\alpha(2,3)d(T2,T3) = 0.1 = 0$, $M(2,3) = 1$ and $N(2,3) = 0$.

Hence, we have

$\alpha(2,3)d(T2,T3) = 0 \leq \frac{2}{3}.1 + 2.0 = \frac{2}{3}$
$= \psi(M(2,3)) + LN(2,3).$

From all the cases considered above, $T$ satisfies the inequality (2.1.1) and hence $T$ satisfies all the hypotheses of Theorem 2.1 and $T$ has two fixed points 0 and 4.

We now illustrate an example in support of Theorem 2.2.

*Example 3.14:* Let $X = [0,\ 4]$ with the usual metric.

We define a partial order $\preceq$ on $X$ by

$\preceq := \{(x,y) : x,\ y \in [0,\ 2), x = y\} \bigcup \{(x,y) : x,\ y \in [2,\ 4]),$
$x \leq y\}$. Then $(X, \preceq)$ is a partially ordered set.

We define $T : X \to X$ and $\alpha : X \times X :\to [0,\ \infty)$ by

$$T(x) = \begin{cases} \frac{x}{3} & if\ 0 \leq x < 1 \\ 2 & if\ 1 \leq x < \frac{8}{3} \\ \frac{3}{2}x - 2 & if\ \frac{8}{3} \leq x \leq 4, \end{cases}$$

$$\alpha(x,\ y) = \begin{cases} 1 & if\ 2 \leq x \leq 4 \text{ and } y = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Here, we note that $T$ is nonincreasing on $X$ and not continuous at 1. Moreover, we choose $x_0 = \frac{8}{3} \in X$, then
$\alpha(x_{0,\ Tx_0}) = \alpha(\frac{8}{3},\ 4) = 1$ and $\frac{8}{3} \leq T\frac{8}{3}$.

Now, we show that $T$ is $\alpha$- admissible.

Case(1): $2 \leq x < \frac{8}{3}$ and $y = 4$.

In this case, $Tx = 2$ and $Ty = T4 = 4$. Therefore, by the definition of $\alpha$ we have $\alpha(Tx,\ Ty) = \alpha(2,\ 4) = 1$.

Case (2): $x = \frac{8}{3}$ and $y = 4$.

Then, we have $T\frac{8}{3} = 2$ and $T4 = 4$ and
$\alpha(Tx,\ Ty) = \alpha(2,\ 4) = 1$.

Case(3): $\frac{8}{3} < x \leq 4$ and $y = 4$.

In this case, $2 < Tx \leq 4$ and $Ty = T4 = 4$. Hence, by the definition of $\alpha$ we have $\alpha(Tx,\ Ty) = \alpha(Tx,\ 4) = 1$. Therefore, $T$ is $\alpha$- admissible.

Now, we verify the inequality (2.2.1) by choosing $\psi \in \Psi$ given by $\psi(t) = \frac{1}{2}t$ for $t \geq 0$ and $L = 1$.

Case(1): $2 \leq x < \frac{8}{3}$ and $y = 4$.

In this case, $\alpha(x,y) = 1$, $Tx = 2$, $Ty = 4$ and $\alpha(x,y)d(Tx,Ty) = 2$,

$M(x,y) = \max\{4-x, x-2, 0, \frac{6-x}{2}\} = \frac{6-x}{2}$ and
$N(x,y) = \min\{4-x, 2\} = 4-x$.

Hence, we have

$2 = \alpha(x,y)d(Tx,Ty) \leq \frac{1}{2}\frac{6-x}{2} + 1.(4-x)$
$= \psi(M(x,y)) + LN(x,y).$

Case(2): $\frac{8}{3} \leq x \leq 4$ and $y = 4$.

Then, $Tx = \frac{3}{2}x - 2$ and $Ty = T4 = 4$ and $\alpha(x,\ y) = 1$.

$\alpha(x,\ y)d(Tx,\ Ty) = 6 - \frac{3}{2}x,$

$M(x,\ y) = \max\{4-x,\ 2-\frac{1}{2}x,\ 0,\ \frac{20-5x}{4}\} = \frac{20-5x}{4}$ and
$N(x,\ y) = \min\{4-x,\ 6-\frac{3}{2}x\} = 4-x$.

Hence, we have

$6 - \frac{3}{2}x = \alpha(x,\ y)d(Tx,\ Ty) \leq \frac{1}{2}\frac{20-5x}{4} + 1.(4-x)$
$= \psi(M(x,\ y)) + LN(x,\ y).$

From all the cases considered above, $T$ satisfies the inequality (2.2.1) and hence $T$ satisfies all the hypotheses of the Theorem 2.2 and T has three fixed points $0,\ 2$ and $4$.

Here, we note that if $L = 0$ in the inequality (2.2.1), then for $x = 2$ and $y = 4$ we have
$\alpha(2,\ 4)d(T2,\ T4) = 2 \leq \psi(M(2,\ 4)) = \psi(2)$, which is absurd for any $\alpha$ admissible function $T$ with $\alpha(2,4) \geq 1$, and any $\psi \in \Psi$. Hence the inequality (2.2.1) fails to hold when $L = 0$ for any $\alpha$- admissible function $T$ with $\alpha(x,y) \geq 1$, $x,y \in X$ and $\psi \in \Psi$. This example illustrates the importance of $L$ in Theorem 2.2.

Here, we observe that the inequality (1.12.1) fails to hold at $x = 2$ and $y = 4$ for any $\delta \in [0,\ 1)$ and $L \geq 0$, for

$d(T2,T4) = 2 \leq 2\delta = \delta M(2,4) + L.0 = \delta M(2,4) +$
$L\min\{d(2,T2), d(4,T4), d(2,T4), d(4,T2)\}$
which is absurd. Hence Theorem 1.12 is not applicable. So by Remark 3.9 and Example 3.14, it follows that Theorem 2.2 is a generalization of Theorem 1.12.

In the following, we give an example in support of Theorem 2.3.

*Example 3.15:* Let $X = \{0,\ 1,\ 2,\ 3,\ 4\}$ with the usual metric.

We define a partial ordering on $X$ as follows

$\preceq := \{(0,\ 0),\ (1,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 4),\ (0,\ 1),\ (0,\ 2),$
$(0,\ 3), (0,\ 4),\ (1,\ 2),\ (1,\ 4)\}.$
Then $(X, \preceq)$ a partially ordered set.

Let $A = \{(0,\ 0), (1,\ 1), (2,\ 2), (3,\ 3), (4,\ 4), (0,\ 1), (0,\ 2),$
$(0,\ 3), (2,\ 1), (4,\ 1), (1,\ 0), (2,\ 0), (3,\ 0), (4,\ 0)\}.$

We define $T : X \to X$ and $\alpha : X \times X \to [0,\ \infty)$ by

$T0 = T1 = T3 = 1$, $T2 = 4$ and $T4 = 2$ .

$$\alpha(x,\ y) = \begin{cases} 1 & if\ (x,\ y) \in A \\ 0 & if\ (x,\ y) \in X \times X \setminus A. \end{cases}$$

Then $T$ is continuous, nondecreasing and $\alpha$- admissible. We choose $x_0 = 1 \in X$, then $x_0 \preceq Tx_0$ and

$\alpha(x_0, Tx_0) = \alpha(1,\ 0) = 1$. Also we choose $z = 0 \in X$, then $z$ is comparable with every $x, y \in X$ and $\alpha(z, Tz) = 1$ and $z \preceq Tz$.

Now, we verify the inequality (2.3.1) by choosing $\psi \in \Psi$ given by $\psi(t) = \frac{2}{3}t$ for $t \geq 0$ and $L = 3$ for comparable elements in $X$.

We check only at $x = 0$ and $y = 2$.

In this case, $\alpha(0,2)d(T0, T2) = 3$, $M(0,2) = \frac{5}{2}$ and $N(0,2) = 1$.

Hence, we have

$$\alpha(0,2)d(T0, T2) = 3 \leq \frac{2}{3}\frac{5}{2} + 3.1 = \frac{14}{3}$$
$$= \psi(M(0,2)) + LN(0,2).$$

At all the remaining points the inequality (2.3.1) holds trivially.

Hence $T$ satisfies the inequality (2.3.1) and hence all the hypotheses of Theorem 2.3 are satisfied and $T$ has a unique fixed point 1.

## IV. Conclusion

In this paper, we proved the existence of fixed points for almost generalized $\alpha$-$\psi$- contractive maps (Theorem 2.1 and Theorem 2.2) in partially ordered sets endowed with a metric.

(i) Example 3.12 and Remark 3.7 suggest that Theorem 2.1 is a generalization of Theorem 1.11; Also, Theorem 2.1 is a generalization of Theorem 1.4 and Theorem 1.7.

(ii) Example 3.14 and Remark 3.9 suggest that Theorem 2.2 is a generalization of Theorem 1.12.

Uniqueness of the fixed points is also studied. Our results generalize the results of Samet, Vetro and Vetro [16], Ciric, Abbas, Saadati and Hussain [6] and Karapinar and Samet [8].

## References


[1] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, "Generalized contractions in partially ordered metric spaces", *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.

[2] I. Altun and H. Simsek, "Some fixed point theorems on ordered metric spaces and application", *Fixed Point Theory and Appl.*, Article ID 621492, 2010.

[3] A. Amini-Harandi and H. Emami, "A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations", *Nonlinear Analysis*, vol. 72, pp. 2238–2242, 2010.

[4] G. V. R. Babu, M. L. Sandhya, M. V. R. Kameswari, "A note on a fixed point theorem of Berinde on weak contractions", *Carpathinen J. Math.*, vol. 24, no. 1, pp. 08-12, 2008.

[5] V. Berinde, "Approximating fixed point theorems of weak contractions using picard iterations", *Nonlinaear Analysis Forum*, vol. 9, no. 1, pp. 43–53, 2004.

[6] L. Ciric, M. Abbas, R. Saadati and N. Hussain, "Common fixed points of almost generalized contractive mappings in ordered metric spaces", *Applied Mathematics and Computation*, vol. 217, pp. 5784–5789, 2011.

[7] L. Ciric, N. Cakic, M. Rajovic and J. S. Ume, "Monotone generalized nonlinear contractions in partially ordered metric spaces", *Fixed point Theory and Applications*, Article ID 131294, 2008.

[8] E. Karapinar and B. Samet, "Generalized $\alpha$-$\psi$- contractive type mappings and related fixed point theorems with applications", *Abstract and Applied Analysis*, Article ID 793486, 2012.

[9] B. Mohammadi, S. Rezapour and N. Shahzad, "Some results on fixed points of $\alpha$-$\psi$ Ciric generalized multifunctions", *Fixed Point Theory and Applications*, 2013.

[10] M. Mursaleen, S. A. Mohiuddine and R. P. Agarwal, "Coupled fixed point theorems for $\alpha$-$\psi$- contractive type mappings in partially ordered metric spaces", *Fixed Point Theory and Applications*, Article ID 228, 2012.

[11] H. K. Nashine and I. Altun, "Fixed point theorems for generalized weakly contractive condition in ordered metric spaces", *Fixed Point Theory and Applications*, Article ID 132367, 2011.

[12] H. K. Nashine, B. Samet, "Fixed point results for mappings satisfying $(\psi, \varphi)$- weakly contractive condition in partially ordered metric spaces", *Nonlinear Analysis*, vol. 74, pp. 2201–2209, 2011.

[13] H. K. Nashine, B. Samet and C. Vetro, "Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces", *Math. Comput. Model.*, vol. 54, pp. 712–720, 2011.

[14] J. J. Nieto, R. Rodriguez- Lopez, "Contraction mapping theorem in partially ordered sets and application to ordinary differential equations", *Order*, vol. 22, pp. 223-239, 2005.

[15] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations", *Proc. of Amer. Math. Soc.*, vol. 132, no. 5, pp. 1435–1443, 2004.

[16] B. Samet, C. Vetro and P. Vetro, "Fixed point theorems for $\alpha$-$\psi$-contractive type mappings", *Nonlinear Analysis*, vol. 75, pp. 2154–2165, 2012.