Dispersion of Oil Spilled Under Ice Covered Sea Using Aris Moment Analysis

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***Abstract*—The mathematical model, presented here, is developed to study the movement and dispersion of oil spilled under solid ice cover. The objective is to establish the equations governing the movement of oil in a rotating environment. These equations are simplified in order to make more pertinent to geophysical flows. The velocity is obtained by employing New Modified Variational Iteration Laplace Transform method and the dispersion of oil is studied using Aris method of moments. The results obtained are discussed graphically illustrating the influence of Ekman number on velocity and dispersion coefficient.**

***Index Terms*—Sea ice, Oil spill, Concentration of oil particle, Aris moment analysis.**

***MSC 2010 Codes* – 76D05, 34A25**

I. Introduction

SPREADING of oil under ice covered sea is a complex process whose description requires large number of various factors such as rapid climate change, remoteness of the area, the low temperatures and seasonal darkness, along with the presence of ice. If oil spill happens in such conditions, it will persist longer as the cold weather slow down the natural weathering and oil is trapped in or under ice. This type of environment poses additional challenges for oil spill response when compared to open waters.

Many researchers reported theoretical and experimental studies to explain the oil spreading mechanism in the presence of an ice cover. Existing research into the behavior of oil spills in sea dates back to the 1970s. In the literature, the results, reports and discussions presented by different authors[1-9], as a combination of analytical studies, laboratory tests and field spills gave a better understanding of the spread of oil under an ice sheet.

Understanding the phenomenon of dispersion under the combined influence of advection and diffusion is of considerable value in many fields. Among the earlier work on dispersion by Taylor[10], Aris[11], Gill and Sankarasubramanian[12], Barton[13] and others, Taylor and Aris model are valid for large time after the injection of solute into the medium. Aris model is an improvement over Taylor model by relaxing some of the assumptions made by Taylor[10]. It is a different approach which shows that the total diffusion coefficient is the sum of the molecular diffusion coefficient and Taylor diffusion coefficient. Following Aris, the basis of our analysis is describing the distribution of oil particles in terms of their moments in the direction of flow.

In this paper, we have developed a model to study the dispersion of oil spilled under solid ice cover. The movement of oil slick with the effect of the coriolis force on its motion are derived from Navier-Stokes equation. The resulting velocity of oil is obtained by employing new modified variational iteration Laplace transform method. The dispersion of oil is studied using Aris method of moments.

II. Mathematical Formulation

Consider the oil slick as a viscous layer of thickness h on a horizontal plane rotating with frequency f, beneath the solid ice cover and lying above the moving water. No layer of water exists in between oil and solid ice. Effect of coriolis force which is significant in sea acts on oil because of its movement. The dimensionless parameter namely Ekman number signifies the coriolis force.

A. Velocity

In the coordinate system rotating about the z-axis, the continuity equation and the Navier-Stokes equations for steady motions depending on the horizontal coordinates x, y and the vertical coordinate z are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+fv \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-fu \tag{3}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-g \tag{4}$$

where, u, v, w are the horizontal and vertical components of velocities along x, y and z directions, respectively, p is the pressure, ρ is the density, ν is the kinematic viscosity and g is the acceleration due to gravity.

We now introduce the following non-dimensional quantities:

$(u^*, v^*, w^*)=\frac{1}{v_0}(u, v, w),\ (x^*, y^*, z^*)=\frac{1}{L}(x, y, z),$

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$p^* = \frac{p}{\rho v_0^2}$
where, v_0 and L are the characteristic velocity and length, respectively.

Dropping the '*' symbol, the reduced non-dimensional governing equations (1) to (4) depending on the horizontal coordinate x and the vertical coordinate z, can be written in the form[14]

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + \frac{v}{Ek} = g_1 \tag{6}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} - \frac{u}{Ek} = 0 \tag{7}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} = Re\frac{\partial p}{\partial z} + \frac{Re}{Fr} \tag{8}$$

where, $g_1 = Re\frac{\partial p}{\partial x}$, $Re = \frac{v_0 L}{\nu}$ is the Reynolds number, $Ek = \frac{\nu}{L^2 f}$ is the Ekman number and $Fr = \frac{v_0^2}{Lg}$ is the Froude number.

The appropriate non-dimensional boundary conditions are written in the form

$$u = u_a,\ v = v_a,\ w = 0\ at\ z = 0,\ \forall x \tag{9}$$

$$u = 0,\ v = 0,\ w = 0\ at\ z = 1,\ \forall x \tag{10}$$

where, $u_a,\ v_a$ are constants.

B. Concentration

In general, the unsteady three dimensional concentration equation is defined by

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) \tag{11}$$

where, c is the concentration of oil and D is the molecular diffusivity.

With Aris statistical method of moments, the spread can be calculated in one direction with the assumption of unidirectional flow that eliminates all convective terms except in the axial one. Thus the concentration of oil, c, which is a function of time, the axial and transverse coordinates (x, z) satisfying the advection-diffusion equation (11) is

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial z^2}\right) \tag{12}$$

Introducing the following non-dimensional quantities
$t^* = \frac{\bar{u}t}{L}, \quad x^* = \frac{x - \bar{u}t}{L}, \quad U^* = \frac{u}{\bar{u}}, \quad c^* = \frac{c}{c_0}$
and dropping the (*) symbol for simplicity, the non-dimensional form of equation (12)becomes

$$\frac{\partial c}{\partial t} + U\frac{\partial c}{\partial x} = \frac{1}{ReSc}\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial z^2}\right) \tag{13}$$

where, $\bar{u}$, c_0 are the average velocity and the characteristic concentration, respectively, U is the non-dimensional relative velocity in the moving coordinate and $Sc = \frac{\nu}{D}$ is the Schmidt number.

The required dimensionless initial and boundary conditions are

$$\begin{aligned} &c(x,z,0) = f(x,z) \\ &\frac{\partial c}{\partial z} = 0\ at\ z = 0,1 \\ &x^p c \to 0,\ x^p c\frac{\partial c}{\partial x} \to 0\ as\ |x| \to \infty,\ p = 0,1,2,\ldots \end{aligned} \tag{14}$$

III. Method of Solution

A. Velocity

The new modified variational iteration Laplace transform method is used to solve the system of momentum equations[15-17]. Taking Laplace transform with respect to z on both sides of equations (6) to (8), the correction functional of the variational iteration method results in the solution for the velocity(neglecting the higher order terms) as

$$u = u_a + \left(g_1 - \frac{v_a}{Ek}\right)\frac{z^2}{2} \tag{15}$$

$$v = v_a + \left(\frac{u_a}{Ek}\right)\frac{z^2}{2} \tag{16}$$

$$w = g_2\frac{z^2}{2} + Re\alpha_0\frac{z^3}{6} \tag{17}$$

This gives the relative velocity of u as

$$U = u - \bar{u} = u_a + \left(g_1 - \frac{v_a}{Ek}\right)\frac{z^2}{2} - g_3 \tag{18}$$

where, the pressure gradient along the z-axis is assumed to be a linear function $\frac{\partial p}{\partial z} = \alpha_0 z$, α_0 being a constant, $g_2 = \frac{Re}{Fr}$ and $g_3 = u_a + \frac{1}{6}\left(g_1 - \frac{v_a}{Ek}\right)$

B. Dispersion Coefficient

Using the Aris method of moments[11], we define the p^{th} moment of concentration distribution through z at time t as

$$c_p(z,t) = \int_{-\infty}^{\infty} x^p c(x,z,t)dx \tag{19}$$

and the p^{th} moment of the distribution over the cross-section of the channel as

$$m_p(t) = \int_0^1 c_p(z,t)dz \tag{20}$$

Using the definitions (19) and (20) in the advection-diffusion equation (13), we get the recursive equations for c_p and m_p as

$$\frac{\partial c_p}{\partial t} - pUc_{p-1} = \frac{1}{ReSc}\left(\frac{\partial^2 c_p}{\partial z^2} + p(p-1)c_{p-2}\right) \tag{21}$$

$$\frac{dm_p}{dt} = p\int_0^1 Uc_{p-1}dz + \frac{1}{ReSc}p(p-1)m_{p-2} \tag{22}$$

The method of statistical moments is used to rewrite the advection-diffusion equation to give additional information about the oil distribution. Since the distribution ultimately tends to normality, the first two moments are ultimately sufficient to describe the distribution. Thus the equations (21) and (22) are solved recursively starting with c_0, c_1,... and

m_0, m_1,...(for p=0,1,...), respectively. Beginning with p=0, equation (22) states $\frac{dm_0}{dt} = 0$, so that m_0 is a constant taken as unity. The zeroth moment m_0 which is a scalar, represents that the total quantity of oil particles is constant.
Now, for p=0 in equation (21) implies

$$\frac{\partial c_0}{\partial t} = g_4 \frac{\partial^2 c_0}{\partial z^2} \tag{23}$$

subject to the initial and boundary conditions

$$c_0(z,0) = c_a e^{\alpha_1 z}$$

$$\frac{\partial c_0}{\partial z} = 0 \ at \ z = 0, 1 \tag{24}$$

where, $g_4 = \frac{1}{ReSc}$, c_a is the resident concentration and α_1 is a constant less than one with its dimension as inverse of space variable z. Using the one dimensional heat equation, the solution for c_0, by the method of separation of variables is obtained as

$$c_0 = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi z) e^{-g_4 n^2 \pi^2 t} \tag{25}$$

where, $A_0 = \frac{2c_a}{\alpha_1}(e^{\alpha_1} - 1)$ and $A_n = \frac{2c_a \alpha_1}{n^2\pi^2}\left[(-1)^n e^{\alpha_1} - 1\right]$
Similarly for p=1, equation (22) reduces to

$$\frac{dm_1}{dt} = \int_0^1 U c_0 dz \tag{26}$$

Solving equation (26), we get

$$\begin{aligned} m_1 = {} & g_5(u_a - g_3)t + \frac{g_5}{6}\left(g_1 - \frac{v_a}{Ek}\right)t \\ & - \left(g_1 - \frac{v_a}{Ek}\right)\sum_{n=1}^{\infty}\frac{(-1)^n A_n e^{-g_4 n^2 \pi^2 t}}{g_4 n^4 \pi^4} \end{aligned} \tag{27}$$

where, $g_5 = \frac{A_0}{2}$. The first moment(m_1) is a vector indicating the position of the time-dependent axial center of mass.
For p=1, equation (21) takes the form

$$\frac{\partial c_1}{\partial t} = g_4 \frac{\partial^2 c_1}{\partial z^2} + U c_0 \tag{28}$$

Using the Laplace transform method and its inverse by Cauchy residue method, we obtain the solution to equation (28) as

$$\begin{aligned} c_1 = {} & g_5(u_a - g_3)t + \frac{g_5}{2}\left(g_1 - \frac{v_a}{Ek}\right)(z^2 + g_4 t)t \\ & + \left(g_1 - \frac{v_a}{Ek}\right)\sum_{n=1}^{\infty}\frac{(-1)^n A_n}{4g_4 n^4\pi^4}\{\, g_4^2 n^6\pi^6 z^2 t^2 - 12 g_4^2 n^4 \pi^4 t^2 \\ & - 4 g_4 n^4 \pi^4 z^2 t + 4n^2\pi^2 z^2 + 28 g_4 n^2 \pi^2 t - 28 \,\} \end{aligned} \tag{29}$$

Substituting p=2 in equation (22), we get

$$\frac{1}{2}\frac{dm_2}{dt} = g_4 + \int_0^1 U c_1 dz \tag{30}$$

According to Aris[11,18], we use the definition of the measure of the effective dispersion coefficient as the variance of the distribution of oil particles about the moving origin, is proportional to $\lim_{t \to \infty} \frac{1}{2}\frac{dm_2}{dt}$. The role of increase of the variance with time equals twice the dispersion coefficient. Thus the measure of the effective dispersion coefficient $\frac{E}{D_e}$,

$$\begin{aligned} \frac{E}{D_e} = {} & g_4 + \int_0^1 U c_1 dz \\ = {} & g_4 + (u_a - g_3)\frac{g_5}{6}\left(g_1 - \frac{v_a}{Ek}\right)(2 + 3g_4 t)t \\ & + g_5(u_a - g_3)^2 t + \frac{g_5}{60}\left(g_1 - \frac{v_a}{Ek}\right)^2 (3 + 5g_4 t)t \\ & + (u_a - g_3)\left(g_1 - \frac{v_a}{Ek}\right)\sum_{n=1}^{\infty}\frac{(-1)^n A_n}{4g_4 n^4\pi^4}\{\, \frac{g_4^2}{3} n^6 \pi^6 t^2 \\ & - 12 g_4^2 n^4\pi^4 t^2 - \frac{4}{3} g_4 n^4 \pi^4 t + 28 g_4 n^2 \pi^2 t + 2n^2\pi^2 - 28 \,\} \\ & + \left(g_1 - \frac{v_a}{Ek}\right)^2 \sum_{n=1}^{\infty}\frac{(-1)^n A_n}{8g_4 n^4\pi^4}\{\, \frac{g_4^2}{5} n^6\pi^6 t^2 - \frac{12}{3} g_4^2 n^4 \pi^4 t^2 \\ & - \frac{4}{5} g_4 n^4 \pi^4 t + \frac{28}{3} g_4 n^2 \pi^2 t + \frac{6}{5} n^2 \pi^2 - \frac{28}{3} \,\} \end{aligned} \tag{31}$$

where, E is the effective dispersion coefficient which is the sum of the particle eddy diffusion coefficient (D_e) and the apparent dispersion coefficient.

IV. Results and Discussion

Numerical evaluation of the analytical results reported in the previous section are displayed graphically. Due to the coriolis effect, the characteristics of geophysical flows vary with the values of Ekman number. Thus the results are obtained to illustrate the influence of Ekman number on velocity and dispersion coefficient, while the values of the other physical parameters are fixed as real constants.

Figure 1 shows the velocity profile for different Ekman number. The figure indicates that an increase in Ekman number results in increasing velocity. Also we see that, as the oil goes deeper, the velocity decreases. This agrees with the results of Yapa and Chowdhury[4] which states that spreading becomes slower for oil of high viscosities.

Dispersion of oil is studied using Aris method of moments. The effective dispersion coefficient E, computed for different Ekman number are discussed in Figure 2. From this figure, we observe that as the Ekman number increases, the dispersion coefficient increases. Also the figure shows that as the oil goes deeper into the layer, there is a decrease in dispersion coefficient. This decrease in dispersibility is because the thicker slick becomes more viscous and the energy required to tear it into small droplets increases. Dispersion coefficient of the under ice slick is much smaller than in open waters because the ice sheet weakens the turbulent wave and the friction also restricts the slick from elongating to some extent.

In general, we know that the smaller values of Ekman number refers the slow frame of earth rotation, increase in Ekman number results in faster earth rotation frame. Thus we observe that for faster frame of rotations the velocity and the dispersion coefficient of oil increases.

V. Conclusion

The physical distributions and conditions of spilled oil under, within or on top of the ice plays a major role in determining the most effective response strategies. The analysis of the proposed model presented here, in particular, is to predict the movement of oil slick under solid ice cover and to determine exact solution for the effective dispersion coefficient. The time dependent dispersion coefficient is evaluated using Aris model which is valid for large time. Graphical results obtained for velocity and dispersion coefficient are presented and discussed.

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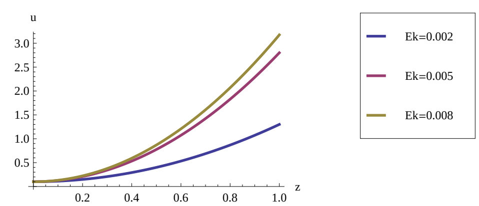

Figure 1: Effect of Ekman number on velocity profile

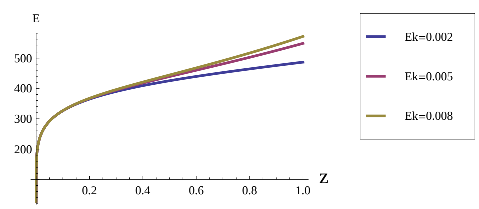

Figure 2:Effect of Ekman number on dispersion coefficient

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