Necessary and Sufficient Conditions for the Existence of Fixed Points of Weak Generalized Geraghty Contractions

G. V. R. Babu, K. K. M. Sarma and P. H. Krishna

***Abstract*—We introduce the notion of (α, φ, β)- weak generalized Geraghty contractions via triangular α-admissible mappings and prove the necessary and sufficient conditions for the existence of fixed points of such maps in complete metric spaces, where φ is an altering distance function and $\beta \in S$ where $S = \{\beta : (0, \infty) \to [0, 1)$ satisfying $\beta(t_n) \to 1 \Rightarrow t_n \to 0\}$. Examples are provided to illustrate our results.**

***Index Terms*—Fixed points, Weak contraction, Altering distance function, Geraghty type contraction.**

***MSC 2010 Codes* – 47H10, 54H25**

I. Introduction

BANACH contraction principle is one of the fundamental results of fixed point theory. Because of its importance in non-linear analysis, a number of researchers have improved and generalized this result. Proving the existence of fixed points of contractive maps is an interesting aspect, since contractive maps are free from contraction constant. In order to establish the existence of fixed points of contractive maps, Geraghty introduced certain type of contraction maps, namely Geraghty contraction maps through which the author developed a technique, where the class of Geraghty contraction maps is one among the generalizations of contraction maps.

In 2012, Samet, Vetro and Vetro [9] introduced the concept of α-ψ-contractive maps where α is an α-admissible mapping which is a new direction in the context of generalization of contraction maps and proved the existence of fixed points of such mappings. In 2013, Karapinar, Kumam and Salimi [7] introduced α-ψ-Meir- Keeler contractive mappings in the setting of complete metric spaces via triangular α-admissible mapping.

In 2013, Cho, Bae and Karapinar [4] introduced the notion of α- Geraghty contraction type maps which are more general than Geraghty contraction maps.

We use the following notation throughout this paper. (X, d) denotes a metric space and we write it as X. Let $T : X \to X$ be a self map of X and $Fix(T)$ denotes the set of all fixed points of T. We denote
$S = \{\beta : (0, \infty) \to [0, 1)/\ \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$, and
$\Phi = \{\varphi : [0, \infty) \to [0, \infty)/\varphi$ is non-decreasing, continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$.
We call the elements of φ as altering distance functions [8].
Further, we use the following notation: for any sequences $\{x_n\}$ and $\{y_n\}$ in X with $x_n \neq y_n$, we write $d_n = d(x_n, y_n)$, $\triangle_n = \frac{d(T(x_n),T(y_n))}{d_n}$ and $\triangle_n^{\varphi} = \frac{\varphi(d(T(x_n),T(y_n)))}{\varphi(d_n)}$ for all n.
We denote the set of all real numbers by R, the set of all nonnegative reals by R^+ and the set of all natural numbers by N.

In complete metric spaces, Geraghty [5] established a criteria for the sequence of Picard iterates defined by $x_0 \in X$, $x_n = Tx_{n-1}$, n = 1, 2, ... to be Cauchy for contractive mappings. If it is Cauchy, it is easy to see that it converges to a unique fixed point of T in X and proved necessary and sufficient condition for a sequence of iterates to be convergent.

Theorem 1.1.[5] Let (X, d) be a complete metric space. Let $T : X \to X$ with

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X,\ x \neq y. \qquad (1.1.1)$$

Let $x_0 \in X$ and set $x_n = Tx_{n-1}$ for n = 1, 2, 3,... . Then $x_n \to x_\infty$ in X, with x_∞ is a unique fixed point of T, if and only if for any two sub-sequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that $\triangle_n \to 1$ only if $d_n \to 0$.

Theorem 1.2. [3] Let T be a self map on a complete metric space $(X,\ d)$. Assume that there exist a $\varphi \in \Phi$ with

$$\varphi(d(Tx,\ Ty)) < \varphi(d(x,\ y)) \text{ for all } x,\ y \text{ in } X \text{ with } x \neq y. \qquad (1.2.1)$$

Let $x_0 \in X$, and set $x_n = Tx_{n-1}$ for n = 1, 2, 3,... . Then $x_n \to z$ in X, with z is a unique fixed point of T if and only if for any two subsequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that $\triangle_n^{\varphi} \to 1$ only if $d_n \to 0$.

Theorem 1.3.[5] Let (X, d) be a complete metric space. Let $T : X \to X$ be a contractive map. Let $x_0 \in X$ and set $x_n = Tx_{n-1}$ for $n > 0$. Then $x_n \to x_\infty$ in X, with x_∞ is a unique fixed point of T, if and only if there exists $\beta \in S$ such that for all $n,\ m \in N$.

$$d(Tx_n, Tx_m) \leq \beta(d(x_n, x_m)).d(x_n, x_m). \qquad (1.3.1)$$

Theorem 1.4.[5] Let (X, d) be a complete metric space. Let $T : X \to X$ be a self mapping. Assume that there exists $\beta \in S$ such that

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \text{ for all } x, y \in X. \qquad (1.4.1)$$

Then T has a unique fixed point $z \in X$ and, for any choice

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of the initial point $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for each $n \geq 1$ converges to the point z.

A selfmap T of a metric space X that satisfies (1.4.1) is called β - Geraghty contraction and these contractions are well known as Geraghty contractions. Here we observe that every contraction is a Geraghty contraction, but its converse need not be true [2].

Definition 1.5. [9] Let $T : X \to X$ be a self map and $\alpha : X \times X \to R$ be a function. Then T is said to be α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Definition 1.6. [7] An α-admissible map T is said to be triangular α-admissible if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ implies $\alpha(x, y) \geq 1$.

For more details and examples on α-admissible and triangular α-admissible maps, we refer [6], [7] and [9].

Lemma 1.7. [7] Let $T : X \to X$ be triangular α-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_{n+1}\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in N$ with $n < m$.

Definition 1.8. [4] Let (X, d) be a metric space, and let $\alpha : X \times X \to R$ be a function. A map $T : X \to X$ is called a 'α-*Geraghty type contraction*' if there exists $\beta \in S$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \text{ for all } x, y \in X. \quad (1.8.1)$$

Theorem 1.9. [4] Let (X, d) be a complete metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:

(i) T is an α-Geraghty type contraction;
(ii) T is triangular α-admissible;
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
(iv) either T is continuous (or) $\liminf\limits_{n\to\infty} \alpha(x_n, x) > 0$ for any cluster point x of a sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq 1$.

Then T has a fixed point u in X and $T^n x_1$ converges to u.

Theorem 1.10. [4] Let (X, d) be a complete metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:

(i) T is an α-Geraghty type contraction;
(ii) T is triangular α-admissible;
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
(iv) either T is continuous (or) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a sub-sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then T has a fixed point u in X and $T^n x_1$ converges to u. In addition to the hypotheses of Theorem 1.10, if for all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then T has a unique fixed point.

Definition 1.11. [4] Let (X, d) be a metric space, and let $\alpha : X \times X \to R$ be a function. A map $T : X \to X$ is called a '*generalized* α-*Geraghty type contraction*' if there exists $\beta \in S$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \beta(M(x, y))M(x, y) \text{ for all } x, y \in X \quad (1.11.1)$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

In 2013, Cho, Bae and Karapinar [4] proved the following existence theorem.

Theorem 1.12. [4] Let (X, d) be a complete metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions hold :

(i) T is generalized α-Geraghty type contraction;
(ii) T is triangular α-admissible;
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;
(iv) either T is continuous (or) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a sub-sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then T has a fixed point u in X and T is a Picard operator, *i.e.*, $T^n x_1$ converges to u.

In addition to the hypotheses of Theorem 1.12, if for all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then T has a unique fixed point.

In Section II, we introduce almost generalized α-contractive maps with an altering distance function and prove the existence of fixed points of such maps in complete metric spaces. Also, we introduce (α, φ, β)-weak generalized Geraghty contraction mappings via triangular α-admissible mappings and prove the existence of fixed points of such maps, where φ is an altering distance function and $\beta : (0, \infty) \to [0, 1)$ satisfying $\beta(t_n) \to 1 \Rightarrow t_n \to 0$. Corollaries and examples in support of our results are given in Section IV.

II. (α, φ, β) - WEAK GENERALIZED GERAGHTY CONTRACTIONS

We now introduce almost generalized α-contractive maps involving an altering distance function.

Definition 2.1. Let (X, d) be a metric space let $T : X \to X$ be a self map. If there exist $\alpha : X \times X \to R$, $\varphi \in \Phi$ and $L \geq 0$ such that

$$\alpha(x, y)\varphi(d(Tx, Ty)) < \varphi(M(x, y)) + L.N(x, y) \quad (2.1.1)$$
$$\text{for all } x,\ y \in X, x \neq y$$

where $M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx))}{2}\}$,
$N(x, y) = min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$
then we say that T is an almost generalized α- contractive map with respect to an altering distance function φ.

Example 2.2. Let $X = \{0, 1, 2, 3, \}$ with the usual metric. We define $T : X \to X$ by $T(0) = 2,\ T(1) = 3,\ T(2) = 2$ and $T(3) = 2$.

Let $A := \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 2), (2, 0), (0, 3), (3, 0), (2, 3), (3, 2)\}$.

Let $B := \{(0, 1), (1, 2), (1, 0)\}$.

Let $C := \{(1, 3), (3, 1), (2, 1)\}$.

We define $\alpha : X \times X \to R$ by

$\alpha(x,y) = \begin{cases} 1 & if\ (x,y) \in A \cup C \\ 2 & otherwise. \end{cases}$

We define $\varphi\ : [0,\infty) \to [0,\infty)$ by $\varphi(t) = 2t, t \geq 0$.

We now verify the inequality (2.1.1).

The inequality (2.1.1) trivially holds for $(x,y) \in A$.

Now we verify the inequality (2.1.1) for $(x,y) \in B$.

Case (i): $(x,\ y) =\ (0,\ 1)$

In this case $\alpha(0,1) = 2, \varphi(d(T(0),T(1)))$ = 2,

M(0, 1) = 2, $\varphi(M(0,1))$ = 4, N(0, 1) = 1

$\alpha(0,1)\varphi(d(T0,T1)) = 4 < 4 + L.1$

$= \varphi(M(0,1)) + L.N(0,1)$

holds with L = 1.

Case (ii):$(x,\ y) =\ (1,\ 2)$

In this case $\alpha(1,2) = 2, \varphi(d(T(1),T(2)))$ = 2,

M(1, 2) = 2, $\varphi(M(1,2))$ = 4, N(1, 2) = 1

$\alpha(1,2)\varphi(d(T1,T2)) = 4 < 4 + L.1$

$= \varphi(M(1,2)) + L.N(1,2)$

holds with L = 1.

Case (iii): $(x,\ y) =\ (1,\ 0)$

In this case $\alpha(1,0) = 2, \varphi(d(T(1),T(0)))$ = 2,

M(1, 0) = 2, $\varphi(M(1,0))$ = 4, N (1, 0) = 1

$\alpha(1,0)\varphi(d(T1,T0)) = 4 < 4 + L.1$

$= \varphi(M(1,0)) + L.N(1,0)$

holds with L = 1.

Now we verify the inequality (2.1.1) for $(x,y) \in C$.

Case (iv): $(x,\ y) =\ (1,\ 3)$

In this case $\alpha(1,3) = 1, \varphi(d(T(1),T(3)))$ = 2,

M(1, 3) = 2, $\varphi(M(1,3))$ = 4, N(1, 3) = 0

$\alpha(1,3)\varphi(d(T1,T3)) = 2 < 4 = \varphi(M(0,1)) + L.N(1,0)$

holds for any $L \geq 0$.

Case (v): $(x,\ y) =\ (3,\ 1)$

In this case $\alpha(3,1) = 1, \varphi(d(T(3),T(1)))$ = 2,

M(3, 1) = 2, $\varphi(M(3,1))$ = 4, N(3, 1) = 0

$\alpha(3,1)\varphi(d(T3,T12)) = 2 < 4 = \varphi(M(3,1)) + L.N(3,2)$

holds for any $L \geq 0$.

Case (vi): $(x,\ y) =\ (2,\ 1)$

In this case $\alpha(2,1) = 1, \varphi(d(T(2),T(1)))$ = 2,

M(2, 1) = 2, $\varphi(M(2,1))$ = 4, N (2, 1) = 0

$\alpha(2,1)\varphi(d(T2,T1)) = 2 < 4 = \varphi(M(2,1)) + L.N(2,1)$

holds for any $L \geq 0$.

Therefore T is an almost generalized α-contractive map with L = 1.

In the following, we introduce (α,φ,β) - weak generalized Geraghty contractions.

Definition 2.3. Let (X,d) be a metric space let $T : X \to X$ be a self map. If there exist $\alpha : X \times X \to R$, $\beta \in S$, $\varphi \in \Phi$ and $L \geq 0$ such that

$$\alpha(x,y)\varphi(d(Tx,Ty)) \leq \beta(\varphi(M(x,y)))\varphi(M(x,y)) + \text{L.N}(x,y) \quad (2.3.1)$$

where $M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$,

$N(x,y) = min\{d(x,Tx), d(x,Ty), d(y,Tx)\}$

for all $x,\ y \in X$ then we say that T is a (α,φ,β) - weak generalized Geraghty contraction.

Remark 2.4 If $\alpha \equiv 1$ in (2.3.1) then we say that T is a (φ,β) - weak generalized Geraghty contraction.

Remark 2.5 If $\alpha \equiv 1$ and φ is the identity map in (2.3.1) then we say that T is a β - weak generalized Geraghty contraction.

Remark 2.6 If φ is the identity map and L = 0 in (2.3.1) then we say that T is an α -β- generalized Geraghty contraction.

Remark 2.7 If $\alpha \equiv 1$ and φ is the identity map and L = 0 in (2.3.1) then we say that T is a β - generalized Geraghty contraction. We call 'β - generalized Geraghty contraction' as 'generalized Geraghty contraction'.

The following is an example of (α,φ,β) - weak generalized Geraghty contraction with $L > 0$.

Example 2.8. Let $X = \{1,2,3,4\}$ with the usual metric. We define $T : X \to X$ by $T(1) = 2,\ T(2) = 3,\ T(3) = 4$ and $T(4) = 4$.

Let $A := \{(1,1),(2,2),(3,3),(4,4),(1,3),(1,4),(2,4),(3,4)\}$.

We define $\beta : (0,\infty) \to [0,1)$ by $\beta(t) = \frac{1}{1+t}\ if\ t > 0$.

We define

$\alpha : X \times X \to R$ by $\alpha(x,y) = \begin{cases} 1 & if\ (x,y) \in A \\ 0 & otherwise. \end{cases}$

We define $\varphi\ : [0,\infty) \to [0,\infty)$ by $\varphi(t) = 2t, t \geq 0$.

We now verify the inequality (2.3.1) for the elements (1, 3), (1, 4), and (2, 4) with L = 4, whereas in the remaining cases the inequality (2.3.1) holds trivially.

Case (i): $(x,\ y) =\ (1,\ 3)$

In this case $\varphi(d(T(1),T(3))$ = 4,

M(1, 3) = 2, N(1, 3) = 1

$\alpha(1,3)\varphi(d(T1,T3)) = 4 \leq \frac{4}{5} + L.1$

$=\beta(\varphi(M(1,3))).\varphi(M(1,3))$ + L.N(1, 3)

holds with L = 4.

Case (ii):$(x,\ y) =\ (1,\ 4)$

In this case $\varphi(d(T(1),T(4)))$ = 4,

M(1, 4) = 3, N(1, 4) = 1

$\alpha(1,4)\varphi(d(T(1),T(4))) = 4 \leq \frac{6}{7} + L.1$

$= \beta(\varphi(M(1,4))).\varphi(M(1,4))$

+ L.N(1, 4) holds with L = 4.

Case (iii): $(x,\ y) =\ (2,\ 4)$

In this case $\varphi(d(T(2),T(4)))$ = 2, M(2, 4) = 2, N(2, 4) = 1

$\alpha(2,4)\varphi(d(T(2),T(4))) = 2 \leq \frac{4}{5} + L.1$

$= \beta(\varphi(M(2,4))).\varphi(M(2,4))$

+ L.N(2, 4) holds with L = 4.

Therefore T is (α,φ,β) - weak generalized Geraghty contraction with L = 4.

The following is an example of α-β - generalized Geraghty contraction.

Example 2.9. Let $X = [0,\ 1]$ with the usual metric. We define $T : X \to X$ by $T(x) = x^2$, and

$\beta : (0,\infty) \to [0,1)$ by $\beta(t) = \frac{1}{1+t}$ if $t > 0$.

We define $\alpha : X \times X \to R$ by

$\alpha(x,y) = \begin{cases} 1 & if\ x,y \in [0,\frac{1}{4}] \\ 0 & otherwise. \end{cases}$

If $x > y$, then $\alpha(x,y)d(Tx,Ty) = x^2 - y^2$, $M(x,y) = x - y$

$\alpha(x,y)d(Tx,Ty) = x^2 - y^2$

$\leq \frac{(x-y)}{1+(x-y)} = \beta((M(x,y))(M(x,y))$

so that T is a α- β-generalized Geraghty contraction. Here we observe that if $\alpha \equiv 1$ then for any $\varphi \in \Phi$ the inequality

(2.3.1) fails to hold whe $x = 1$ and $y = 0$.

The following lemma is useful in proving our main results.

Lemma 2.10. [1] Let $(X,\ d)$ be a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ and

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon$
(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$
(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$ and
(iv) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon$.

III. MAIN RESULTS

First we prove the existence of fixed points of almost contractive maps in complete metric spaces by using an altering distance function via an α-admissible function.

Theorem 3.1. Let T be a self map on a complete metric space X. Let $\alpha : X \times X \to R$ be a function. Assume that there exist $\varphi \in \Phi$, and $L \geq 0$ such that

$$\alpha(x, y)\varphi(d(Tx, Ty)) < \varphi(M(x, y)) + L.N(x, y) \quad \text{for all } x,\ y \in X,\ x \neq y \quad (3.1.1)$$

where $M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$,
$N(x, y) = min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$.
Further, assume that

(i) T is α-admissible, and
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, ...$.

Then the sequence $\{x_n\}$ converges to z and z is a unique fixed point of T in X if and only if for any two sub-sequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ of $\{x_n\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that $\triangle_n^{\varphi} \to 1$ implies $d_n \to 0$, provided that T is continuous at z.

Proof: : First we assume that $x_n \to z$ and z is a unique fixed point of T. Let $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ be any two sub-sequences of $\{x_n\}$ with $x_{h(n)} \neq x_{k(n)}$. Suppose that $\triangle_n^{\varphi} \to 1$ as $n \to \infty$. Now

$$\varphi(d(x_{h(n)},\ x_{k(n)})) = \frac{\varphi(d(x_{h(n)},\ x_{k(n)}))}{\varphi(d(Tx_{h(n)},\ Tx_{k(n)}))} . \varphi(d(Tx_{h(n)},\ Tx_{k(n)})) = \frac{1}{\triangle_n^{\varphi}}.\varphi(d(Tx_{h(n)},\ Tx_{k(n)})).$$

On letting $n \to \infty$ and then by using the continuity of φ, we have that

$$\lim_{n\to\infty} \varphi(d(x_{h(n)},\ x_{k(n)})) = \lim_{n\to\infty} \frac{1}{\triangle_n^{\varphi}}.\varphi(d(Tx_{h(n)},\ Tx_{k(n)})) = \lim_{n\to\infty} \varphi(d(x_{h(n)+1},\ x_{k(n)+1})) = \varphi(d(z,\ z)) = 0.$$

Hence $\varphi(\lim_{n\to\infty} d(x_{h(n)},\ x_{k(n)})) = 0$
so that $\lim_{n\to\infty} d(x_{h(n)},\ x_{k(n)}) = 0$.
Hence $\lim_{n\to\infty} d_n = 0$. *i.e.,* $d_n \to 0$ as $n \to \infty$.

Conversely, assume that $\triangle_n^{\varphi} \to 1$ implies $d_n \to 0$ as $n \to \infty$.

Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$ by (ii). We define $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for each $n = 0, 1, 2, 3, ...$

If $x_n = x_{n+1}$ for some $n \in N$, then $x_n = Tx_n$ and hence x_n is a fixed point of T. Hence, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in N$.

By using the α- admissibility of T, we have
$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$.
Now, by mathematical induction, it is easy to see that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$. (3.1.2)

By taking $x = x_{n-1}$ and $y = x_n$ in the inequality (3.1.1), we get

$$\begin{aligned}\varphi(d(x_n, x_{n+1})) &= \varphi(d(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\varphi(d(Tx_{n-1}, Tx_n)) \\ &< \varphi(M(x_{n-1}, x_n)) + LN(x_{n-1}, x_n), \quad (3.1.3)\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max\ \{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \tfrac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}\} \\ &= \max\ \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2}\} \\ &= \max\ \{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1},x_{n+1})}{2}\} \\ &\leq \max\ \{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\} \\ &= \max\ \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},\end{aligned}$$

and

$$\begin{aligned}N(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}),\ 0\} \\ &= 0 \quad (3.1.4)\end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ for some $n \in N$ then from (3.1.3) and (3.1.4), we have
$\varphi(d(x_n, x_{n+1})) < \varphi(M(x_{n-1}, x_n)) = \varphi(d(x_n, x_{n+1}))$,
a contradiction.

Thus, we have

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$$

for all $n \in N$ and hence
$\varphi(d(x_n, x_{n+1})) < \varphi(d(x_{n-1}, x_n))$ for all $n \in N$.

Thus it follows that $\{\varphi(d(x_n, x_{n+1}))\}$ is a non-negative, decreasing sequence of real numbers.

Suppose that $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = r$, $r \geq 0$.
Now we prove that $r = 0$.

Assume that $r > 0$.

By choosing $h_n = n$ and $k_n = n + 1$
we have $\triangle_n^{\varphi} = \frac{\varphi(d(Tx_n,Tx_{n+1}))}{\varphi(d(x_n,x_{n+1}))}$
on taking limits as $n \to \infty$, then we have $\lim_{n\to\infty} \triangle_n^{\varphi} = 1$.

Hence by our assumption $d_n \to 0$ as $n \to \infty$ so that $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0$,
i.e., $r = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X.

Suppose that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 2.10, there exists some $\epsilon > 0$, such that we can find sub-sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that $d(x_{n(k)}, x_{m(k)}) \geq \epsilon$ and $d(x_{n(k)-1}, x_{m(k)}) < \epsilon$ and we have the following identities.
$(i) \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon$, $(ii) \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$.

Hence $\lim_{k\to\infty} \triangle_k^\varphi = \lim_{k\to\infty} \frac{\varphi(d(Tx_{m(k)}, Tx_{n(k)}))}{\varphi(d(x_{m(k)}, x_{n(k)}))}$
$= \frac{\varphi(\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}))}{\varphi(\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}))}$
$= \frac{\varphi(\epsilon)}{\varphi(\epsilon)} = 1$.

Now, by our assumption we have that $d_k \to 0$ as $k \to \infty$.
i.e., $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = 0$, a contradiction to (ii).

Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Now, we show that z is a fixed point of T.

We consider

$$\begin{aligned}\varphi(d(x_n, Tx_n)) &= \varphi(d(x_n, x_{n+1})) \\ &= \varphi(d(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\varphi(d(Tx_{n-1}, Tx_n)) \\ &< \varphi(M(x_{n-1}, x_n))L.N(x_{n-1}, x_n). \quad (3.1.5)\end{aligned}$$

On letting $n \to \infty$, by using the continuity of T, we get
$\varphi(d(z, Tz)) \leq \varphi(M(z, z)) + L.N(z, z) = 0$.

Hence $\varphi(d(z, Tz)) = 0$.

Since $\varphi \in \Phi$, it follows that $z = Tz$. Hence z is a fixed point of T in X. ■

If $L = 0$ in the inequality (3.1.1) we have the following corollary.

Corollary 3.2. Let T be a self map on a complete metric space X. Let $\alpha : X \times X \to R$ be a function. Assume that there exists $\varphi \in \Phi$ such that

$\alpha(x, y)\varphi(d(Tx, Ty)) < \varphi(M(x, y))$
for all $x,\ y = Tx \in X$. (3.2.1)

Further, assume that

(i) T is α-admissible, and
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and set $x_n = Tx_{n-1}$ for n = 1, 2, 3, ...

Then the sequence $\{x_n\}$ converges to z and z is a unique fixed point of T in X if and only if for any two sub-sequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ of $\{x_n\}$ with $x_{h(n)} \neq x_{k(n)}$, we have that $\triangle_n^\varphi \to$1 implies $d_n \to 0$, provided T is continuous at z.

Theorem 3.3. Let T be a self map on a complete metric space X. Let $\alpha : X \times X \to R$ be a function. Assume that there exist $\varphi \in \Phi$ and $k \in [0,\ 1)$such that

$$\alpha(x, y)\varphi(d(Tx, Ty)) \leq k\varphi(d(x, y)) \text{ for all } x,\ y \in X. \quad (3.3.1)$$

Further, assume that

(i) T is triangular α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and set $x_n = Tx_{n-1}$ for n = 1, 2, 3, ... and
(iii) for any two sub-sequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with $x_{h(n)} \neq x_{k(n)}$, we have $\lim_{n\to\infty} \alpha(x_{h(n)}, x_{k(n)}) = 1$.

Then the sequence $\{x_n\}$ converges to z and z is a unique fixed point of T in X if and only if there exists $\beta \in S$ such that for all $n,\ m \in N$.

$$\alpha(x_n, x_m)\varphi(d(Tx_n, Tx_m)) \leq \beta(d(x_n, x_m))\varphi(d(x_n, x_m)). \quad (3.3.2)$$

Proof: Since the inequality (3.3.1) implies (3.2.1), by the Corollary 3.2, it is sufficient to show that there exists β in S such that (3.3.2) holds if and only if $\triangle_n^\varphi \to$1 implies $d_n \to 0$.
Suppose that such β exists in S. Let $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ be two sub-sequences with $x_{h(n)} \neq x_{k(n)}$.
Now we assume that $\triangle_n^\varphi \to$1.
From (3.3.1), we have
$\alpha(x_{h(n)}, x_{k(n)})\frac{\varphi(d(Tx_{h(n)}, Tx_{k(n)}))}{\varphi(d(x_{h(n)}, x_{k(n)}))} \leq \beta(d(x_{h(n)}, x_{k(n)})) < 1$.
On letting $n \to \infty$, we get
$1 \leq \lim_{n\to\infty} \beta(d(x_{h(n)}, x_{k(n)})) \leq 1$
Hence $\lim_{n\to\infty} \beta(d(x_{h(n)}, x_{k(n)})) = 1$.
Since β is in S, it follows that $d(x_{h(n)}, x_{k(n)}) \to 0$ as $n \to \infty$.
i.e., $d_n \to 0$ as $n \to \infty$.

Conversely, assume that the sequential condition holds. *i.e.,* $\triangle_n^\varphi \to$1 implies $d_n \to 0$ as $n \to \infty$.
We define $\beta : R^+ \mapsto R$ as follows:
$\beta(t)$ = sup $\{\alpha(x_n, x_m)\frac{\varphi(d(Tx_n, Tx_m))}{\varphi(d(x_n, x_m))}/d(x_n, x_m) \geq t\}$.
from the inequality (3.2.1), we have
$\alpha(x_n, x_m)\frac{\varphi(d(Tx_n, Tx_m))}{\varphi(d(x_n, x_m))} \leq k$ for all $n,\ m$ with $x_n \neq x_m$.
Hence $\beta(t) \leq k$, but $k < 1$ implies that $\beta(t) < 1$ for all $t > 0$. So β is defined for all $t > 0$ and $\beta(t) \leqq 1$. Also $\beta(d(x_n, x_m)) \geq \alpha(x_n, x_m)\frac{\varphi(d(Tx_n, Tx_m))}{\varphi(d(x_n, x_m))}$. Hence $\alpha(x_n, x_m)\varphi(d(Tx_n, Tx_m)) \leq \beta(d(x_n, x_m))\varphi(d(x_n, x_m))$.
Now assume that $\beta(t_n) \to 1$ as $n \to \infty$ for a sequence $\{t_n\}$ in $(0,\ \infty)$.
Without loss of generality, we assume that there exist a sequence $\{s_n\}$ in $(0,\ \infty)$ such that $\lim_{n\to\infty} s_n = 0$ and $1 - s_n < \beta(t_n) < 1$.
Now we have to show that $t_n \to 0$.
Since $\beta(t_n)$ is the above least upper bound so there is for each $n > 0$, there exist two sub-sequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ with d$(x_{h(n)},\ x_{k(n)}) \geq t_n$ and
$1 - s_n < \alpha(x_{h(n)}, x_{k(n)})\frac{\varphi(d(Tx_{h(n)}, Tx_{k(n)}))}{\varphi(d(x_{h(n)}, x_{k(n)}))} \leq \beta(t_n) < 1$.
On letting $n \to \infty$, we get
$\lim_{n\to\infty}(1 - s_n) \leq \lim_{n\to\infty} \alpha(x_{h(n)}, x_{k(n)})\triangle_n^\varphi \leq \lim_{n\to\infty} \beta(t_n) = 1$.
Hence $\lim_{n\to\infty} \triangle_n^\varphi = 1$.
i.e., Hence by our assumption, we have $d_n \to 0$ as $n \to \infty$ so that $t_n \to 0$. ■

In the following, we prove the existence of fixed points of (α, φ, β) - weak generalized Geraghty contraction type maps in a complete metric space.

Theorem 3.4. Let (X, d) be a complete metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a self map. Suppose that the following conditions hold:

(i) T is (α, φ, β) - weak generalized Geraghty contraction;
(ii) T is triangular α-admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and set

$x_n = Tx_{n-1}$ for n = 1, 2, 3, . . . ;

(iv) either (a) T is continuous (or)
(b) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a sub-sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then T has a fixed point u in X, provided that β is continuous on $(0, \infty)$.

Proof: Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. We define $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for each n.

If $x_n = x_{n+1}$ for some $n \in N$, then $x_n = Tx_n$ and hence x_n is a fixed point of T. Hence, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in N$.
Since $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$, by using α-admissible property of T, we have $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$.
By mathematical induction, it follows

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in N. \quad (3.4.1)$$

We consider

$$\begin{aligned}\varphi(d(x_{n+1}, x_{n+2})) &= \varphi(d(Tx_n, Tx_{n+1})) \\ &\leq \alpha(x_n, x_{n+1})\varphi(d(T(x_n, Tx_{n+1}))) \\ &\leq \beta(\varphi(M(x_n, x_{n+1})))\varphi(M(x_n, x_{n+1})) \\ &\quad + L.N(x_n, x_{n+1}) \quad (3.4.2)\end{aligned}$$

where

$$\begin{aligned} &= \max\ \{d(x_n, x_{n+1}), d(x_n, Tx_n), \\ &\quad d(x_{n+1}, Tx_{n+1}), \tfrac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\} \\ &= \max\ \{d(x_n, x_{n+1}), d(x_n, x_{n+1}), \\ &\quad d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1}))}{2}\} \\ &= \max\ \{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+2})}{2}\} \\ &\leq \max\ \{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\quad \tfrac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\} \\ &= \max\ \{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}\end{aligned}$$

and

$$\begin{aligned} N(x_n, x_{n+1}) &= \min\ \{d(x_n, Tx_n), d(x_n, Tx_{n+1}), \\ &\quad d(x_{n+1}, Tx_n)\} \\ &= \min\ \{d(x_n, x_{n+1}), d(x_n, x_{n+2}), \\ &\quad d(x_{n+1}, x_{n+1})\} \\ &= \min\ \{d(x_n, x_{n+1}), d(x_n, x_{n+2}), 0\} \\ &= 0.\end{aligned}$$

If $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$ for some $n \in N$ then from (3.2.2), we have

$$\begin{aligned}\varphi(d(x_{n+1}, x_{n+2})) &\leq \beta(\varphi(M(x_n, x_{n+1})))\varphi(M(x_n, x_{n+1})) \\ &\leq \beta(\varphi(M(x_n, x_{n+1})))\varphi(d(x_{n+1}, x_{n+2})) \\ &< \varphi(d(x_{n+1}, x_{n+2})),\end{aligned}$$

a contradiction.
So we have

$$max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1})$$

for all $n \in N$ and hence

$$\begin{aligned}\varphi(d(x_{n+1}, x_{n+2})) &\leq \beta(\varphi(M(x_n, x_{n+1})))\varphi(d(x_n, x_{n+1})) \\ &\quad + \text{L}.0 \\ &< \varphi(d(x_n, x_{n+1})) \text{ for all } n \in N.\end{aligned}$$

Thus it follows that $\{\varphi(d(x_n, x_{n+1}))\}$ is non-negative, decreasing sequence of real numbers and so $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1}))$ exists and it is r (say).
So $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = r \geq 0$.
We now show that r = 0.
If $r > 0$ then from (3.2.2) we have

$$\begin{aligned}\varphi(d(x_{n+1}, x_{n+2})) &= \varphi\ (d(Tx_n, Tx_{n+1})) \\ &\leq \beta(\varphi(M(x_n, x_{n+1})))\varphi(M(x_n, x_{n+1})) \\ &\quad + LN(x_n, x_{n+1}) \\ &\leq \beta(\varphi(M(x_n, x_{n+1})))\varphi(d(x_n, x_{n+1})) \\ &\quad + LN(x_n, x_{n+1}), \text{ and hence}\end{aligned}$$

$$\frac{\varphi(d(x_{n+1}, x_{n+2}))}{\varphi(d(x_n,\ x_{n+1}))} \leq \beta(\varphi(M(x_n, x_{n+1}))) + L\frac{N(x_n, x_{n+1})}{\varphi(d(x_n,\ x_{n+1}))}\ .$$

On letting $n \to \infty$, we get

$$1 = \lim_{n\to\infty} \frac{\varphi(d(x_{n+1}, x_{n+2}))}{\varphi(d(x_n, x_{n+1}))} \leq \lim_{n\to\infty} \beta(\varphi(M(x_n, x_{n+1}))) \leq 1$$

so that $\beta(\varphi(M(x_n,\ x_{n+1}))) \to 1$ as $n \to\ \infty$. This implies that $\lim_{n\to\infty} \varphi(M(x_n, x_{n+1}))$ = 0.
Hence $\varphi(\lim_{n\to\infty} M(x_n, x_{n+1})) = 0$.
Now by the property of φ, we have $\lim_{n\to\infty} M(x_n, x_{n+1}) = 0$.
Now, it is clear that $r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
Now, we show that $\{x_n\}$ is a Cauchy sequence in X.
Suppose that $\{x_n\}$ is not a Cauchy sequence. Hence, by Lemma 2.10, there exist $\epsilon > 0$ and sequences of positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ satisfying (i) to (iv) of Lemma 2.10.
Now, taking $x = x_{m(k)-1}, y = x_{n(k)}$ in (2.3.1), it follows that

$$\begin{aligned}\varphi(d(x_{m(k)},\ x_{n(k)+1})) &= \varphi(d(Tx_{m(k)-1}, Tx_{n(k)})) \\ &\leq \alpha(x_{m(k)-1}, x_{n(k)})\varphi(d(T(x_n, Tx_{n+1}))) \\ &\leq \beta(\varphi(M(x_{m(k)-1}, x_{n(k)})) \\ &\quad \varphi(M(x_{m(k)-1}, x_{n(k)}))+ \\ &\quad L.N(x_{m(k)-1}, x_{n(k)})\end{aligned}$$

where

$$\begin{aligned}M(x_{m(k)-1}, x_{n(k)}) = max\{&d(x_{m(k)-1}, x_{n(k)}), \\ &d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), \\ &\tfrac{d(x_{m(k)-1},\ x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})}{2}\}.\end{aligned}$$

On letting $k \to \infty$ and from Lemma 2.10 we get
$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}) = max\{\epsilon,\ 0,\ \epsilon\} = \epsilon$.
Also, we have

$$\begin{aligned}N(x_{m(k)-1}, x_{n(k)}) = min\{&d(x_{m(k)-1},\ x_{m(k)}), \\ &d(x_{m(k)-1},\ x_{n(k)+1}),\ d(x_{n(k)}\ , x_{m(k)})\}.\end{aligned}$$

On letting $k \to \infty$ and from Lemma 2.10, we get
$\lim_{k\to\infty} N(x_{m(k)-1}, x_{n(k)}) = min\{0,\ \epsilon,\ \epsilon\} = 0$.
Now, we have

$$\begin{aligned}\varphi(d(x_{m(k)},\ x_{n(k)+1})) \leq \beta(&\varphi(M(x_{m(k)-1}, x_{n(k)}))) \\ &\varphi(M(x_{m(k)-1}, x_{n(k)})) + \\ &LN(x_{n(k)}, x_{m(k)-1})\end{aligned}$$

and hence

$$\frac{\varphi(d(x_{m(k)},\ x_{n(k)+1}))}{\varphi(M(x_{m(k)-1},\ x_{n(k)}))} \leq \beta(\varphi(M(x_{m(k)-1}, x_{n(k)}))) + L\frac{N(x_{n(k)}, x_{m(k)-1})}{\varphi(M(x_{m(k)-1},\ x_{n(k)}))}.$$

On letting $k \to \infty$ and from Lemma 2.10, we get
$1 = \frac{\varphi(\epsilon)}{\varphi(\epsilon)} \leq \lim_{k\to\infty} \beta(\varphi(M(x_{m(k)-1}, x_{n(k)}))) \leq 1$ so that
$\beta(\varphi(M(x_{m(k)-1}, x_{n(k)}))) \to 1$ as k $\to \infty$.
Since $\beta \in S$, $\varphi(M(x_{n(k)}, x_{m(k)-1})) \to 0$ as k $\to \infty$.
i.e., $\varphi(\epsilon)$ = 0, since φ is continuous.

Hence it follows that ϵ = 0, a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence in X, and since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now, we show that u is a fixed point of T.

First suppose that T is continuous. In this case we have $x_{n+1} = Tx_n \to Tu$ as $n \to \infty$.

By the uniqueness of the limit, we get $u = Tu$ so that u is a fixed point of T in X.

Now, we suppose that the condition (iv) (b) holds. Since we have $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, we have that there exists a sub-sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Now, we show that u is a fixed point of T. Otherwise *i.e.*, if $d(u, Tu) > 0$ then by taking $x = x_{n(k)}$ and $y = u$ in (2.3.1), we have

$$\begin{aligned}\varphi(d(x_{n(k)+1}, Tu)) &= \varphi(d(Tx_{n(k)}, Tu)) \\ &\leq \alpha(x_{n(k)}, u)\varphi(d(Tx_{n(k)}, Tu)) \\ &\leq \beta(\varphi(M(x_{n(k)}, u)))\varphi(M(x_{n(k)}, u)) \\ &\quad + L.N(x_{n(k)}, u) \qquad (3.4.4)\end{aligned}$$

where

$$M(x_{n(k)}, u) = max\{d(x_{n(k)}, u), d(x_{n(k)}, x_{n(k)+1}), d(u, Tu), \frac{d(x_{n(k)}, Tu) + d(u, x_{n(k)+1})}{2}\}.$$

On letting $k \to \infty$, we get

$$\begin{aligned}\lim_{k\to\infty} M(x_{n(k)}, u) &= max\{d(u, u), d(u, u), d(u, Tu), \frac{d(u, Tu) + d(u, u)}{2}\} \\ &= d(u, Tu).\end{aligned}$$

Also

$$N(x_{n(k)}, u) = min\{d(x_{n(k)}, x_{n(k)+1}), d(x_{n(k)}, Tu), d(u, x_{n(k)+1})\}, \text{ and}$$

$$\begin{aligned}\lim_{k\to\infty} N(x_{n(k)}, u) &= min\{d(u, u), d(u, Tu), d(u, u)\} \\ &= 0.\end{aligned}$$

From (3.4.4), we have

$$\lim_{k\to\infty} \varphi(d(x_{n(k)+1}, Tu)) \leq \lim_{k\to\infty} \beta(\varphi(M(x_{n(k)}, u))) \lim_{k\to\infty} \varphi(M(x_{n(k)}, u)) + \lim_{k\to\infty} L.N(x_{n(k)}, u).$$

Since β is continuous, it follows that

$$\begin{aligned}\varphi(d(u, Tu)) &\leq \beta(\varphi(d(u, Tu)))\varphi(d(u, Tu)) \\ &< \varphi(d(u, Tu)),\end{aligned}$$

a contradiction.

Hence it follows that $u = Tu$.

This completes the proof of the theorem. ■

IV. Corollaries and Examples

Remark 4.1. If we choose $\alpha \equiv 1$ and φ is the identity map then Theorem 1.1 (Geraghty Theorem) follows as a corollary to Theorem 3.1, since the inequality (1.1.1) implies the inequality (3.1.1) with $L = 0$.

Remark 4.2. If we choose $\alpha \equiv 1$ then Theorem 1.2 follows as a corollary to Theorem 3.1, since the inequality (1.2.1) implies the inequality (3.1.1) with $L = 0$.

Corollary 4.3. Let (X, d) be a complete metric space, let $T : X \to X$ be a map. Assume that

(i) T is a β- weak generalized Geraghty contraction;

(ii) T is continuous.

Then T has a fixed point in X, provided that β is continuous on $(0, \infty)$.

Proof: Since T is a β - weak generalized Geraghty contraction, the inequality (2.3.1) holds with $\alpha \equiv 1$ and φ the identity map.

For any $x_0 \in X$, we define $\{x_n\} \subset X$ by $x_n = Tx_{n-1}$ for $n = 1, 2, 3, ...$ then all the hypotheses of Theorem 3.4 hold and T has a fixed point. ■

Remark 4.4. Choosing φ the identity map and $L = 0$ in the inequality (2.3.1), we have T is a generalized α- Geraghty contraction and hence Theorem 1.12 follows as a corollary to Theorem 3.4.

Example 4.6. Let $X = [0, \frac{1}{2}]$ with the usual metric.

We define $T : X \to X$ by $Tx = \frac{1}{2} - x$.

We define $\alpha : X \times X \to R$ by

$$\alpha(x, y) = \begin{cases} 1 & if \quad 0 < x < \frac{1}{2} \; and \; y = \frac{1}{4} \\ 0 & otherwise \end{cases}$$

and define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = t.e^{(1-t)}$ if $t \geq 0$.

If $x \in (0, \frac{1}{2})$ and $y = \frac{1}{4}$, then $Tx = \frac{1}{2} - x \in (0, \frac{1}{2})$, $T\frac{1}{4} = \frac{1}{4}$ $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$. Therefore T is α admissible.

Choosing $x_0 = \frac{1}{4}$ we have $Tx_0 = \frac{1}{4}$ and $\alpha(x_0, Tx_0) \geq 1$ and hence condition (ii) of Theorem 3.1 is satisfied.

Now, we verify the inequality

$\alpha(x, y)\varphi(d(Tx, Ty)) < \varphi(M(x, y)) + L.N(x, y)$ for all $x, \ y \in X, \, x \neq y$.

Let $x, y \in [0, \frac{1}{2}]$ we have that $\alpha(x, y) = 0$ for $x = 0$ or $\frac{1}{2}$ and $y \neq \frac{1}{4}$. Hence the inequality holds trivially in this case. So it is enough to verify the inequality when $(x, y) \in (0, \frac{1}{2})$ and $y = \frac{1}{4}$.

Let $x, y \in (0, \frac{1}{2})$ and $y = \frac{1}{4}$

$\alpha(x, \frac{1}{4})\varphi(d(T(x), T(\frac{1}{4}))) = 1.\varphi(d(\frac{1}{2} - x, \frac{1}{4})) = \varphi(| \, x - \frac{1}{4} \, |)$

$M(x, \frac{1}{4}) = 2 \mid x - \frac{1}{4} \mid$ and $N(x, \frac{1}{4}) = \mid x - \frac{1}{4} \mid$

For $x \neq \frac{1}{4}$

$$\begin{aligned}\alpha(x, \tfrac{1}{4})\varphi(d(T(x), T(\tfrac{1}{4}))) = e^{1-|x-\frac{1}{4}|} &< 2.e^{1-2|x-\frac{1}{4}|} \\ &\quad + L. \mid x - \tfrac{1}{4} \mid \\ &= \varphi(M(x, \tfrac{1}{4})) + LN(x, \tfrac{1}{4})\end{aligned}$$

Therefore the inequality (3.1.1) holds for any $L \geq 0$.

$\triangle_n^{\varphi} = \frac{\varphi(d(T(x_n), T(y_n)))}{\varphi(d_n)}$

$\triangle_n^{\varphi} = \frac{\varphi(d(\frac{1}{2} - x_{h(n)}, \frac{1}{2} - x_{k(n)}))}{\varphi(d(x_{h(n)}, \ x_{k(n)}))} \to 1$ as $n \to \infty$. Also $d_n = d(x_{h(n)}, \ x_{k(n)}) \to 0$ as $n \to \infty$. Therefore T satisfy all the hypotheses of Theorem 3.1, and T has a unique fixed point $\frac{1}{4}$.

In the following, we present an example in support of Theorem 3.3.

Example 4.7. Let $X = [0, 2]$ with the usual metric.

We define $T : X \to X$ by $Tx = \begin{cases} x^2 & if \; 0 \leq x < \frac{1}{4} \\ 2x & if \; \frac{1}{4} < \; x \leq 1 \\ \frac{x+1}{2} & if \; 1 < \; x \leq 2. \end{cases}$

We define $\alpha : X \times X \to R$ by

$$\alpha(x, y) = \begin{cases} 1 & if \quad 0 \leq x, y \leq \frac{1}{4} \\ 0 & otherwise. \end{cases}$$

We define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = t^2, t \geq 0$, and $\beta : (0, \infty) \to [0, 1)$ by $\beta(t) = \frac{3}{4}, t > 0$.

Then $\varphi \in \Phi$ and $\beta \in S$.

(i) If $x, \ y \in [0, \frac{1}{4}]$, then

$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) = \alpha(x^2, y^2) \geq 1$ and

$\alpha(x, y) \geq 1, \ \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$.

Therefore T is a triangular α-admissible map.

(ii) We choose $x_0 = \frac{1}{8}$. In this case $\alpha(x_0, Tx_0) = \alpha(\frac{1}{8}, \frac{1}{64}) \geq 1$.
(iii) For $x_0 = \frac{1}{8}$, the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ is in $[0, \frac{1}{4}]$ and for any subsequences $\{x_{h(n)}\}$ and $\{x_{k(n)}\}$ of $\{x_n\}$ in $[0, \frac{1}{4}]$ with $x_{h(n)} \neq x_{k(n)}$ we have $\lim_{n\to\infty} \alpha(x_{h(n)}, x_{k(n)}) = 1$.
Now we verify the inequality (3.3.1)
If $x,\ y \in [0, \frac{1}{4}]$, we have
$\alpha(x,y)\varphi(d(Tx,Ty)) = (x^2-y^2)^2 \leq \frac{1}{4}(x-y)^2 = k\varphi(d(x,y))$ holds with $k = \frac{1}{4}$.
Now we verify the inequality (3.3.2)
$\alpha(x_n, x_m)\varphi(d(Tx_n, Tx_m)) = (x_n + x_m)^2(x_n - x_m)^2 \leq \frac{3}{4}(x_n - x_m)^2 = \beta(d(x_n, x_m))\varphi(d(x_n, x_m))$
holds with $\beta(t) = \frac{3}{4}$.
When $x,\ y \in (\frac{1}{4}, 2]$ the inqualities (3.3.1) and (3.3.2) hold trivially. Therefore T satisfies all the conditions of Theorem 3.3 and T has a unique fixed point 0.
Now we present an example in support of Theorem 3.4.
Example 4.8. Let $X = [0, 5]$ with the usual metric.

We define $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{3} & if\ 0 \leq x < 1 \\ 2x - \frac{5}{3} & if\ 1 \leq\ x \leq \frac{10}{3} \\ 5 & if\ \ \frac{10}{3} <\ x \leq 5. \end{cases}$$

We define $\alpha : X \times X \to R$ by

$$\alpha(x,y) = \begin{cases} 1 & if\ \ 2 \leq x \leq 5\ and\ y = 5 \\ 0 & otherwise. \end{cases}$$

We define $\varphi\ : [0,\infty) \to [0,\infty)$ by $\varphi(t) = 2t, t \geq 0$, and define $\beta : (0,\infty) \to [0,1)$ by $\beta(t) = \frac{1}{1+t}, t > 0$.
Then $\varphi \in \Phi$ and $\beta \in S$. We now verify the inequality (2.3.1).
If $x \in [2, \frac{10}{3}]$ and $y = 5$, then $Tx = 2x - \frac{5}{3} \in [\frac{7}{3}, 5] \subseteq [2, 5]$ and $T5 = 5$
If $x \in [\frac{10}{3}, 5]$ and $y = 5$, then $Tx = 5$ and $T5 = 5$.
Therefore T is α- admissible and also triangular α- admissible.
By choosing $x_0 = \frac{10}{3}$ we have $\alpha(x_0, Tx_0) = \alpha(\frac{10}{3}, 5) = 1$.
Therefore condition (ii) of Theorem (3.2) is satisfied.
Taking $x_n = 5$ for all n, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, and $x_n \to 5$.
Therefore condition (iii) of Theorem (3.2) is satisfied.
Now we verify the inequality (2.3.1)
Case (i): If $x \in [2, \frac{10}{3}]$ and $y = 5$
$\alpha(x,5)\varphi(d(T(x),T(5))) = 1.\varphi(d(2x - \frac{5}{3}, 5)) = \varphi(\frac{20}{3} - 2x)$
$M(x,5) = 5 - x$, $\varphi(M(x,5)) = 10 - 2x$,
$\beta(\varphi(M(x,5))) = \frac{10-2x}{11-2x}$
Sub case (i): If $2 \leq x < \frac{25}{9}$, $N(x,5) = (x - \frac{5}{3})$

$$\begin{aligned} \alpha(x,5)\varphi(d(T(x),T(5))) &= (\tfrac{40}{6} - 4x) \\ &\leq \tfrac{10-2x}{11-2x} + 20.\ (x - \tfrac{5}{3}) \\ &= \beta(\varphi(M(x,5))(\varphi(M(x,5)) \\ &\quad + L.N(x,5) \end{aligned}$$

holds with L = 20.
Sub case (ii): If $\frac{25}{9} \leq x < \frac{10}{3}$, $N(x,5) = (\frac{20}{3} - 2x)$

$$\begin{aligned} \alpha(x,5)\varphi(d(T(x),T(5))) &= (\tfrac{40}{6} - 4x) \\ &\leq \tfrac{10-2x}{11-2x} + 20.\ (\tfrac{20}{3} - 2x) \\ &= \beta(\varphi(M(x,5))(\varphi(M(x,5)) \\ &\quad +L.N(x,5) \end{aligned}$$

holds with L = 20.
Therefore the inequality (2.3.1) holds for $L = 20$.
Case (ii): If $x \in [\frac{10}{3}\ , 5]$ and $y = 5$,
then Tx = 5 and $T5$ = 5,
the inequality (2.1.1) trivially holds.
Therefore T satisfies all the conditions of Theorem 3.1 with L = 20 and T has three fixed points $\frac{1}{3}, \frac{5}{3}$ and 5.
$Importance\ of\ \alpha$: If $\alpha(x\ ,y)$ = 1 for all $(x,y) \in X \times X$ then the inequality (2.3.1) fails to hold for x= $\frac{5}{3}$ and y = 5. For

$$\begin{aligned} \alpha(\tfrac{5}{3},5)\varphi(d(T(\tfrac{5}{3}),T(5))) &= \varphi(\tfrac{10}{3}) \nleq \beta(\varphi(\tfrac{10}{3}))(\varphi(\tfrac{10}{3})) + L.0 \\ &= \beta(\varphi(M(\tfrac{5}{3},5))(\varphi(M(\tfrac{5}{3},5)) \\ &\quad + L.N(\tfrac{5}{3},5). \end{aligned}$$

Therefore the inequality (2.3.1) fails to hold for any $\varphi \in \Phi$ and for any $L \geq 0$ when $\alpha(x,y)$ = 1 for all $x,\ y \in X$.
Inparticular, if φ is the identity map then also the inequality (2.3.1) fails to hold for any L.

V. Conclusion

In this paper, we proved necessary and sufficient conditions for the existence of fixed points of almost contractive maps using an altering distance function via α-admissible function (Theorem 3.1). Further we extended this result for triangular α-admissible map with Geraghty type contractions (Theorem 3.3). Also, we proved the existence of fixed points of (α, φ, β)-weak generalized Geraghty contraction type maps in a complete metric spaces (Theorem 3.4). We derived some known results as corollaries from our theorems in Section IV. Examples are provided in support of our results to show the importance of α.

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